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COINCIDENCE AND FIXED POINT THEOREMS FOR A NEW CONTRACTION PRINCIPLE IN PARTIALLY ORDERED G-METRIC SPACES AND APPLICATIONS TO ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The purpose of this article is to present some coincidence and fixed point theorems for generalized contraction in partially ordered complete G-metric spaces. As an application, we give an existence and uniqueness for the solution of some initial-boundary-value problems. Our result generalizes and improves some theorems in the literature.

1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity, see [14]–[18], [21, 22], [24]–[27]. The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [1, 2]. Recently, Mustafa and Sims [30, 31, 32, 34] have shown that most of the results concerning Dhage's D-metric spaces are invalid. In [31, 32], they introduced a improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces, one can refer to the papers [3]–[12], [19, 23, 28, 29, 34, 35, 36]. Subsequently, several authors proved fixed point

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results in these spaces, some of them have given some applications to matrix equations, ordinary differential equations, and integral equations.

2. Preliminaries

Definition 2.1. ([29]) Let X be a non-empty set, $G : X \times X \times X \to R_+$ be a function satisfying the following properties

- $(G1)$ $G(x, y, z) = 0$ if $x = y = z$.
- $(G2)$ $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.
- $(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
- $(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (symmetry in all three variables).
- $(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 2.2. ([29]) Let (X, G)be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x; x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x; x_n, x_m) < \varepsilon$, for all $n; m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.3. ([29]) *Let (X, G) be a G-metric space. The following are equivalent:*

1. *(x_n) is G-convergent to x.*
2. *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
3. *$G(x_n, x, x) \to 0$ as $n \to \infty$.*
4. *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Definition 2.4. ([29]) Let (X, G)be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.5. ([29]) *Let (X, G)be a G-metric space. Then the following are equivalent:*

1. *The sequence (x_n) is G-Cauchy.*
2. *For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Proposition 2.6. ([29]) *Let (X,G)be a G-metric space. A mapping $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.*

Proposition 2.7. ([29]) *Let (X,G)be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous all three of its variables.*

Definition 2.8. ([29]) A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Definition 2.9. (weakly compatible mappings ([29])) Two mappings $f,g : X \to X$ are weakly compatible if they commute at their coincidence points, that is $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

Definition 2.10. ($g-$Non decreasing Mapping ([29])) Suppose $(X,\preceq)$ is a partially ordered set and $f,g : X \to X$ are mappings. f is said to be $g-$Non decreasing if for $x,y \in X$, $gx \preceq gy$ implies $fx \preceq fy$.

Now, we are ready to state and prove our results.

Let Ψ denotes the class of the functions $\psi : [0,+\infty[\to [0,+\infty[$ which satisfies the following conditions:

(1) ψ is nondecreasing.
(2) ψ is continuous.
(3) $\psi(t) = 0 \iff t = 0$.

The elements of Ψ are called altering distance functions.

Remark 2.11. ([37]) If $\psi \in \Psi$ and if $\phi : [0,+\infty[\to [0,+\infty[$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.

3. Main Results

Lemma 3.1. *Let (X,G) be a G-metric space and (x_n) be a sequence in X such that $G(x_{n+1},\, x_{n+1},\, x_n)$ is decreasing and*

$$\lim_{n\to\infty} G(x_{n+1}, x_{n+1}, x_n) = 0. \tag{3.1}$$

If (x_{2n}) is not a Cauchy sequence, then there exists $\varepsilon > 0$ and two sequences (m_k) and (n_k) of positive integers such that the following four sequences tends to ε as $k \to \infty$:

$$\begin{gathered} G(x_{2m_k}, x_{2m_k}, x_{2n_k}), \quad G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}), \\ G(x_{2m_{k-1}}, x_{2m_{k-1}}, x_{2n_k}), \quad G(x_{2m_{k-1}}, x_{2m_{k-1}}, x_{2n_{k+1}}). \end{gathered} \tag{3.2}$$

Proof. If (x_{2n}) is not a Cauchy sequence, then there exists $\varepsilon > 0$ and two sequences (m_k) and (n_k) of positive integers such that

$$n_k > m_k > k\ ;\ \ G(x_{2m_k}, x_{2m_k}, x_{2n_k-2}) < \varepsilon,\ \ G(x_{2m_k}, x_{2m_k}, x_{2n_k}) \geq \varepsilon$$

for all integer k. Then

$$\begin{aligned}\varepsilon &\leq G(x_{2m_k}, x_{2m_k}, x_{2n_k}) \\ &\leq G(x_{2m_k}, x_{2m_k}, x_{2n_k-2}) + G(x_{2m_{k-2}}, x_{2n_{k-2}}, x_{2n_k-1}) \\ &\quad + G(x_{2n_{k-1}}, x_{2m_{k-1}}, x_{2n_k}) \\ &< \varepsilon + G(x_{2n_{k-2}}, x_{2n_{k-2}}, x_{2n_k-1}) + G(x_{2n_{k-1}}, x_{2n_{k-1}}, x_{2n_k}).\end{aligned}$$

Using (3.1), we conclude that

$$\lim_{k\to\infty} G(x_{2m_k}, x_{2m_k}, x_{2n_k}) = \varepsilon. \tag{3.3}$$

Further,

$$G(x_{2m_k}, x_{2m_k}, x_{2n_k}) \leq G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}) + G(x_{2n_{k+1}}, x_{2n_{k+1}}, x_{2n_k})$$

and

$$G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}) \leq G(x_{2m_k}, x_{2m_k}, x_{2n_k}) + G(x_{2n_k}, x_{2n_k}, x_{2n_{k+1}}).$$

Passing to the limit when $k \to \infty$ and using (3.1) and (3.3), we obtain

$$\lim_{k\to\infty} G(x_{2m_k}, x_{2m_k}, x_{2n_{k+1}}) = \varepsilon.$$

The remaining two sequences in (b) tend to ε can be proved in a similar way. □

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist a continuous function $\phi : [0, +\infty[\to [0, +\infty[$ with the condition $\psi(t) > \phi(t)$ for all $t > 0$ and $\psi \in \Psi$ such that*

$$\psi(G(fx, fy, fz)) \leq \phi(G(gx, gy, gz)) \tag{3.4}$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$. Assume that X is such that if an increasing sequence

$$x_n \text{ converges to } x, \text{ then } x_n \preceq x \text{ for each } n \geq 0. \tag{3.5}$$

If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Proof. By the condition of the theorem there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$. Since $f(X) \subseteq g(X)$, we can define $x_1 \in X$ such that $gx_1 = fx_0$, then $gx_0 \preceq fx_0 = gx_1$. Since f is $g-$nondecreasing, we have $fx_0 \preceq fx_1$. In this way we construct the sequence (x_n) recursively as

$$fx_n = gx_{n+1}, \quad \text{for all } n \geq 1 \tag{3.6}$$

for which

$$\begin{aligned} gx_0 &\preceq fx_0 = gx_1 \preceq fx_1 = gx_2 \preceq fx_2 \preceq \cdots \\ &\preceq fx_{n-1} = gx_n \preceq fx_n = gx_{n+1} \preceq \cdots . \end{aligned} \tag{3.7}$$

