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COMMON FIXED POINT RESULTS FOR CONTRACTIVE MAPPINGS IN COMPLEX VALUED b-METRIC SPACES

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Abstract. In this article, we extend and generalize the results of Ahmad *et al.*, and to establish the existence and the uniqueness of common fixed points for a pair of self mappings on a closed ball in complex valued b-metric space. Our results generalize well-known results in the literature.

1. Introduction

The first important result on fixed points for contractive type mapping was the well-known Banach's contraction principle [10] which was published in 1922. After this classical result, several authors have proved various extensions and generalizations of that result by considering contractive mappings on many different metric spaces. In 1989, Bakhtin [11] introduced the concept of b-metric space as a generalization of metric spaces.

Recently, Azam *et al.*, [1] first introduced the concept of complex valued metric spaces which is more general than well-known metric space and also established the common fixed results for a pair of contractive type mappings satisfying some rational expressions.

Subsequently, many authors have obtained the existence and uniqueness of fixed points and common fixed points of self-mappings in the context of complex-valued metric spaces [4, 6, 12, 13, 16, 18, 19, 20].

In 2013, Rao *et al.*, [17] introduced the notion of complex valued b-metric space which was more general than the well known complex valued metric

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spaces [1]. Afterwards, A.A. Mukheimer [14, 15] proved some common fixed point theorems of two self mappings satisfying some contraction condition on complex valued b-metric spaces.

The purpose of this paper is to extend and generalize the results of Ahmad *et al.*, [5] and obtain common fixed points for a pair of self mappings on a closed ball in complex valued b-metric space. The results given in this paper substantially extend and strengthen the results given in [1, 5, 13, 14, 16, 17].

2. Preliminaries

The following definitions and results will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex number and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

$$0 \precsim z_1 \precnsim z_2 \quad \Rightarrow \quad |z_1| < |z_2|,$$
$$z_1 \precsim z_2, z_2 \prec z_3 \quad \Rightarrow \quad z_1 \prec z_3.$$

Definition 2.1. ([17]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

Then pair (X, d) is called a complex valued b-metric space.

Example 2.2. ([17]) Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 2.3. ([17]) Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$, where

$B(x,r)$is an open ball. Then $\overline{B(x,r)} = \{y \in X : d(x,y) \precsim r\}$ is a closed ball.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}, B(x,r) \cap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family $F = \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Definition 2.4. ([17]) Let (X,d) be a complex valued b-metric space, $\{x_n\}$be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$converges to x and x is the limit point of $\{x_n\}$. We denote this by $lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x\, as\, n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 2.5. ([17]) *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 2.6. ([17]) *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

Theorem 3.1. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D and E are nonnegative reals such that $A + B + C + 2sD + 2sE < 1$. Let $S, T : X \to X$ are mappings satisfying:*

$$\begin{aligned} d(Sx,Ty) \precsim Ad(x,y) &+ B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} \\ &+ D\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + E\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)} \end{aligned} \tag{3.1}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$|d(x_0, Sx_0)| \leq (1-\lambda)|r| \tag{3.2}$$

where $\lambda = max\left\{\frac{A+sD}{1-B-sD}, \frac{A+sE}{1-B-sE}\right\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Su = Tu$.

Proof. Let x_0 be an arbitrary point in X and define $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$, where $n = 0, 1, 2, \cdots$. We will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$ by mathematical induction. Using inequality (3.2) and the fact that $\lambda = max\left\{\frac{A+sD}{1-B-sD}, \frac{A+sE}{1-B-sE}\right\} < 1$, we have $|d(x_0, Sx_0)| \leq |r|$. It implies that $x_1 \in \overline{B(x_0, r)}$. Let $x_2, x_3, \cdots x_k \in \overline{B(x_0, r)}$ for some $k \in \mathbb{N}$.

