Nonlinear Functional Analysis and Applications
Vol. 17, No. 3 (2012), pp. 301-306

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FIXED POINT THEOREM IN ORDERED METRIC SPACES FOR GENERALIZED CONTRACTION MAPPINGS SATISFYING RATIONAL TYPE EXPRESSIONS

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Abstract. In this paper, we prove a fixed point result in the framework of partially ordered metric spaces satisfying a generalized contractive condition of rational type. The result generalize and extend some known results in the literature.

1. Introduction and Preliminaries

In [9], Jaggi and Dass proved the following fixed point theorem.

Theorem 1.1. *Let T be a continuous self map defined on a complete metric space (X, d). Suppose that T satisfies the following contractive condition:*

$$d(Tx, Ty) \quad \leq \quad \alpha\left(\frac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)}\right) + \beta(d(x, y)) \qquad (1.1)$$

for all $x, y \in X$, $x \geq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, then T has a unique fixed point in X.

The aim of this paper is to give a version of Theorem 1.1 in partially ordered metric spaces.

[0]Received May 7, 2012. Revised July 5, 2012.
[0]2000 Mathematics Subject Classification: 46T99, 41A50, 47H10, 54H25.
[0]Keywords: Fixed point, rational type generalized contraction mappings, ordered metric spaces.

The Banach contraction mapping is one of the pivotal results of analysis. It is very popular tool for solving existence problems in many different fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature (see [1]-[11] and references cited therein).

Ran and Reurings [11] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodŕiguez-López [10] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point results in metric spaces and partially ordered metric spaces. The purpose of this paper is to establish a fixed point result satisfying a generalized contraction mappings of rational type in partially ordered metric spaces.

2. Main Results

Definition 2.1. Suppose $(X, \leq)$ is a partially ordered set and $T : X \to X$. T is said to be *monotone nondecreasing* if for all $x, y \in X$,

$$x \leq y \quad \text{implies} \quad Tx \leq Ty. \tag{2.1}$$

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T is a continuous self-mapping on X, T is monotone nondecreasing mapping and*

$$d(Tx, Ty) \;\leq\; \alpha \left(\frac{d(x, Tx)d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)} \right) + \beta(d(x, y)) \tag{2.2}$$

for all $x, y \in X$, $x \geq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. If $Tx_0 = x_0$, then we have the result. Suppose that $x_0 < Tx_0$. Since T is a monotone nondecreasing mapping, we obtain by induction that

$$x_0 < Tx_0 \leq T^2x_0 \leq \ldots \leq T^n x_0 \leq T^{n+1}x_0 \leq \ldots. \tag{2.3}$$

By induction, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, for every $n \geq 0$. Since T is monotone nondecreasing mapping, we obtain

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_{n+1} \leq \ldots.$$

If there exists $n \geq 1$ such that $x_{n+1} = x_n$, then from $x_{n+1} = Tx_n = x_n$, x_n is a fixed point and the proof is finished. Suppose that $x_{n+1} \neq x_n$, for all $n \geq 1$.

Since $x_n > x_{n-1}$, for all $n \geq 1$, from (2.2), we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d(Tx_{n+1}, Tx_n) \\ &\leq \alpha\left(\frac{d(x_{n+1}, Tx_{n+1})d(x_n, Tx_n)}{d(x_{n+1}, x_n) + d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})}\right) \\ &\quad +\beta(d(x_{n+1}, x_n)) \\ &= \alpha\left(\frac{d(x_{n+1}, x_{n+2})d(x_n, x_{n+1})}{d(x_{n+1}, x_n) + d(x_n, x_{n+2})}\right) + \beta(d(x_{n+1}, x_n)) \\ &\leq \alpha\left(\frac{d(x_{n+1}, x_{n+2})d(x_n, x_{n+1})}{d(x_{n+1}, x_{n+2})}\right) + \beta(d(x_{n+1}, x_n)) \\ &= \alpha(d(x_n, x_{n+1})) + \beta(d(x_{n+1}, x_n)) \\ &= (\alpha + \beta)d(x_{n+1}, x_n), \end{aligned} \tag{2.4}$$

which implies that

$$d(x_{n+2}, x_{n+1}) \leq (\alpha + \beta)d(x_{n+1}, x_n). \tag{2.5}$$

Using, mathematical induction we have

$$d(x_{n+2}, x_{n+1}) \leq (\alpha + \beta)^{n+1}d(x_1, x_0). \tag{2.6}$$

Put $k = \alpha + \beta < 1$. Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. For $m \geq n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n) \\ &\leq \left(k^{m-1} + k^{m-2} + ... + k^n\right) d(x_1, x_0) \\ &\leq \left(\frac{k^n}{1-k}\right) d(x_1, x_0), \end{aligned} \tag{2.7}$$

which implies that $d(x_m, x_n) \to 0$, as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Therefore, there exits $u \in X$ such that $\lim_{n\to\infty} x_n = u$. By the continuity of T, we have

$$Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u.$$

Hence u is a fixed point of T. $\square$

In what follows, we prove that Theorem 2.2 is still valid for T, not necessarily continuous, assuming the following hypothesis in X.

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$.

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose* *that T is a self-mapping on X, T is monotone nondecreasing mapping and*

$$d(Tx,Ty) \leq \alpha\left(\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}\right)+\beta(d(x,y)) \quad (2.8)$$

for all $x, y \in X$, $x \geq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha+\beta<1$.

Assume that $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. Following the proof of Theorem 2.2, we have $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to u$, then $u = \sup\{x_n\}$. Particularly, $x_n \leq u$ for all $n \in \mathbb{N}$.