First, we suppose that there exists $n_0 \in \mathbb{N}$ such that $\psi(G(fx_{n_0}, fx_{n_0}, fx_{n_0+1})) = 0$, then it follows from the properties of ψ, $G(fx_{n_0}, fx_{n_0}, fx_{n_0+1}) = 0$, so, $fx_{n_0} = fx_{n_0+1}$ we have $gx_{n_0+1} = fx_{n_0+1}$ and x_{n_0+1} is a coincidence point of f and g. Now we suppose $\psi(G(fx_{n_0}, fx_{n_0}, fx_{n_0+1})) \neq 0$. The elements gx_n and gx_{n+1} are comparable, substituting $x = y = x_n$ and $z = x_{n+1}$ in (3.4), using (3.5)and (3.7), we have

$$\begin{aligned} \psi(G(fx_n, fx_n, fx_{n+1})) &\leq \phi(G(gx_n, gx_n, gx_{n+1})) \\ &\leq \phi(G(fx_{n-1}, fx_{n-1}, fx_n)). \end{aligned} \tag{3.8}$$

Using the condition of the Theorem 3.2, we obtain

$$G(fx_n, fx_n, fx_{n+1}) < G(fx_{n-1}, fx_{n-1}, fx_n).$$

Hence the sequence $(G(fx_n, fx_n, fx_{n+1}))$ is decreasing and consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} G(fx_n, fx_n, fx_{n+1}) = r \geq 0$.
By going to the limit in (3.8), we get

$$\psi(r) \leq \varphi(r).$$

By using the condition of the Theorem 3.2, we obtain $r = 0$ and hence

$$\lim_{n\to\infty} G(fx_n, fx_n, fx_{n+1}) = 0.$$

Now in what follows we show that (fx_n) is a Cauchy sequence. Suppose that (fx_n) is not a Cauchy sequence. Using Lemma, we know that there exist $\varepsilon > 0$ and two sequences (m_k) and (n_k) of positive integers such that the following four sequences tend to ε as k goes to infinity:

$$\begin{aligned} &G(fx_{2m_k}, fx_{2m_k}, fx_{2n_k}), \quad G(fx_{2m_k}, fx_{2m_k}, fx_{2n_{k+1}}), \\ &G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}), \quad G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_{k+1}}). \end{aligned}$$

Putting in the contractive condition $x = y = x_{2m_k}$ and $z = fx_{2n_{k+1}}$, using (3.5) and (3.7), it follows that

$$\psi(G(fx_{2m_k}, fx_{2m_k}, fx_{2n_{k+1}})) \leq \phi(G(fx_{2m_{k-1}}, fx_{2m_{k-1}}, fx_{2n_k}))$$

by going to the limit, we have

$$\psi(\varepsilon) \leq \phi(\varepsilon).$$

By the condition of the Theorem 3.2, we get $\varepsilon = 0$, which contradicts $\varepsilon > 0$. This shows that (fx_n) is a Cauchy sequence in (X, G). Since (X, G) is a complete metric space, there exists $a \in X$ such that $\lim_{n\to\infty} fx_n = a$. Since $g(X)$ is closed, then $a = gz$, for some $z \in X$. Using (3.5) we get

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = gz. \tag{3.9}$$

Now we prove that z is a coincidence point of f and g. From (3.7), we have (gx_n) is a non-decreasing sequence in X. By (3.5) and by (3.9) we have

$$gx_n \preceq gz. \tag{3.10}$$

Putting $x = y = x_n$ in (3.4), by the virtue of (3.10), we get

$$\psi(G(fx_n, fx_n, fz)) \leq \phi(G(gx_n, gx_n, gz) \quad \text{for each } n \geq 1.$$

Taking $n \to \infty$ in the above inequality, using (3.9) , we obtain

$$\psi(G(gz, gz, fz)) \leq \varphi(G(gz, gz, gz)) = \varphi(0) = 0.$$

Therefore, we get $G(gz, gz, fz) = 0$ and so we have

$$fz = gz. \tag{3.11}$$

This proves that z is a coincidence point. This completes the proof. □

Theorem 3.3. *If in Theorem* 3.2*, it is additionally assumed that*

$$gz \preceq ggz, \tag{3.12}$$

where z is as in the condition of theorem and f and g are weakly compatible, then f and g have a common fixed point in X.

Proof. Following the proof of the Theorem 3.2, we have (3.9), that is, a non decreasing sequence (gx_n) converging to gz. Then by (3.12) we have $gz \preceq ggz$. Since f and g are weakly compatible, by (3.11), we have $fgz = gfz$. We set

$$w = gz = fz. \tag{3.13}$$

Therefore, we have

$$gz \preceq ggz = gw. \tag{3.14}$$

Also

$$fw = fgz = gfz = gw. \tag{3.15}$$

If $z = w$, then z is a common fixed point. If $z \neq w$, then, by (3.4) and by (3.10), we have

$$\psi(G(fx_n, fx_n, fw)) \leq \phi(G(gx_n, gx_n, gw).$$

Taking $n \to \infty$ in the above inequality, using (3.9) , we obtain

$$\begin{aligned}\psi(G(gz,gz,fw)) &\leq \varphi(G(gz,gz,gw)) \\ &\leq \varphi(G(gz,gz,fw)).\end{aligned}$$

From the condition of the Theorem 3.2, we get $\psi\left(G(gz,gz,fw)\right) = 0$ which implies $G(gz,gz,fw) = 0$, so $gz = fw$. Then, by (3.13) and (3.15), we have $w = gw = fw$. This completes the proof. □

Theorem 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f : X \to X$ be a nondecreasing function. Suppose that there exist a continuous function $\phi : [0, +\infty[\to [0, +\infty[$ with the condition $\psi(t) > \phi(t)$ for all $t > 0$ and $\psi \in \Psi$ such that*

$$\psi(G(fx, fy, fz)) \leq \phi(G(x, y, z)) \tag{3.16}$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$. Assume that f is continuous or X is such that if a nondecreasing sequence

$$x_n \text{ converges to } x, \text{ then } x_n \preceq x \quad \text{for each } n \geq 0. \tag{3.17}$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Following the proof of Theorem 3.2, with $g = id_X$, we have from (3.9) a nondecreasing sequence (x_n) converging to z. Now we show that z is a fixed point of f. If f is continuous, then

$$z = \lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} f^{n+1}(x_0) = f(\lim_{n\to\infty} f^n(x_0)) = f(z)$$

and hence $f(z) = z$.