If $k = 2n+1$, where $n = 0, 1, 2, \cdots, \frac{k-1}{2}$, or $k = 2n+2$, where $n = 0, 1, 2, \cdots, \frac{k-2}{2}$, we obtain by using inequality (3.1)

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})\\
&\precsim Ad(x_{2n}, x_{2n+1}) + B\frac{d(x_{2n+1}, Tx_{2n+1})d(x_{2n}, Sx_{2n})}{1+d(x_{2n}, x_{2n+1})}\\
&\quad + C\frac{d(x_{2n}, Tx_{2n+1})d(x_{2n+1}, Sx_{2n})}{1+d(x_{2n}, x_{2n+1})} + D\frac{d(x_{2n}, Tx_{2n+1})d(x_{2n}, Sx_{2n})}{1+d(x_{2n}, x_{2n+1})}\\
&\quad + E\frac{d(x_{2n+1}, Tx_{2n+1})d(x_{2n+1}, Sx_{2n})}{1+d(x_{2n}, x_{2n+1})}\\
&\precsim Ad(x_{2n}, x_{2n+1}) + B\frac{d(x_{2n+1}, x_{2n+2})d(x_{2n}, x_{2n+1})}{1+d(x_{2n}, x_{2n+1})}\\
&\quad + D\frac{d(x_{2n}, x_{2n+2})d(x_{2n}, x_{2n+1})}{1+d(x_{2n}, x_{2n+1})}.
\end{aligned}$$

This implies that

$$\begin{aligned}
|d(x_{2n+1}, x_{2n+2})| &\leq A|d(x_{2n}, x_{2n+1})| + B\frac{|d(x_{2n+1}, x_{2n+2})||d(x_{2n}, x_{2n+1})|}{|1+d(x_{2n}, x_{2n+1})|}\\
&\quad + D\frac{|d(x_{2n}, x_{2n+2})||d(x_{2n}, x_{2n+1})|}{|1+d(x_{2n}, x_{2n+1})|}.
\end{aligned}$$

Since $|1+d(x_{2n}, x_{2n+1})| > |d(x_{2n}, x_{2n+1})|$, we have

$$\begin{aligned}
|d(x_{2n+1}, x_{2n+2})| &\leq A|d(x_{2n}, x_{2n+1})| + B|d(x_{2n+1}, x_{2n+2})| + D|d(x_{2n}, x_{2n+2})|\\
&\leq A|d(x_{2n}, x_{2n+1})| + B|d(x_{2n+1}, x_{2n+2})|\\
&\quad + sD\left\{|d(x_{2n}, x_{2n+1})| + |d(x_{2n+1}, x_{2n+2})|\right\}.
\end{aligned}$$

This implies that

$$|d(x_{2n+1}, x_{2n+2})| \leq \frac{A+sD}{1-B-sD}|d(x_{2n}, x_{2n+1})|. \tag{3.3}$$

Similarly, we get

$$|d(x_{2n+2}, x_{2n+3})| \leq \frac{A + sE}{1 - B - sE}|d(x_{2n+1}, x_{2n+2})|. \tag{3.4}$$

Putting $\lambda = max\left\{\frac{A+sD}{1-B-sD}, \frac{A+sE}{1-B-sE}\right\}$, we obtain

$$|d(x_k, x_{k+1})| \leq \lambda^k |d(x_0, x_1)| \tag{3.5}$$

for all $k \in \mathbb{N}$.

$$\begin{aligned}
&|d(x_0, x_{k+1})| \\
&\leq s|d(x_0, x_1)| + s|d(x_1, x_{k+1})| \\
&\leq s|d(x_0, x_1)| + s^2|d(x_1, x_2)| + s^2|d(x_2, x_{k+1})| \\
&\leq s|d(x_0, x_1)| + s^2|d(x_1, x_2)| + s^3|d(x_2, x_3)| + \cdots + s^{k+1}|d(x_k, x_{k+1})| \\
&\leq s|d(x_0, x_1)| + s^2\lambda|d(x_0, x_1)| + s^3\lambda^2|d(x_0, x_1)| + \cdots + s^{k+1}\lambda^k|d(x_0, x_1)| \\
&= |d(x_0, x_1)|[s + s^2\lambda + s^3\lambda^2 + \cdots + s^{k+1}\lambda^k] \\
&\leq (1 - \lambda)|r|\frac{1 - (s\lambda)^{k+1}}{1 - s\lambda} \leq |r|
\end{aligned}$$

