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ON COMMON FIXED POINT THEOREMS UNDER IMPLICIT RELATION IN 2-BANACH SPACES

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Abstract. In this paper, we establish some common fixed point theorems using implicit relation which is a weaker contraction condition in the framework of 2-Banach spaces. Our results extend and generalize some recent results from the existing literature.

1. Introduction

The concept of 2-Banach space and some basic fixed point results in such spaces are initially given by Gahler ([3], [4]) during $1960's$. Later on some fixed point results have been obtained in such spaces by Iseki [5], Khan *et al.* [6], Rhoades [7] and many others extending the fixed point results for non expansive mappings from Banach space to 2-Banach space. In 2011, Choudhury and Som [2] (J. Indian Acad. Math. 33(2) (2011), 411-418) have established common fixed point and coincidence fixed point results for a pair of non-linear mappings in 2-Banach space which generalize the results of Som [9], Cho *et al.* [1] and Zhao [10] in turn. Recently, Saluja [8] (International J. Math. Combin. 1(2014), 13-18) established some unique fixed point theorems satisfying the contractive type condition in 2-Banach spaces.

In this paper, we establish some common fixed point theorems under implicit relation which is a weaker contractive condition in the framework of 2-Banach spaces. Our results extend and improve some previous work from the existing literature.

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2. Preliminaries

Here we give some preliminary definitions related to 2-Banach spaces which are needed in the sequel.

Definition 2.1. ([1]) Let X be a linear space and $\|\cdot,\cdot\|$ be a real valued function defined on X satisfying the following conditions:

(i) $\|x,\, y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x,\, y\| = \|y,\, x\|$ for all $x, y \in X$;
(iii) $\|x,\, ay\| = |a|\|x,\, y\|$ for all $x, y \in X$ and real a;
(iv) $\|x,\, y+z\| = \|x,\, y\| + \|x,\, z\|$ for all $x, y, z \in X$.

$\|\cdot,\cdot\|$ is called a 2-norm and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space.

Some of the basic properties of the 2-norms are that they are non negative and

$$\|x,\, y+ax\| = \|x,\, y\|$$

for all $x, y \in X$ and all real number a.

Definition 2.2. ([1]) A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is called a Cauchy sequence if $\lim_{m,\,n\to\infty} \|x_m - x_n,\, y\| = 0$ for all $y \in X$.

Definition 2.3. ([1]) A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to a point x in X if $\lim_{n\to\infty} \|x_n - x,\, y\| = 0$ for all $y \in X$.

Definition 2.4. ([1]) A linear 2-normed space $(X, \|\cdot,\cdot\|)$ in which every Cauchy sequence is convergent is called a 2-Banach space.

Definition 2.5. ([1]) Let X be a 2-Banach space and T be a self mapping of X. T is said to be continuous at x if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$.

Definition 2.6. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be **2-Banach contraction** if there is $a \in [0, 1)$ such that

$$\|Tx - Ty,\, u\| \leq a\, \|x - y,\, u\|$$

for all $x, y, u \in X$.

Definition 2.7. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be **2-Kannan contraction** if there is

$b \in [0, \frac{1}{2})$ such that

$$\|Tx - Ty,\, u\| \leq b\,\big[\|x - Tx,\, u\| + \|y - Ty,\, u\|\big]$$

for all $x, y, u \in X$.

Definition 2.8. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be **2-Chatterjea contraction** if there is $c \in [0, \frac{1}{2})$ such that

$$\|Tx - Ty,\, u\| \leq c\,\big[\|x - Ty,\, u\| + \|y - Tx,\, u\|\big]$$

for all $x, y, u \in X$.

Definition 2.9. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and T be a self mapping of X. A mapping T is said to be **2-Zamfirescu operator** if there are real numbers $0 \leq a < 1$, $0 \leq b < 1/2$, $0 \leq c < 1/2$ such that for all $x, y, u \in X$ at least one of the conditions is true:

(z_1) $\|Tx - Ty,\, u\| \leq a\,\|x - y,\, u\|$;
(z_2) $\|Tx - Ty,\, u\| \leq b\,\big[\|x - Tx,\, u\| + \|y - Ty,\, u\|\big]$;
(z_3) $\|Tx - Ty,\, u\| \leq c\,\big[\|x - Ty,\, u\| + \|y - Tx,\, u\|\big]$.

Definition 2.10. (**Implicit Relation**) Let Φ be the class of real valued continuous functions $\varphi\colon (R^+)^3 \to R^+$ non-decreasing in the first argument and satisfying the following condition: for $x,\, y \geq 0$,

$$\text{(i)} \quad x \leq \varphi\Big(y,\, \frac{x+y}{2},\, \frac{x+y}{2}\Big) \qquad \text{or} \qquad \text{(ii)} \quad x \leq \varphi\Big(x,\, 0,\, x\Big)$$

there exists a real number $0 < h < 1$ such that $x \leq h\,y$.

Example 2.11. $\varphi(t_1, t_2, t_3) = t_1 - a\,t_2,$ where $a > 1$.