If the second condition of the theorem holds, then we have As (x_n) is a nondecreasing sequence in X and $\lim\limits_{n\to\infty} x_n = x$. The condition (3.8) gives us that $x_n \leq x$ for every $n \geq 0$, consequently,

$$\psi\left(G\left(x_{n+1}, f(z), f(z)\right)\right) = \psi\left(G\left(f(x_n), f(z), f(z)\right)\right) \leq \phi\left(G(x_n, x_n, z)\right).$$

Letting $n \to \infty$ and taking into account that $\psi \in \Psi$, we have by using Remark 2.11

$$\psi(G(z, fz, fz)) \leq \phi(0) = 0,$$

which implies that $\psi(G(z, fz, fz)) = 0$. Thus $G(z, fz, fz) = 0$ or equivalently, $z = fz$. □

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorem 3.3 and in Theorem 3.4. This condition is as follows:

$$\text{For } x, y \in X, \text{ there exists a lower bound or an upper bound.} \tag{3.18}$$

In [13], it is proved that the condition (3.18) is equivalent to

$$\text{For } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{3.19}$$

Theorem 3.5. *Adding the condition* (3.19) *to the hypothesis of Theorem* 3.3 *(resp. Theorem* 3.4*), we obtain the uniqueness of the fixed point of* f*.*

Proof. Suppose that there exist x, y which are fixed points. We distinguish the following two cases:

Case 1. If y is comparable to z, then $f^n(y) = y$ is comparable to $f^n(z) = z$ for $n \geq 0$ and

$$\begin{aligned} \psi\left(G\left(z, z, f^n x\right)\right) &= \psi\left(G\left(f^n z, f^n z, f^n x\right)\right) \\ &\leq \varphi\left(G\left(f^{n-1} z, f^{n-1} z, f^{n-1} x\right)\right) \\ &\leq \varphi\left(G\left(z, z, f^{n-1} x\right)\right) \end{aligned} \tag{3.20}$$

Hence, $\psi \in \Psi$, then $(G\left(z, z, f^n x\right))$ is a nonnegative decreasing sequence, and consequently, there exists γ such that

$$\lim_{n \to \infty} G\left(z, z, f^n x\right) = \gamma.$$

Letting $n \to \infty$ in (3.20) and taking account that ψ and ϕ are continuous functions, we obtain

$$\psi\left(\gamma\right) \leq \phi\left(\gamma\right).$$

This and the condition of Theorem 3.2 implies $\phi\left(\gamma\right) = 0$ and consequently, $\gamma = 0$. Analogously, it can be proved that

$$\lim_{n \to \infty} G\left(y, y, f^n x\right) = 0.$$

Finally, as

$$\lim_{n \to \infty} G\left(z, z, f^n x\right) = 0 = \lim_{n \to \infty} G\left(y, y, f^n x\right).$$

The uniqueness of the limit gives us $y = z$.

Case 2. If y is not comparable to z, then there exists $x \in X$ comparable to y and z. Monotonicity of f implies that $f^n(x)$ is comparable to $f^n(y)$ and $f^n(z)$ for $n \geq 0$. Moreover

$$\begin{aligned} \psi\left(G\left(z, z, f^n(x)\right)\right) &= \psi\left(G\left(f^n(z), f^n(z), f^n(x)\right)\right) \\ &\leq \varphi\left(G\left(f^{n-1}(z), f^{n-1}(z), f^{n-1}(x)\right)\right) \\ &\leq \varphi\left(G\left(z, z, f^{n-1}(x)\right)\right). \end{aligned} \tag{3.21}$$

Hence, by the same way as above, we obtain

$$\lim_{n \to \infty} G\left(z, z, f^n(x)\right) = 0.$$

Analogously, it can be proved that

$$\lim_{n\to\infty} G\left(y, y, f^{n}(x)\right) = 0.$$

Finally, as

$$\lim_{n\to\infty} G\left(z, z, f^{n}(x)\right) = \lim_{n\to\infty} G\left(y, y, f^{n}(x)\right) = 0.$$

The uniqueness of the limit gives us $y = z$. This finishes the proof. $\square$

Remark 3.6. Under the assumption of Theorem 3.2, it can be proved that for every $x \in X$, $\lim_{n\to\infty} f^{n}x = z$, where z is the fixed point.

Let S denotes the class of the functions β: $[0;+\infty) \to [0;1)$ which satisfies the condition $\beta\left(t_n\right) \to 1$ implies $t_n \to 0$ and continuous.

Corollary 3.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G- complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that*

$$\psi(G(fx, fy, fz)) \leq \beta\left(\psi(G(gx, gy, gz))\right) \psi(G(gx, gy, gz)) \tag{3.22}$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$. Assume that X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Proof. It follows from Theorem 3.2, by choosing $\phi\left(x\right) = \beta\left(\psi\left(x\right)\right)\psi\left(x\right)$. $\square$

Later, from the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type as an application of theorem 3.2 above. For this purpose, let

$$Y = \left\{ \begin{array}{l} \chi, \chi : \mathbb{R}^{+} \to \mathbb{R}^{+}, \text{ satisfies that } \chi \text{ is Lebesgue integrable,} \\ \qquad\quad \text{summable on each compact of subset of } \mathbb{R}^{+} \\ \qquad\quad \text{and } \int_0^{\epsilon} \chi\left(t\right) dt > 0 \quad \text{for each} \quad \varepsilon > 0 \end{array} \right\}.$$

Theorem 3.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist a continuous function $\phi : [0, +\infty[\to [0, +\infty[$ with the condition $\psi\left(t\right) > \phi\left(t\right)$ for all $t > 0$ and $\psi \in \Psi$ such that for $\chi \in Y$*

$$\int_0^{\psi(G(fx,fy,fz))} \chi\left(t\right) dt \leq \int_0^{\varphi(G(gx,gy,gz))} \chi\left(t\right) dt, \tag{3.23}$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$. Assume that X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Proof. For $\chi \in Y$, consider the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Lambda(x) = \int_0^x \chi(t)\,dt$ we note that $\Lambda \in \Psi$. Thus the inequality (3.23) becomes

$$\Lambda(\psi(G(fx, fy, fz))) \leq \Lambda(\varphi(G(gx, gy, gz))). \tag{3.24}$$

Setting $\Lambda \circ \psi = \psi_1, \psi_1 \in \Psi$, $\Lambda \circ \varphi = \psi_1, \varphi_1 \in \Psi$ and so we obtain

$$\psi_1(G(fx, fy, fz)) \leq \varphi_1(G(gx, gy, gz)).$$

Therefore by Theorem 3.2 above, f and g have a coincidence point. □

Corollary 3.9. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f : X \to X$ be a nondecreasing function. Suppose that there exist a continuous function $\phi : [0, +\infty[\to [0, +\infty[$ with the condition $\psi(t) > \phi(t)$ for all $t > 0$ and $\psi \in \Psi$ such that for $\chi \in Y$*

$$\int_0^{\psi(G(fx,fy,fz))} \chi(t)\,dt \leq \int_0^{\varphi(G(x,y,z))} \chi(t)\,dt, \tag{3.25}$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$. Assume that either f is continuous or X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 3.10. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist $\psi, \varphi \in \Psi$ with the condition $\psi(t) > \phi(t)$ for all $t > 0$ such that*

$$\psi(G(fx, fy, fz)) \leq \psi(G(fx, fy, fz)) - \phi(G(gx, gy, gz))$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$. Assume that X is such that if an increasing sequence x_n converges to x, then $x_n \preceq x$ for each $n \geq 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point.