gives $x_{k+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$ and

$$|d(x_n, x_{n+1})| \leq \lambda^n |d(x_0, x_1)| \tag{3.6}$$

for all $n \in \mathbb{N}$. Without loss of generality, we take $m > n$, then

$$\begin{aligned}
|d(x_n, x_m)| &\leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\
&\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)| \\
&\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + \cdots \\
&\quad + s^{m-n-1}|d(x_{m-2}, x_{m-1})| + s^{m-n}|d(x_{m-1}, x_m)|.
\end{aligned}$$

By using (3.6), we get

$$\begin{aligned}
|d(x_n, x_m)| &\leq s\lambda^n|d(x_0, x_1)| + s^2\lambda^{n+1}|d(x_0, x_1)| + s^3\lambda^{n+2}|d(x_0, x_1)| + \cdots \\
&\quad + s^{m-n-1}\lambda^{m-2}|d(x_0, x_1)| + s^{m-n}\lambda^{m-1}|d(x_0, x_1)| \\
&= \sum_{i=1}^{m-n} s^i\lambda^{i+n-1}|d(x_0, x_1)|, \\
|d(x_n, x_m)| &\leq \frac{(s\lambda)^n}{1 - s\lambda}|d(x_0, x_1)| \to 0 \quad \text{as} \quad m, n \to \infty.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence in $\overline{B(x_0, r)}$. Therefore, there exists a point $u \in \overline{B(x_0, r)}$ with $\lim_{n\to\infty} x_n = u$.

We prove that that $u = Su$. Let us consider

$$\begin{aligned}
&|d(u,Su)| \\
&\leq s|d(u,x_{2n+2})| + s|d(Su,Tx_{2n+1})| \\
&\leq s|d(u,x_{2n+2})| + sA|d(x_{2n+1},u)| + sB\frac{|d(x_{2n+1},Tx_{2n+1})||d(u,Su)|}{|1+d(u,x_{2n+1})|} \\
&\quad + sC\frac{|d(x_{2n+1},Su)||d(u,Tx_{2n+1})|}{|1+d(u,x_{2n+1})|} + sD\frac{|d(u,Tx_{2n+1})||d(u,Su)|}{|1+d(u,x_{2n+1})|} \\
&\quad + sE\frac{|d(x_{2n+1},Tx_{2n+1})||d(x_{2n+1},Su)|}{|1+d(u,x_{2n+1})|}.
\end{aligned}$$

Notice that,

$$\lim_{n\to\infty} |d(u,x_{2n+2})| = \lim_{n\to\infty} |d(x_{2n+1},u)| = |d(x_{2n+1},Su)| = 0.$$

Hence $|d(u,Su)| = 0$, that is, $u = Su$. It follows similarly that $u = Tu$.

For uniqueness, assume that $u^\star$ in $\overline{B(x_0,r)}$ is a another common fixed point of S and T. Then

$$\begin{aligned}
|d(u,u^\star)| \leq A|d(u,u^\star)| &+ B\frac{|d(u,Su)||d(u^\star,Tu^\star)|}{|1+d(u,u^\star)|} + C\frac{|d(u^\star,Su)||d(u,Tu^\star)|}{|1+d(u,u^\star)|} \\
&+ D\frac{|d(u,Su)||d(u,Tu^\star)|}{|1+d(u,u^\star)|} + E\frac{|d(u^\star,Su)||d(u^\star,Tu^\star)|}{|1+d(u,u^\star)|}.
\end{aligned}$$

Since $|1+d(u,u^\star)| > |d(u,u^\star)|$, so we have

$$|d(u,u^\star)| \leq (A+C)|d(u,u^\star)|.$$

This is contradiction because $A + C < 1$. Hence $u^\star = u$. Therefore, u is a unique common fixed point of S and T. This completes the proof of the Theorem. □

