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ON THE QUASIMODULES AND NORMED QUASIMODULES

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Abstract. In this paper, we give definition of quasiring as a new concept. Also, we introduce the notions of quasimodule and normed quasimodule defined on a quasiring. We should immediately note that a quasimodule is a generalization of quasilinear spaces. Similarly, normed quasimodule is a generalization of normed quasilinear spaces defined by Aseev, [3]. Moreover, we obtain some results about the relationships between these concepts. We think that investigations on quasimodules may provide some important contributions to improvement of some branches of quasilinear functional analysis such as the duality theory of quasilinear spaces. Also we recognize that the notion of quasimodule is more suitable backdrop as regards theory of quasilinear spaces in examination of quasilinear functional.

1. Introduction

As is known, many problems in real world are characterizated by differantial equations containing of single valued functions. But some problems can not be represented by means of these equations. Such these problems can be characterizated with multivalued differential equations (known as set differential equations) generated by multivalued differential inclusions. Researches on characterization of these problems have been presented in references such as [1], [4], [8] and so on. In 1986, Aseev [3] introduced the concept of quasilinear

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spaces both including classical linear spaces and modelling nonlinear spaces of subsets and multivalued mappings from a single point of view. Then he proceeded a similar way to linear functional analysis on quasilinear spaces by introducing the notions of the norm, quasilinear operators and functionals. Further, he presented some results which are quasilinear counterparts of fundamental definitions and theorems in linear functional analysis and differential and integral calculus in Banach spaces. This pioneering work has motivated a lot of author to introduce some new results concerning with quasilinear spaces, [6], [7], [10], [11].

Section 2 has a preparatory character. It collects the definitions of some algebraic structures which will be needed further on, such as ordered monoid, ordered ring. Also, in this section, we present some definitions and preliminaries results about quasilinear spaces and normed quasilinear spaces given in [3] and [11]. One can see that, as different from linear spaces, Aseev used the partial order relation when he defined quasilinear spaces and he gave coherent counterparts of results in linear spaces.

In final section, firstly we introduce the concept of "quasiring" which is a generalization of ring and semiring. Also this new concept is different from ordered semiring. Moreover, in this section, we give the relation between quasiring and field. Then we introduce "quasimodules" and "normed quasimodules" defined on a quasiring. We also obtain some results related to these concepts.

We write $\Omega(\mathbb{R})$ and $\Omega_C(\mathbb{R})$ to denote the family of all nonempty, closed-bounded and nonempty, convex, closed-bounded subsets of $\mathbb{R}$, respectively.

The purpose of this study is to give foundations of mathematical structures of quasiring and quasimodules. Also this paper especially constitutes the basics of investigations concerning with quasimodules on the quasiring $\Omega_C(\mathbb{R})$. We think that investigations on quasimodules may provide some important contributions to improvement of some branches of quasilinear functional analysis such as the duality theory of quasilinear spaces. We recognized that the notion of quasimodules is more suitable backdrop as regards theory of quasilinear spaces in examination of quasilinear functional.

2. Preliminaries and Some Results About Quasilinear Spaces and Normed Quasilinear Spaces

A *semigroup* is a set G together with a binary operation " $\cdot$ " (that is, a function $\cdot : G \times G \to G$) that satisfies the associative property: for all $a, b, c \in G$, the equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

An *ordered semigroup* is a semigroup $(G, \cdot)$ together with a partial order " $\leq$ " that is compatible with the semigroup operation, meaning that $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. for all a, b, c in G.

A *monoid* is a semigroup with an identity element $e \in G$ such that $e \cdot a = a \cdot e = a$.

An *ordered monoid* G is a monoid equipped with a partial order " $\leq$ ", such that $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ if $a \leq b$, for all $a, b, c \in G$.

A *group* G is a monoid with an inverse namely such that for every element $x \in G$ there exists an element $y \in G$ such that $xy = yx = e$. An inverse is unique and is denoted by x^{-1}.

An *ordered group* is a group $(G, \cdot)$ equipped with a partial order " $\leq$ " that is translation-invariant; in other words, " $\leq$ " has the property that, for all a, b and c in G, if $a \leq b$ then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

A *semiring* is a set R equipped with two binary operations " $+$ " and " $\cdot$ " called addition and multiplication with the following properties; $(R, +)$ is a commutative monoid with identity element θ and $(R, \cdot)$ is a monoid with identity element 1 and further multiplication left and right distributes over addition. Also multiplication by θ annihilates R, that is $\theta \cdot a = a \cdot \theta = \theta$.