Proof. It results by taking in Theorem 3.23, $\phi(x) = \psi(x) - \varphi(x)$. □

Corollary 3.11. *Let $(X, \preceq)$ be a partially ordered set and suppose that (X, G) be a G-complete metric space. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$, f is $g-$nondecreasing, $g(X)$ is closed. Suppose that there exist $\psi \in \Psi$ and $\beta \in S$, with the condition $\psi(t) > \beta(t)\,t$ for all $t > 0$ such that*

$$\psi(G(fx, fy, fz)) \leq \beta(G(fx, fy, fz))G(gx, gy, gz)$$

for all $x, y, z \in X$ *with* $gx \preceq gy \preceq gz$*. Assume that* X *is such that if an increasing sequence* x_n *converges to* x*, then* $x_n \preceq x$ *for each* $n \geq 0$*. If there exists* $x_0 \in X$ *such that* $gx_0 \preceq fx_0$ *, then* f *and* g *have a coincidence point.*

Proof. It results by taking in Theorem 3.2, $\phi(x) = \beta(x)x$. □

4. Application to ordinary differential equations

In this section, we study the existence of solution for the following first-order periodic problem:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T], \\ u(0) = u(T), \end{cases} \tag{4.1}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Previously, we considered the space $C(I = [0, T])$ of continuous functions defined on I. Obviously, this space with the metric given by

$$G(x, y, z) = \sup_{t \in I} |x(t) - y(t)| + \sup_{t \in I} |y(t) - z(t)| + \sup_{t \in I} |z(t) - x(t)|,$$

for $x, y, z \in C(I)$ is a complete metric space. $C(I)$ can also be equipped with a partial order given by

$$x, y \in C(I),\ x \leq y \iff x(t) \leq y(t) \quad \text{for } t \in I.$$

Clearly, $(C(I), \leq)$ satisfies the condition (3.18) since for $x, y \in C(I)$, the function $\max\{x, y\}$ and $\min\{x, y\}$ are the least upper and the greatest lower bounds of x and y, respectively. Moreover, it is proved in [13] that $(C(I), \leq)$ with the above mentioned metric satisfies the condition (3.17).

Now we give the following definition.

Definition 4.1. A lower solution for (4.1) is a function $\alpha \in C^{(1)}(I)$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), & \text{for } t \in [0, T], \\ \alpha(0) \leq \alpha(T). \end{cases}$$

Theorem 4.2. *Consider the problem* (4.1) *with* $f : I \times \mathbb{R} \to \mathbb{R}$ *continuous, suppose that there exist* $\lambda, \alpha > 0$ *with*

$$\alpha \leq \left(\frac{2}{3} \frac{\lambda \left(e^{\lambda T} - 1\right)}{T \left(e^{\lambda T} + 1\right)} \right)^{\frac{1}{2}}$$

such that for $x, y \in \mathbb{R}$ *with* $x \geq y$

$$0 \leq f(t,x) + \lambda x - [f(t,y) + \lambda y] \leq \alpha \sqrt{\ln\left[(x-y)^2 + 1\right]}.$$

Then the existence of a lower solution for (4.1) *provides the existence of a unique solution of* (4.1).

Proof. The problem (4.1) can be written as

$$\begin{cases} u^{'}(t) + \lambda u(t) &= f(t,u(t)) + \lambda u(t), \text{ for } t \in [0,T], \\ u(0) &= u(T). \end{cases}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)\left[f(s,u(s)) + \lambda u(s)\right] ds,$$

where $G(t,s)$ is the Green function given by

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s < t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t < s \leq T. \end{cases}$$

Define $F : C(I) \to C(I)$ by

$$(Fu)(t) = \int_0^T G(t,s)\left[f(s,u(s)) + \lambda u(s)\right] ds.$$

Note that if $u \in C(I)$ is a fixed point of F, then $u \in C^{'}(I)$ is a solution of (4.1). In what follows, we check that the hypotheses in Theorems 3.4, and 3.5 are satisfied. The mapping F is nondecreasing for $u \geq v$. Using our assumption, we can obtain

$$f(t,u(t)) + \lambda u(t) \geq f(t,v(t)) + \lambda v(t)$$

which implies, since $G(t,s) > 0$, that for $t \in I$,

$$\begin{aligned} (Fu)(t) &= \int_0^T G(t,s)\left[f(s,u(s)) + \lambda u(s)\right] ds \\ &\geq \int_0^T G(t,s)\left[f(s,v(s)) + \lambda v(s)\right] ds = (Fv)(t). \end{aligned}$$

Besides, for $u \geq v$,we have

$$\sup_{t\in I} |(Fu)(t) - (Fv)(t))| \tag{4.2}$$
$$= \sup_{t\in I} \int_0^T G(t,s)\left[f(s,u(s)) + \lambda u(s) - f(s,v(s)) - \lambda v(s)\right] ds$$
$$\leq \sup_{t\in I} \int_0^T G(t,s)\alpha \sqrt{\ln\left[(u(s)-v(s))^2+1\right]}ds.$$

Using the Cauchy-Schwartz inequality in the last integral, we get

$$\int_0^T G(t,s)\alpha \sqrt{\ln\left[(u(s)-v(s))^2+1\right]}ds \tag{4.3}$$
$$\leq \left(\int_0^T G(t,s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^T \alpha^2 \ln\left[(u(s)-v(s))^2+1\right] ds\right)^{\frac{1}{2}}.$$

The first integral gives us

$$\begin{aligned}\int_0^T G(t,s)^2 ds &= \int_0^t G(t,s)^2 ds + \int_t^T G(t,s)^2 ds \\ &= \int_0^t \frac{e^{2\lambda(T+s-t)}}{(e^{\lambda T}-1)^2} ds + \int_t^T \frac{e^{2\lambda(s-t)}}{(e^{\lambda T}-1)^2} ds \\ &= \frac{e^{2\lambda(T-1)}}{2\lambda\left(e^{\lambda T}-1\right)^2} = \frac{e^{2\lambda T}+1}{2\lambda\left(e^{\lambda T}-1\right)}.\end{aligned} \tag{4.4}$$

The second integral in (4.2) gives us the following estimate:

$$\int_0^T \alpha^2 \ln\left[(u(s)-v(s))^2+1\right] ds \leq \alpha^2 \ln\left[||u-v||^2+1\right].T \tag{4.5}$$
$$\leq \alpha^2 \ln\left[G(u,v,w)^2+1\right].T \tag{4.6}$$