(i) Let $x,\, y \geq 0$. We have $\varphi\Big(y,\, \frac{x+y}{2},\, \frac{x+y}{2}\Big) = y - a\left(\frac{x+y}{2}\right) \geq 0$. Hence $x \leq h\,y$ with $h = \frac{2-a}{a} < 1$.

Example 2.12. $\varphi(t_1, t_2, t_3) = t_1 - b\,\max\{t_2, t_3\},$ where $b > 1$.

(i) Let $x,\, y \geq 0$. We have $\varphi\Big(y,\, \frac{x+y}{2},\, \frac{x+y}{2}\Big) = y - a\,\max\left\{\frac{x+y}{2},\, \frac{x+y}{2}\right\} = y - b\,\frac{x+y}{2} \geq 0$. Hence $x \leq h_1\,y$ with $h_1 = \frac{2-b}{b} < 1$.

Example 2.13. $\varphi(t_1, t_2, t_3) = t_1 - c\,(t_2 + t_3),$ where $0 < c < 1$.

(i) Let $x,\, y \geq 0$. We have $\varphi\Big(y,\, \frac{x+y}{2},\, \frac{x+y}{2}\Big) = y - c\,(x + y) \geq 0$. Hence $x \leq h_2\,y$ with $h_2 = \frac{1-c}{c} < 1$.

Condition (A) Let X be a 2-Banach space (with $dim\, X \geq 2$) and let S and T be two self mappings of X such that for all x, y, u in X satisfying the condition:

$$\|Sx - Ty,\, u\| \leq \varphi\Big(\|x-y,\, u\|, \frac{\|x-Sx,\, u\|+\|y-Ty,\, u\|}{2}, \frac{\|x-Ty,\, u\|+\|y-Sx,\, u\|}{2}\Big). \quad (2.1)$$

3. Main Results

In this section we shall prove some common fixed point theorems using condition (A) in the setting of 2-Banach spaces.

Theorem 3.1. *Let X be a 2-Banach space (with $dim\, X \geq 2$) and let S and T be two continuous self mappings of X satisfying the condition (A), then S and T have a unique common fixed point in X.*

Proof. For given each $x_0 \in X$ and $n \geq 1$, we choose $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general we define sequence of elements of X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$. Now for all $u \in X$, using (2.1), we have

$$\begin{aligned}
\|x_{2n+1} - x_{2n},\, u\| &= \|Sx_{2n} - Tx_{2n-1},\, u\| \\
&\leq \varphi\Big(\|x_{2n} - x_{2n-1},\, u\|, \frac{\|x_{2n} - Sx_{2n},\, u\| + \|x_{2n-1} - Tx_{2n-1},\, u\|}{2}, \\
&\qquad \frac{\|x_{2n} - Tx_{2n-1},\, u\| + \|x_{2n-1} - Sx_{2n},\, u\|}{2}\Big) \\
&= \varphi\Big(\|x_{2n} - x_{2n-1},\, u\|, \frac{\|x_{2n} - x_{2n+1},\, u\| + \|x_{2n-1} - x_{2n},\, u\|}{2}, \\
&\qquad \frac{\|x_{2n} - x_{2n},\, u\| + \|x_{2n-1} - x_{2n+1},\, u\|}{2}\Big) \\
&= \varphi\Big(\|x_{2n} - x_{2n-1},\, u\|, \frac{\|x_{2n} - x_{2n+1},\, u\| + \|x_{2n-1} - x_{2n},\, u\|}{2}, \\
&\qquad \frac{\|x_{2n-1} - x_{2n+1},\, u\|}{2}\Big) \\
&\leq \varphi\Big(\|x_{2n} - x_{2n-1},\, u\|, \frac{\|x_{2n} - x_{2n+1},\, u\| + \|x_{2n-1} - x_{2n},\, u\|}{2}, \\
&\qquad \frac{\|x_{2n-1} - x_{2n},\, u\| + \|x_{2n} - x_{2n+1},\, u\|}{2}\Big). \qquad (3.1)
\end{aligned}$$

Hence from definition 2.10(i), we have

$$\|x_{2n+1} - x_{2n},\, u\| \;\leq\; h\, \|x_{2n} - x_{2n-1},\, u\|, \text{ where } 0 < h < 1. \qquad (3.2)$$

Similarly, we have

$$\|x_{2n} - x_{2n-1},\, u\| \quad \leq \quad h\,\|x_{2n-1} - x_{2n-2},\, u\|. \tag{3.3}$$

Hence form (3.2) and (3.3), we have

$$\|x_{2n+1} - x_{2n},\, u\| \quad \leq \quad h^2\,\|x_{2n-1} - x_{2n-2},\, u\|. \tag{3.4}$$

On continuing this process, we get

$$\|x_{2n+1} - x_{2n},\, u\| \quad \leq \quad h^{2n}\,\|x_1 - x_0,\, u\|. \tag{3.5}$$

Also for $n > m$, we have

$$\begin{aligned} \|x_n - x_m,\, u\| &\leq \|x_n - x_{n-1},\, u\| + \|x_{n-1} - x_{n-2},\, u\| + \dots \\ &\quad + \|x_{m+1} - x_m,\, u\| \\ &\leq \big(h^{n-1} + h^{n-2} + \cdots + h^m\big)\|x_1 - x_0,\, u\| \\ &\leq \Big(\frac{h^m}{1-h}\Big)\|x_1 - x_0,\, u\|. \end{aligned} \tag{3.6}$$

Since $0 < h < 1$, by Definition 2.10, $\left(\frac{h^m}{1-h}\right) \to 0$ as $m \to \infty$. Hence $\|x_n - x_m,\, u\| \to 0$ as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exist a point z in X such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of S and T that $Sz = Tz = z$. Thus z is a common fixed point of S and T.