Since T is monotone nondecreasing mapping $Tx_n \leq Tu$, for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \leq Tu$, for all $n \in \mathbb{N}$. Moreover, as $x_n < x_{n+1} \leq Tu$ and $u = \sup\{x_n\}$, we get $u \leq Tu$.

Construct a sequence $\{y_n\}$ as $y_0 = u$, $y_{n+1} = Ty_n$, for all $n \geq 0$. Since $y_0 \leq Ty_0$, arguing like above part, we obtain that $\{y_n\}$ is a non-decreasing sequence and $\lim_{n\to\infty} y_n = y$ for certain $y \in X$, so we have $y = \sup\{y_n\}$. Since $x_n < u = y_0 \leq Tu = Ty_0 \leq y_n \leq y$, for all n, using (2.8), we have

$$\begin{aligned} d(x_{n+1},y_{n+1}) &= d(Tx_n,Ty_n) \\ &\leq \alpha\left(\frac{d(x_n,Tx_n)d(y_n,Ty_n)}{d(x_n,y_n)+d(x_n,Ty_n)+d(y_n,Tx_n)}\right) \\ &\quad +\beta(d(x_n,y_n)) \\ &= \alpha\left(\frac{d(x_n,x_{n+1})d(y_n,y_{n+1})}{d(x_n,y_n)+d(x_n,y_{n+1})+d(y_n,x_{n+1})}\right) \\ &\quad +\beta(d(x_n,y_n)). \end{aligned} \quad (2.9)$$

Letting $n \to \infty$, we have $d(u,y) \leq \beta d(u,y)$. As $\beta < 1$, we have $d(u,y) = 0$. Particularly, $u = y = \sup\{y_n\}$, and consequently, $u \leq Tu \leq u$. Hence we conclude that u is a fixed point of T. □

Now, we shall prove the uniqueness of the fixed point.

Theorem 2.4. *In addition to the hypotheses of Theorem 2.2 (or Theorem 2.3), suppose that for every $x, y \in X$, there exists $z \in X$ that is comparable to x and y, then T has a unique fixed point.*

Proof. From Theorem 2.2 (or Theorem 2.3), the set of fixed points of T is non-empty. Suppose that $x, y \in X$ are two fixed points of T. We distinguish two cases:

Case 1. If x and y are comparable and $x \neq y$, then using (2.2), we have

$$\begin{aligned} d(x,y) &= d(Tx,Ty) \\ &\leq \alpha\left(\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}\right)+\beta(d(x,y)) \\ &= \beta(d(x,y)), \end{aligned}$$

which implies that $d(x,y)=0$, as $\beta<1$. Hence $x=y$.

Case 2. If x is not comparable to y, there exists $z \in X$ that is comparable to x and y. Monotonicity implies that that $T^n z$ is comparable to $T^n x = x$ and $T^n y = y$ for $n = 0,1,2,\ldots$. If there exists $n_0 \geq 1$ such that $T^{n_0} z = x$, then as x is a fixed point, the sequence $\{T^n z : n \geq n_0\}$ is constant, and, consequently, $\lim_{n\to\infty} T^n z = x$. On the other hand, if $T^n z \neq x$ for $n \geq 1$, using the contractive condition, we obtain, for $n \geq 2$,

$$\begin{aligned} d(T^n z,x) &= d(T^n z,T^n x) \\ &\leq \alpha\left(\frac{d(T^{n-1}x,T^n x)d(T^{n-1}z,T^n z)}{d(T^{n-1}x,T^{n-1}z)+d(T^{n-1}x,T^n z)+d(T^{n-1}z,T^n x)}\right) \\ &\quad +\beta(d(T^{n-1}x,T^{n-1}z)) \\ &= \alpha\left(\frac{d(x,x)d(T^{n-1}z,T^n z)}{d(x,T^{n-1}z)+d(x,T^n z)+d(T^{n-1}z,x)}\right)+\beta(d(x,T^{n-1}z)) \\ &= \beta(d(x,T^{n-1}z)), \end{aligned}$$

which implies that $d(T^n z,x) \leq \beta(d(T^{n-1}z,x))$. Using mathematical induction, we have $d(T^n z,x) \leq \beta^n(d(z,x))$, for $n \geq 2$, and as $\beta < 1$, we have $\lim_{n\to\infty} T^n z = x$.

Using a similar argument, we can prove that $\lim_{n\to\infty} T^n z = y$. Now, the uniqueness of the limit implies $x=y$. Hence T has a unique fixed point. $\square$

Other consequences of our results are the following for the mappings involving contractions of integral type.

Denote by Λ the set of functions $\mu : [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;
(h2) for any $\epsilon > 0$, we have $\int_0^\epsilon \mu(t)dt > 0$.

Corollary 2.5. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that T is a continuous self-mapping on X, T is monotone nondecreasing mapping and*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad \alpha \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(y,Tx)+d(x,Ty)}} \psi(t)dt + \beta \int_0^{d(x,y)} \psi(t)dt$$

for all $x, y \in X$ *for which* x *and* y *are comparable,* $\psi \in \Lambda$ *and for some* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$.

If there exists $x_0 \in X$ *with* $x_0 \leq Tx_0$*, then* T *has a fixed point.*

Corollary 2.6. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose that* T *is a self-mapping on* X*,* T *is monotone nondecreasing mapping and*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad \alpha \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}} \psi(t)dt + \beta \int_0^{d(x,y)} \psi(t)dt,$$

for all $x, y \in X$ *for which* x *and* y *are comparable,* $\psi \in \Lambda$ *and for some* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$.

Assume that $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $x_n \to x$*, then* $x = \sup\{x_n\}$.

If there exists $x_0 \in X$ *with* $x_0 \leq Tx_0$*, then* T *has a fixed point.*

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