Taking into account (4.4) and (4.5), we obtain

$$\sup_{t\in I} |(Fu)(t) - (Fv)(t))| \tag{4.7}$$
$$\leq \sup_{t\in I} \left(\frac{e^{2\lambda T}+1}{2\lambda\left(e^{\lambda T}-1\right)}\right)^{\frac{1}{2}} \alpha.\sqrt{T} \ln\left[G(u,v,w)^2+1\right].$$

With the same way for $v \geq w$, we have

$$\sup_{t\in I} |(Fv)(t) - (Fw)(t))| \tag{4.8}$$
$$\leq \sup_{t\in I} \left(\frac{e^{2\lambda T}+1}{2\lambda\left(e^{\lambda T}-1\right)}\right)^{\frac{1}{2}} \alpha.\sqrt{T} \ln\left[G(u,v,w)^2+1\right]$$

and for $u \geq w$, we have

$$\sup_{t\in I} |(Fu)(t) - (Fw)(t))| \tag{4.9}$$
$$\leq \sup_{t\in I} \left(\frac{e^{2\lambda T+1}}{2\lambda\left(e^{\lambda T}-1\right)}\right)^{\frac{1}{2}} \alpha.\sqrt{T}\ln\left[G(u,v,w)^2+1\right].$$

From the inequalities above, we obtain

$$G(Fu,Fv,Fw)^2 \leq 3\frac{e^{2\lambda T}+1}{2\lambda\left(e^{\lambda T}-1\right)}.T\alpha^2\ln\left[G(u,v,w)^2+1\right]$$

or equivalently

$$2\lambda\left(e^{\lambda T}-1\right)G(Fu,Fv,Fw)^2 \leq 3\left(e^{2\lambda T}+1\right).T\alpha^2\ln\left[G(u,v,w)^2+1\right].$$

By our assumption, as

$$\alpha \leq \left(\frac{2}{3}\frac{\lambda\left(e^{\lambda T}-1\right)}{T\left(e^{\lambda T}+1\right)}\right)^{\frac{1}{2}}.$$

The last inequality gives us

$$2\lambda\left(e^{\lambda T}-1\right)G(Fu,Fv,Fw)^2 \leq 2\lambda\left(e^{\lambda T}-1\right)\ln\left[G(u,v,w)^2+1\right]$$

and hence

$$G(Fu,Fv,Fw)^2 \leq \ln\left[G(u,v,w)^2+1\right]. \tag{4.10}$$

Put $\psi(x) = x^2$ and $\phi(x) = \ln\left(1+x^2\right)$. Obviously, $\psi \in \Psi$, ψ and ϕ satisfy the condition of $\psi(x) > \phi(x)$ for $x > 0$. From (4.10), we obtain for $u \geq v \geq w$

$$\psi\left(G(Fu,Fv,Fw)\right) \leq \phi\left(G(u,v,w)\right).$$

Finally, let $\alpha(t)$ be a lower solution for (4.1). We claim that $\alpha \leq F(\alpha)$. In fact

$$\alpha^{'}(t) + \lambda\alpha(t) \leq f(t,\alpha(t)) + \lambda\alpha(t), \quad \text{for } t \in I$$

multiplying by $e^{\lambda t}$

$$\alpha(t)\,e^{\lambda t'} \leq \left[f(t,\alpha(t)) + \lambda\alpha(t)\right]e^{\lambda t}, \quad \text{for } t \in I$$

we get

$$\alpha(t)\,e^{\lambda t} \leq \alpha(0) + \int_0^t \left[f(s,\alpha(s)) + \lambda\alpha(s)\right]e^{\lambda s}ds, \text{ for } t \in I. \tag{4.11}$$

As $\alpha(0) \leq \alpha(T)$, the last inequality gives us

$$\alpha(0)\,e^{\lambda t'} \leq \alpha(T)\,e^{\lambda t'} \leq \alpha(0) + \int_0^T \left[f(s,\alpha(s)) + \lambda\alpha(s)\right]e^{\lambda s}ds$$

and so

$$\alpha(0) \le \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right] ds.$$

This and (4.10) give us

$$\begin{aligned} \alpha(t)\, e^{\lambda t} &\le \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right] e^{\lambda s} ds \\ &\quad + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right] ds \end{aligned}$$

and consequently

$$\begin{aligned} \alpha(t) &\le \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right] ds \\ &= \int_0^T G(t,s)\left[f(s,\alpha(s)) + \lambda\alpha(s)\right] ds \\ &= (F\alpha)(t), \quad \text{for } t \in I. \end{aligned}$$

Finally, Theorems 3.4 and 3.5 give that F has a unique fixed point. □

The second example where our results can be applied is the following two-point boundary value problem of the second order differential equation

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t,x), & x \in [0,\infty),\ t \in [0,1], \\ x(0) = x(1) = 0. \end{cases} \tag{4.12}$$

It is well known that $x \in C^2([0,1])$, a solution of (4.12), is equivalent to $x \in C([0,1])$, a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) f(s,x(s)) ds, \quad \text{for } t \in [0,1],$$

where $G(t,s)$ is the green function given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{4.13}$$

Theorem 4.3. *Consider the problem* (4.12) *with* $f : I \times \mathbb{R} \to [0,\infty)$ *continuous and nondecreasing with respect to the second variable, and suppose that there exists* $0 \le \alpha \le \frac{8}{3}$ *such that for* $x, y \in \mathbb{R}$ *with* $y \ge x$

$$f(t,y) - f(t,x) \le \alpha \sqrt{\ln\left[(y-x)^2 + 1\right]}. \tag{4.14}$$

Then our problem (4.12) *has a unique nonnegative solution.*

Proof. Consider the cone

$$P = \{x \in C([0,1]) : x(t) \geq 0\}.$$

Obviously, (P, G) with

$$G(x,y,z) = \sup_{t\in I}|x(t)-y(t)| + \sup_{t\in I}|y(t)-z(t)| + \sup_{t\in I}|z(t)-x(t)|, \quad \text{for } x,y,z \in C(I)$$

is a complete metric space. Consider the operator given by

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))ds, \quad \text{for } x \in P,$$

where $G(t,s)$ is the Green function appearing in (4.13).