Uniqueness

Let v be another common fixed point of S and T, that is, $Sv = Tv = v$. Then, we have

$$\begin{aligned} &\|z - v,\, u\| = \|Sz - Tv,\, u\| \\ &\leq \varphi\Big(\|z - v,\, u\|, \frac{\|z - Sz,\, u\| + \|v - Tv,\, u\|}{2}, \frac{\|z - Tv,\, u\| + \|v - Sz,\, u\|}{2}\Big) \\ &\leq \varphi\Big(\|z - v,\, u\|, 0, \|z - v,\, u\|\Big). \end{aligned} \tag{3.7}$$

From Definition 2.10(ii), (3.7) gives

$$\|z - v,\, u\| \quad \leq \quad 0. \tag{3.8}$$

Hence $z = v$ and for all $u \in X$. Thus z is a unique common fixed point of S and T. This completes the proof. □

If we take $S = T$ in Theorem 3.1, then we have the following result as corollary.

Corollary 3.2. *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a self mapping of X satisfying the condition:*

$$\begin{aligned} &\|Tx - Ty,\, u\| \\ &\leq \varphi\Big\{\|x - y,\, u\|, \frac{\|x - Tx,\, u\| + \|y - Ty,\, u\|}{2}, \frac{\|x - Ty,\, u\| + \|y - Tx,\, u\|}{2}\Big\} \end{aligned}$$

for all $x, y, u \in X$. Then T has a unique fixed point in X.

Proof. The proof of Corollary 3.2 immediately follows from Theorem 3.1 by taking $S = T$. This completes the proof. □

From Theorem 3.1, we obtain the following results as special cases.

Theorem 3.3. *Let X be a 2-Banach space (with $\dim X \geq 2$) and let S and T be two continuous self mappings of X satisfying the condition:*

$$\|Sx - Ty,\, u\| \leq k\,\|x - y,\, u\|$$

for all $x, y, u \in X$ and $k \in (0, 1)$, then S and T have a unique common fixed point in X.

Theorem 3.4. *Let X be a 2-Banach space (with $\dim X \geq 2$) and let S and T be two continuous self mappings of X satisfying the condition:*

$$\|Sx - Ty,\, u\| \leq b\,(\|x - Sx,\, u\| + \|y - Ty, u\|)$$

for all $x, y, u \in X$ and $b \in (0, \frac{1}{2})$, then S and T have a unique common fixed point in X.

Theorem 3.5. *Let X be a 2-Banach space (with $\dim X \geq 2$) and let S and T be two continuous self mappings of X satisfying the condition:*

$$\|Sx - Ty,\, u\| \leq c\,(\|x - Ty,\, u\| + \|y - Sx, u\|)$$

for all $x, y, u \in X$ and $c \in (0, \frac{1}{2})$, then S and T have a unique common fixed point in X.

Remark 3.6. Our results extend, improve and generalize some previous work from the current existing literature.

References

[1] Y.J. Cho, N. Huang and X. Long, *Some fixed point theorems for linear mappings in 2-Banach spaces*, Far East Jour. Math. Sci., **3**(2) (1995), 125–133.

[2] A. Choudhury and T. Som, *2-Banach space and some fixed point results*, J. Indian Acad. Math., **33**(2) (2011), 411–418.

[3] S. Gahler, *Metricsche Räume and their topologische structure*, Math. Nachr., **26** (1963), 115–148.

[4] S. Gahler, *Linear 2-normietre Räume*, Math. Nachr., **28** (1965), 1–43.

[5] K. Iseki, *Fixed point theorems in Banach spaces*, Math. Sem. Notes, Kobe Univ., **2** (1976), 11–13.

[6] M.S. Khan and M.D. Khan, *Involutions with fixed points in 2-Banach spaces*, Int. J. Math. Math. Sci., **16** (1993), 429–434.

[7] B.E. Rhoades, *Contraction type mappings on 2-metric space*, Math. Nachr., **91** (1979), 151–155.

[8] G.S. Saluja, *Existence results of unique fixed point in 2-Banach spaces*, Int. J. Math. Combin., **1** (2014), 13–18.

[9] T. Som, *Some fixed point results in 2-Banach space*, Int. J. Math. Sci., **4**(2) (2005), 323–328.

[10] H.B. Zhao, *Existence theorems of fixed point of average non expansive mappings in Banach spaces*, Acta Math. Sinica, **22** (1979), 459–470.