Corollary 3.2. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A,B,C,D and E are nonnegative reals such that $A + B + C + 2sD + 2sE < 1$. Let $T : X \to X$ satisfy:*

$$\begin{aligned}
d(Tx,Ty) \precsim Ad(x,y) &+ B\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Tx)d(x,Ty)}{1+d(x,y)} \\
&+ D\frac{d(x,Tx)d(x,Ty)}{1+d(x,y)} + E\frac{d(y,Tx)d(y,Ty)}{1+d(x,y)}
\end{aligned}$$

for all $x,y \in \overline{B(x_0,r)}$. If

$$|d(x_0,Tx_0)| \leq (1-\lambda)|r|,$$

where $\lambda = max\left\{\frac{A+sD}{1-B-sD}, \frac{A+sE}{1-B-sE}\right\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$ *.*

Proof. We can prove this result by applying Theorem 3.1 by setting $S = T$. □

Remark 3.3. The result of Theorem 3.1 remains true if the condition (3.2) is replaced by the condition $|d(x_0, Tx_0)| \leq (1 - \lambda)|r|$.

Corollary 3.4. *Suppose that* (X, d) *is a complete complex valued b-metric space with the coefficient* $s \geq 1$ *and* $x_0 \in X$*. Let* $0 \prec r \in \mathbb{C}$ *and* A, B, C, D *be four nonnegative reals such that* $A + B + C + 2sD < 1$*. Let* $S, T : X \to X$ *are mappings satisfying:*

$$d(Sx, Ty) \precsim Ad(x, y) + B\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} + D\frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$*. If*

$$|d(x_0, Sx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = max\left\{\frac{A+sD}{1-B-sD}, \frac{A}{1-B}\right\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Su = Tu$ *.*

Proof. We can prove this result by applying Theorem 3.1 by setting $E = 0$. □

Corollary 3.5. *Suppose that* (X, d) *is a complete complex valued b-metric space with the coefficient* $s \geq 1$ *and* $x_0 \in X$*. Let* $0 \prec r \in \mathbb{C}$ *and* A, B, C, D *be four nonnegative reals such that* $A + B + C + 2sD < 1$*. Let* $T : X \to X$ *satisfy:*

$$d(Tx, Ty) \precsim Ad(x, y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)} + D\frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$*. If*

$$|d(x_0, Tx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = max\left\{\frac{A+sD}{1-B-sD}, \frac{A}{1-B}\right\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$*.*

Proof. We can prove this result by applying Corollary 3.4 by setting $S = T$. □

Corollary 3.6. *Suppose that (X, d) is a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C and E be four nonnegative reals such that $A + B + C + 2sE < 1$. Let $S, T : X \to X$ are mappings satisfying:*

$$d(Sx, Ty) \precsim Ad(x, y) + B\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} + E\frac{d(y, Sx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. If

$$|d(x_0, Sx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = max\left\{\frac{A}{1-B}, \frac{A+sE}{1-B-sE}\right\}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Su = Tu$.

Proof. We can prove this result easily by applying Theorem 3.1 by setting $D = 0$. □

Corollary 3.7. *Suppose that (X, d) is a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C and E be four nonnegative reals such that $A + B + C + 2sE < 1$. Let $T : X \to X$ satisfy:*

$$d(Tx, Ty) \precsim Ad(x, y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)} + E\frac{d(y, Tx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. If

$$|d(x_0, Tx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = max\left\{\frac{A}{1-B}, \frac{A+sE}{1-B-sE}\right\}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Proof. By setting $S = T$ in Corollary 3.6, we get the required result of Corollary 3.7. □

Corollary 3.8. *Suppose that (X, d) is a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C be three nonnegative reals such that $sA + B + C < 1$. Let $S, T : X \to X$ are* *mappings satisfying:*

$$d(Sx,Ty) \precsim Ad(x,y) + B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$|d(x_0, Sx_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Su = Tu$.