As f is nondecreasing with respect to the second variable, then for $x, y \in P$ with $y \geq x$ and $t \in [0,1]$, we have

$$(Ty)(t) = \int_0^1 G(t,s)f(s,y(s))ds \geq \int_0^1 G(t,s)f(s,x(s))ds \geq (Tx)(t)$$

and this proves that T is a nondecreasing operator.
Besides, for $z \geq y \geq x$ and taking into account (4.13), we obtain

$$\begin{aligned} G(Tz,Ty,Tx) &= \sup_{t\in I}|T(x(t))-T(y(t))| + \sup_{t\in I}|T(y(t))-T(z(t))| \\ &\quad +\sup_{t\in I}|T(z(t)-T(x(t))| \\ &= \sup_{t\in I}(T(x(t))-T(y(t))) + \sup_{t\in I}(T(y(t))-T(z(t))) \\ &\quad +\sup_{t\in I}(T(z(t)-T(x(t))), \end{aligned}$$

$$\begin{aligned} \sup_{t\in I}(T(x(t))-T(y(t))) &= \sup_{t\in I}\int_0^1 G(t,s)(f(s,x(s))-f(s,y(s)))\,ds \\ &\leq \sup_{t\in I}\int_0^1 G(t,s)\alpha\sqrt{\ln\left[\|y-x\|^2+1\right]}ds \\ &= \alpha\sqrt{\ln\left[\|y-x\|^2+1\right]}\sup_{t\in I}\int_0^1 G(t,s)ds. \end{aligned} \tag{4.15}$$

It is easy to verify that

$$\int_0^1 G(t,s)ds = -\frac{-t^2}{2} + \frac{t}{2}$$

and that

$$\sup_{t\in I}\int_0^1 G(t,s)ds = \frac{1}{8}.$$

These facts, the inequality (4.15), and the hypothesis $0 < \alpha < 8$ give us

$$\begin{aligned}\sup_{t\in I}(T(x(t)) - T(y(t))) &\leq \frac{\alpha}{8}\sqrt{\ln\left[\|y-x\|^2+1\right]}\\ &\leq \frac{\alpha}{8}\sqrt{\ln\left[G(x,y,z)^2+1\right]}.\end{aligned}$$

With the same way we get

$$\sup_{t\in I}(T(z(t)) - T(y(t))) \leq \frac{\alpha}{8}\sqrt{\ln\left[G(x,y,z)^2+1\right]}$$

and

$$\sup_{t\in I}(T(z(t) - T(x(t))) \leq \frac{\alpha}{8}\sqrt{\ln\left[G(x,y,z)^2+1\right]}$$

from the above inequalities, we obtain

$$G(Tx,Ty,Tz)^2 \leq \frac{3\alpha}{8}\sqrt{\ln\left[G(x,y,z)^2+1\right]} \leq \ln\left[G(x,y,z)^2+1\right].$$

Put $\psi(x) = x^2$ and $\phi(x) = \ln\left(1+x^2\right)$. Obviously, $\psi \in \Psi$, ψ and ϕ satisfy the condition of $\psi(x) > \phi(x)$ for $x > 0$. From the last inequality, we have

$$\psi\left(G(Fu,Fv,Fw)\right) \leq \phi\left(G(u,v,w)\right).$$

Finally, as f and G are non negative functions,

$$T0 = \int_0^1 G(t,s)f(s,0)ds \geq 0.$$

Theorem 3.4 and 3.5 tell us that F has a unique nonnegative solution. □

In the third example, We show the existence of solution for the following initial-value problem by using Theorems 3.5 and 3.7.

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + F(x,t,u,u_x), & -\infty < x < \infty,\ 0 < t < T, \\ u(x,t) = \varphi(x), & -\infty < x < \infty. \end{cases} \tag{4.16}$$

Where we assumed that φ is continuously differentiable and that φ and $\varphi'$are bounded and $F(x,t,u,u_x)$ is a continuous function.

Definition 4.4. We mean a solution of an initial-boundary-value problem for any $u_t(x,t) = u_{xx}(x,t) + F(x,t,u,u_x)$ in $\mathbb{R}\times I$, where $I = [0,\ T]$ a function $u = u(x,\ t)$ defined in $\mathbb{R}\times I$, such that

(a) $u \in C(\mathbb{R}\times I)$,
(b) $u_t,\ u_x,\ u_{xx} \in C(\mathbb{R}\times I)$,
(c) u_t and u_x are bounded in $\mathbb{R}\times I$,
(d) $u_t(x,\ t) = u_{xx}(x,\ t) + F(x,\ t,\ u(x,t),\ u_x(x,\ t))$ for all $(x,t) \in \mathbb{R}\times I$.

Now we consider the space $\Omega = \{v\,(x,\,t) : v, v_x \in C\,(\mathbb{R} \times I)\ \text{ and }\ \|v\| < \infty\}$, where

$$\|v\| = \sup_{x\in\mathbb{R},\ t\in I} |v\,(x,t)| + \sup_{x\in\mathbb{R},\ t\in I} |v_x\,(x,t)|\,.$$

The set Ω with the norm $\|\cdot\|$ is a Banach space. Obviously, the space with the G-metric given by

$$\begin{aligned} G(u,v,w) \ &= \sup_{x\in\mathbb{R},\ t\in I} |u\,(x,t) - v\,(x,t)| + \sup_{x\in\mathbb{R},\ t\in I} |u_x\,(x,t) - v_x\,(x,t)| \\ &\quad + \sup_{x\in\mathbb{R},\ t\in I} |v\,(x,t) - w\,(x,t)| + \sup_{x\in\mathbb{R},\ t\in I} |v_x\,(x,t) - w_x\,(x,t)| \\ &\quad + \sup_{x\in\mathbb{R},\ t\in I} |u\,(x,t) - w\,(x,t)| + \sup_{x\in\mathbb{R},\ t\in I} |u_x\,(x,t) - w_x\,(x,t)| \end{aligned}$$

is a complete G-metric space. The set Ω can also equipped with the a partial order given by

$$u, v \in \Omega, u \preceq v \iff u\,(x,t) \le v\,(x,t)\,, u_x\,(x,t) \le v_x\,(x,t)$$

for any $x \in \mathbb{R}$ and $t \in I$. Obviously, $(\Omega, \preceq)$ satisfies the condition (ii) since, for any $u, v \in \Omega$, $\max\{u, v\}$ and $\min\{u, v\}$ are the least and greatest lower bounds of u and v, respectively. Taking a monotone nondecreasing sequence $\{v_n\} \subseteq \Omega$ converging to v in Ω, for any $x \in \mathbb{R}$ and $t \in I$,

$$v_1\,(x,t) \le v_2\,(x,t) \le \cdots \le v_n\,(x,t) \le \cdots$$

and

$$v_{1x}\,(x,t) \le v_{2x}\,(x,t) \le \cdots \le v_{nx}\,(x,t) \le \cdots\,.$$

Further, since the sequences $\{v_n\,(x,t)\}$ and $\{v_{nx}\,(x,t)\}$ of real numbers converge to $v\,(x,t)$ and $v_x\,(x,t)$, respectively, it follows that, for all $x \in \mathbb{R}$, $t \in I$ and $n \ge 1$, $v_n\,(x,t) \le v\,(x,t)$ and $v_{nx}\,(x,t) \le v_x\,(x,t)$. Therefore, $v_n \le v$ for all $n \ge 1$ and so $(\Omega, \preceq)$ with the above mentioned metric satisfies the condition (I).

Definition 4.5. A lower solution of the initial-value problem (4.16) is a function $u \in \Omega$ such that

$$\begin{cases} u_t\,(x,t) \ = \ u_{xx}\,(x,t) + F\,(x,t,u,u_x)\,, & -\infty < x < \infty,\ 0 < t < T, \\ \\ u\,(x,t) \ = \varphi\,(x)\,, & -\infty < x < \infty, \end{cases}$$

where we assume that φ is continuously differentiable and that φ and φ' are bounded, the set Ω is defined in above and $F\,(x,t,u,u_x)$ is a continuous function.

Theorem 4.6. *Consider the problem* (4.16) *with* $F : \mathbb{R} \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *continuous and assume the following:*

(1) *For any $c > 0$ with $|s| < c$ and $|p| < c$, the function $F(x,t,s,p)$ is uniformly Holder continuous in x and t for each compact subset of $\mathbb{R} \times I$.*
(2) *There exists a constant $c_F \leq \frac{1}{3}(T + 2\pi^{\frac{-1}{2}}T^{\frac{1}{2}})^{-1}$ such that*

$$0 \leq F(x,t,s_2,p_2) - F(x,t,s_1,p_1) \leq c_F \ln(s_2 - s_1 + p_2 - p_1 + 1)$$

for all (s_1,p_1) and (s_2,p_2) in $\mathbb{R} \times \mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$.
(3) *F is bounded for bounded s and p.*

Then the existence of a lower solution for the initial-value problem (4.16) *provides the existence of the unique solution of the problem* (4.16).

Proof. The problem (4.16) is equivalent to the integral equation

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\ &+ \int_0^t \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))\,d\xi d\tau \end{aligned}$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-x^2}{4t}\right\}$$

for all $x \in \mathbb{R}$ and $t > 0$. The initial-value (4.16) possesses a unique solution if and only if the above integral differential equation possesses a unique solution u such that u and u_x are continuous and bounded for all $x \in \mathbb{R}$ and $0 < t \leq T$. Define a mapping $f : \Omega \to \Omega$ by

$$\begin{aligned} (fu)(x,t) &= \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\ &+ \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))\,d\xi d\tau \end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in I$. Note that, if $u \in \Omega$ is a fixed point of f, then u is a solution of the problem (4.16).

Now, we show that the hypothesis in Theorems 3.5 and 3.6 are satisfied. The mapping f is nondecreasing since, by hypothesis, for $u \geq v$,

$$F(x,t,u(x,t),u_x(x,t)) \geq F(x,t,v(x,t),v_x(x,t)).$$

By using that $k(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times (0,T]$, we conclude that

$$\begin{aligned}
(fu)(x,t) &= \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))\,d\xi d\tau \\
&\geq \int_{-\infty}^{+\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)F(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau))\,d\xi d\tau \\
&= (fv)(x,t)
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in I$. Besides, we have

$$\begin{aligned}
&|(fu)(x,t) - (fv)(x,t)| \\
&\leq \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)|F(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)) \\
&\qquad -F(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau))|d\xi d\tau \\
&\leq \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau).c_F \ln(u(\xi,\tau)-v(\xi,\tau) \\
&\qquad +u_x(\xi,\tau)-v_x(\xi,\tau)+1)\,d\xi d\tau \\
&\leq c_F \ln(G(u,v,w)+1) \int_0^t \int_{-\infty}^{+\infty} k(x-\xi,t-\tau)d\xi d\tau \\
&\leq c_F \ln(G(u,v,w)+1).T.
\end{aligned} \tag{4.17}$$

With the same way, we obtain

$$|(fv)(x,t) - (fw)(x,t)| \leq c_F \ln(G(u,v,w)+1).T \tag{4.18}$$

and

$$|(fu)(x,t) - (fw)(x,t)| \leq c_F \ln(G(u,v,w)+1).T \tag{4.19}$$

for all $u \geq v \geq w$. Similarly, we have

$$\begin{aligned}
&\left|\frac{\partial fu}{\partial x}(x,t) - \frac{\partial fu}{\partial x}(x,t)\right| \\
&\leq c_F \ln(G(u,v,w)+1) \int_0^t \int_{-\infty}^{+\infty} \left|\frac{\partial k}{\partial x}(x-\xi,t-\tau)\right| d\xi d\tau \\
&\leq c_F \ln(G(u,v,w)+1)\, 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}},
\end{aligned} \tag{4.20}$$

$$\left|\frac{\partial fv}{\partial x}(x,t)-\frac{\partial fw}{\partial x}(x,t)\right| \tag{4.21}$$
$$\leq c_F \ln\left(G(u,v,w)+1\right)\int_0^t\int_{-\infty}^{+\infty}\left|\frac{\partial k}{\partial x}(x-\xi,t-\tau)\right|d\xi d\tau$$
$$\leq c_F \ln\left(G(u,v,w)+1\right)2\pi^{\frac{-1}{2}}T^{\frac{1}{2}},$$

$$\left|\frac{\partial fu}{\partial x}(x,t)-\frac{\partial fw}{\partial x}(x,t)\right| \tag{4.22}$$
$$\leq c_F \ln\left(G(u,v,w)+1\right)\int_0^t\int_{-\infty}^{+\infty}\left|\frac{\partial k}{\partial x}(x-\xi,t-\tau)\right|d\xi d\tau$$
$$\leq c_F \ln\left(G(u,v,w)+1\right)2\pi^{\frac{-1}{2}}T^{\frac{1}{2}}.$$

Combining $(4.17), (4.18), (4.19)$ with $(4.20), (4.21), (4.22)$, we obtain

$$G(fu,fv,fw)\leq 3c_F(T+2\pi^{\frac{-1}{2}}T^{\frac{1}{2}})\ln\left(G(u,v,w)+1\right)\leq \ln\left(G(u,v,w)+1\right)$$

which implies

$$\ln(G(fu,fv,fw)+1)\leq \ln\left(\ln\left(G(u,v,w)+1\right)+1\right).$$

Put $\psi(x)=\ln(x+1)$ and $\varphi(x)=\ln\left[\ln(x+1)+1\right]$. Obviously, $\psi:[0,\infty)\to[0,\infty)$ is continuous, nondecreasing and ψ is positive in $(0,\infty)$ with, $\psi(0)=0$ and also $\psi(x)>\phi(x)$ for any $x>0$.

Finally, let $\alpha(x,t)$ be a lower solution for (4.16). Then we show that $\alpha\leq f\alpha$ integrating the following:

$$\begin{aligned}&\left(\alpha(\xi,\tau)\,k(x-\xi,t-\tau)\right)_\tau-\left(\alpha_\xi(\xi,\tau)\,k(x-\xi,t-\tau)\right)_\xi\\&\quad+\left(\alpha(\xi,\tau)\,k_\xi(x-\xi,t-\tau)\right)_\xi\\&\leq F\left(\xi,\tau,\alpha(\xi,\tau),\alpha_\xi(\xi,\tau)\right)k(x-\xi,t-\tau)\end{aligned}$$

for $-\infty<\xi<\infty$ and $0<\tau<t$, we obtain the following:

$$\begin{aligned}\alpha(x,t)\leq&\int_{-\infty}^{+\infty}k(x-\xi,t)\,\varphi(\xi)\,d\xi\\&+\int_0^t\int_{-\infty}^{+\infty}k(x-\xi,t-\tau)\,F\left(\xi,\tau,\alpha(\xi,\tau),\alpha_\xi(\xi,\tau)\right)d\xi d\tau\\=&\,(f\alpha)(x,t)\end{aligned}$$

for all $x\in\mathbb{R}$ and $t\in(0,T]$. Therefore, by Theorems 3.4 and 3.5, f has a unique fixed point. This completes the proof. $\square$

References

[1] B.C. Dhage, *Generalized metric space and mapping with fixed point*, Bull. Calcutta Math. Soc., **84** (1992), 329–336.