Proof. We can prove this result by applying Theorem 3.1 by setting $D = E = 0$. Our result is the extension of the Theorem 2.1 of [16] to the closed ball in complex valued b-metric space. □

Corollary 3.9. *Suppose that* (X, d) *is a complete complex valued b-metric space with the coefficient* $s \geq 1$ *and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B, C *be three nonnegative reals such that* $sA + B + C < 1$. *Let* $T : X \to X$ *satisfy:*

$$d(Tx,Ty) \precsim Ad(x,y) + B\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Tx)d(x,Ty)}{1+d(x,y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$|d(x_0, Tx_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$.

Proof. We can prove this result by applying Corollary 3.8 by setting $S = T$. □

Corollary 3.10. *Suppose that* (X, d) *is a complete complex valued b-metric space with the coefficient* $s \geq 1$ *and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B *be nonnegative reals such that* $sA + B < 1$. *Let* $S, T : X \to X$ *are mappings satisfying:*

$$d(Sx,Ty) \precsim Ad(x,y) + B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$|d(x_0, Sx_0)| \leq (1-\lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Su = Tu$.

Proof. By setting $C = D = E = 0$ in Theorem 3.1, we get the required result of Corollary 3.10. Our result is the extension of the Theorem 4 of [1] to the closed ball in complex valued b-metric space. □

Corollary 3.11. *Suppose that (X, d) is a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B be nonnegative reals such that $sA + B < 1$. Let $T : X \to X$ satisfy:*

$$d(Tx, Ty) \precsim Ad(x, y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. If

$$|d(x_0, Tx_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Proof. We can prove this result by applying Corollary 3.10 by setting $S = T$. □

Corollary 3.12. *Suppose that (X, d) is a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D and E be five nonnegative reals such that $A + B + C + 2sD + 2sE < 1$. Let $T : X \to X$ satisfy:*

$$\begin{aligned} d(T^n x, T^n y) \precsim Ad(x, y) &+ B\frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)} + C\frac{d(y, T^n x)d(x, T^n y)}{1 + d(x, y)} \\ &+ D\frac{d(x, T^n x)d(x, T^n y)}{1 + d(x, y)} + E\frac{d(y, T^n x)d(y, T^n y)}{1 + d(x, y)} \end{aligned}$$

for all $x, y \in \overline{B(x_0, r)}$ and

$$|d(x_0, T^n x_0)| \leq (1 - \lambda)|r|,$$

where $\lambda = max\left\{\frac{A+sD}{1-B-sD}, \frac{A+sE}{1-B-sE}\right\}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Proof. For some fixed n, we obtain $u \in \overline{B(x_0, r)}$ such that $T^n u = u$. The uniqueness follows from

$$\begin{aligned} &d(Tu, u) = d(TT^n u, T^n u) = d(T^n Tu, T^n u) \\ &\precsim Ad(Tu, u) + B\frac{d(Tu, T^n Tu)d(u, T^n u)}{1 + d(Tu, u)} + C\frac{d(u, T^n Tu)d(Tu, T^n u)}{1 + d(Tu, u)} \\ &\quad + D\frac{d(Tu, T^n Tu)d(Tu, T^n u)}{1 + d(Tu, u)} + E\frac{d(u, T^n Tu)d(u, T^n u)}{1 + d(Tu, u)} \\ &\precsim Ad(Tu, u) + C\frac{d(u, TT^n u)d(Tu, u)}{1 + d(Tu, u)} + D\frac{d(Tu, TT^n u)d(Tu, u)}{1 + d(Tu, u)} \\ &\precsim Ad(Tu, u) + C\frac{d(u, Tu)d(Tu, u)}{1 + d(Tu, u)}. \end{aligned}$$

Taking modulus in above, we get

$$|d(Tu,u)| \leq A|d(Tu,u)| + C\frac{|d(u,Tu)||d(Tu,u)|}{|1+d(Tu,u)|}.$$

Since $|1+d(Tu,u)| > |d(Tu,u)|$, so we get

$$|d(Tu,u)| \leq (A+C)|d(Tu,u)|,$$

a contradiction. So $u = Tu$.

Hence $Tu = T^n u = u$. Therefore, the fixed point of T is unique. This completes the proof. □

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