[2] B.C. Dhage, *Generalized metric spaces and topological structure I*, Annalele Stintifice ale Universitatii Al.I. Cuza, **46**(1) (2000), 3–24.

[3] M. Abbas, T. Nazir and S. Radenovic, *Some periodic point results in generalized metric spaces*, Appl. Math. and Comp., **217** (2010), 4094–4099.

[4] Z. Kadelburg, H.K. Nasine and S. Radenovic, *Common coupled fixed point results in partially ordered G-metric spaces*, Bull. of Math. Anal. and Appl., **4**(2) (2012), 51–63.

[5] S. Radenovic, S. Pantelic, P. Salimi and J. Vujakovic, *A note on some tripled coincidence point results in G-metric spaces*, Inter. J. of Math. Sci. and Engg. Appls. (IJMSEA), **6**(VI) (November, 2012).

[6] W. Long, M. Abbas, T. Nazir and S. Radenovic, *Common Fixed Point for Two Pairs of Mappings Satisfying (E.A) Property in Generalized Metric Spaces*, Abst. and Appl. Anal., **2012**, Article ID 394830, 15 pages, doi: 10.1155/2012/394830.

[7] B.S. Choudhury and P. Maity, *Coupled fixed point results in generalized metric spaces*, Math. Comput. Modelling, **54**(1-2) (2011), 73–79.

[8] H. Aydi, B. Damjanovic, B. Samet and W. Shatanawi, *Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces*, Math. Comput. Modelling, **54** (2011), 2443–2450.

[9] H. Aydi, W. Shatanawi and C. Vetro, *On generalized weakly G-contraction mapping in G-metric spaces*, Comput. Math. Appl., **62** (2011), 4222–4229.

[10] H. Aydi, *A fixed point result involving a generalized weakly contractive condition in G-metric spaces*, Bull. of Math. Anal. and Appl., **3**(4) (2011), 180–188.

[11] H. Aydi, W. Shatanawi and M. Postolache, *Coupled fixed point results for* (ψ,ϕ)*-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl., **63** (2012), 298–309.

[12] H. Aydi, *A common fixed point of integral type contraction in generalized metric spaces*, J. of Advan. Math. Studies, **5**(1) (2012), 111–117.

[13] JJ. Nieto, R. Rodriguez-L.Opez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Diffe. Equa. J. Order, **22** (2005), 223–239.

[14] J.Y. Cho, R. Saadati and Sh. Wang, *Common fixed point theorems on generalized distance in ordered cone metric spaces*, Comput. Math. Appl., **61** (2011), 1254–1260.

[15] H.K. Nashine, Z. Kadelburg, S. Radenovic and J.K. Kim, *Fixed point theorems under Hardy-Rogers contractive conditions on 0-complete ordered partial metric spaces*, Fixed Point Theory and Appl., **2012:180** (2012), doi:10.1186/1687-1812-2012-180.

[16] L. Gajic and Z.L. Crvenkovic, *On mappings with contractive iterate at a point in generalized metric spaces*, Fixed Point Theory Appl., **2010** (2010), doi:10.1155/2010/458086. Article ID 458086, 16 pages.

[17] M. Abbas and B.E. Rhoades, *Common fixed point results for non-commuting mappings with-out continuity in generalized metric spaces*, Appl. Math. Comput., **215** (2009), 262–269.

[18] M. Abbas, A.R. Khan and T. Nazir, *Coupled common fixed point results in two generalized metric spaces*, Appl. Math. Comput., **217** (2011), 6328–6336.

[19] M.E. Gordji, M. Ramezani, Y.J. CHo and S. Pirbavafa, *A generalization of Geraghty's theorem in partially ordered metric space and application to ordinary differential equations*, Fixed Point Theory and Appl., **2012:74** (2012), doi:10.1186/1687-1812-2012-74.

[20] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608.
[21] NV. Luong and NX. Thuan, *Coupled fixed point in partially ordered metric spaces and applications*, Nonlinear Anal: Theory Methods Appl., **74** (2011), 983–992, doi:10.1016/j.na.2010.09.055.
[22] R. Saadati, S.M. Vaezpour and Lj.B. Ciric, *Generalized distance and some common fixed point theorems*, J. Comput. Anal. Appl., **12**(1A) (2010), 157–162.
[23] R. Saadati, S.M. Vaezpour, P. Vetro and B.E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Modelling., **52** (2010), 797–801.
[24] S. Gahler, *2-Metrische Raume and ihre Topologische Struktur*, Math. Nachr., **26** (1963), 115–148.
[25] S. Gahler, *Zur Geometric 2-Metrische Raume*, Rev. Roumaine Math. Pures Appl., **11** (1966), 665–667.
[26] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393.
[27] V. Lakshmikantham and Lj. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70** (2009), 4341–4349.
[28] W. Shatanawi, *Coupled fixed point theorems in generalized metric spaces*, Hacet. J. Math. Stat., **40** (2011), 441–447.
[29] W. Shatanawi, *Fixed point theory for contractive mappings satisfying ϕ-maps in G-metric spaces*, Fixed Point Theory and Appl., **2010** (2010), Article ID 181650, 9 pages.
[30] Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory*, Ph.D. Thesis, The University of Newcastle, Callaghan, Australia, (2005).
[31] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297.
[32] Z. Mustafa and B. Sims, *Some remarks concerning D-metric spaces*, in: Proc. Int. Conf. on Fixed Point Theory and Appl., Valencia, Spain, July 2003, pp. 189–198.
[33] Z. Mustafa and B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory and Appl., **2009** (2009), Article ID 917175, 10 pages.
[34] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory and Appl., **2008** (2008), Article ID 189870, 12 pages.
[35] Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of fixed point results in G-metric spaces*, Int. J. Math. Sci., **2009** (2009), Article ID 283028, 10 pages.
[36] M. Bousselsal and Z. Mostefaoui, *(ψ, α, β)-weak contraction in partially ordered G-metric spaces*, Thai Journal of Math., **12**(1) (2014), 71–80.
[37] F. Yan, Y. Su and Q. Feng, *A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations*, Fixed Point Theory and Appl., **2012**(152) (2012), doi:10. 1186/1697-1812-2012-152.