An *ordered semiring* is a semiring equipped with a partial order relation " $\leq$ " such that

$$\begin{gathered} a + c \leq b + c \quad \text{and} \quad c + a \leq c + b \quad \text{if} \quad a \leq b, \\ \theta \leq a \cdot b \quad \text{if} \quad \theta \leq a \quad \text{and} \quad \theta \leq b, \end{gathered}$$

for all $a, b, c \in R$.

An *ordered ring* is a ring $(R, +, \cdot)$, together with a compatible partial order, i.e., a partial order on the underlying set A that is compatible with the ring operations in the sense that it satisfies: $a \leq b$ implies $a + c \leq b + c$ and $\theta \leq a$ and $\theta \leq b$ imply that $0 \leq a \cdot b$ for all $a, b, c \in R$.

Let we recall some definitions and auxiliary facts introduced in [3], [6], [11].

Definition 2.1. ([3]) A set X is called a *quasilinear space (qls,* for short*)*, if a partial order relation "$\leq$", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any real numbers α, β:

$$x \leq x, \qquad \text{(QLS 1)}$$

$$x \leq z \text{ if } x \leq y \text{ and } y \leq z, \qquad \text{(QLS 2)}$$

$$x = y \text{ if } x \leq y \text{ and } y \leq x, \tag{QLS 3}$$

$$x + y = y + x, \tag{QLS 4}$$

$$x + (y + z) = (x + y) + z, \tag{QLS 5}$$

$$\text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \tag{QLS 6}$$

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x, \tag{QLS 7}$$

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \tag{QLS 8}$$

$$1 \cdot x = x, \tag{QLS 9}$$

$$\theta \cdot x = \theta, \tag{QLS 10}$$

$$(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x, \tag{QLS 11}$$

$$x + z \leq y + v \text{ if } x \leq y \text{ and } z \leq v, \tag{QLS 12}$$

$$\alpha \cdot x \leq \alpha \cdot y \text{ if } x \leq y. \tag{QLS 13}$$

A linear space is a qls with the partial order relation "=". The most popular example of quasilinear spaces which is not a linear space is the set of all closed intervals of real numbers with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}.$$

We denote this set by $\Omega_C(\mathbb{R})$. Another one is $\Omega(\mathbb{R})$ which is the set of all compact subsets of real numbers. In general, $\Omega(E)$ and $\Omega_C(E)$ stand for the space of all nonempty, closed, bounded and nonempty, convex and closed bounded subsets of any normed linear space E, respectively. Also $\Omega(E)$ and $\Omega_C(E)$ are quasilinear spaces with the inclusion relation and with a slight modification of addition defined by

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}}$$

and with the real scalar multiplication above.

In a qls X, the element θ is minimal, i.e., $x = \theta$ if $x \leq \theta$. The element x' is called *inverse* of x if $x + x' = \theta$. If the inverse of an element exists, then it is unique.

An element x having inverse is called *regular*, otherwise is called *singular*. Note that the minimality is not only a property of θ but also is shared by the other regular elements, [11]. X_r and X_s stand for the sets of all regular and singular elements in X, respectively.

It will be assumed that $-x = (-1)x$. An element x is regular if and only if $x - x = \theta$ equivalently $x' = -x$.

Suppose that any element x has inverse element x'. Then the partial order in X is determined by equality, the distributivity conditions hold and consequently, X is a linear space. In a real linear space, equality is the only way to define a partial order such that the conditions (QLS 1)-(QLS 13) hold.

Suppose that X is a qls and $Y \subseteq X$. Then Y is called a *subspace* of X if Y is a qls with the same partial order and the restriction of the operations on X. Y is subspace of a qls X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \cdot x + \beta \cdot y \in Y$.

Let X be a qls and Y be a subspace of X. Suppose that each element x in Y has inverse element $x' \in Y$ then the partial order on Y is determined by the equality. In this case, the distibutivity conditions hold and Y is a linear subspace of the qls X.

An element $x \in X$ is called symmetric if $-x = x$ and X_d denotes the set of all symmetric elements.

X_r, X_d and $X_s \cup \{0\}$ are subspaces of X. X_r, X_d and $X_s \cup \{0\}$ are called regular, symmetric and singular subspaces of X, respectively. For example, let $X = \Omega_C(\mathbb{R})$ and $Z = \{0\} \cup \{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$. Z is singular subspace of X. On the other hand, the set of all singletons of real numbers $\{\{a\} : a \in \mathbb{R}\}$ is regular subspace of X.

In a qls X, every regular element is minimal. For example, let $X = \Omega_C(\mathbb{R})$ and V be singular subspace of X. Then V is a set containing all closed intervals in addition to $\{0\}$ and so V is a qls with the partial order and operations on X. $\{0\}$ is the only minimal element in V, [11].

Definition 2.2. ([3]) Let X be a qls. A real function $\|\cdot\|_X : X \longrightarrow \mathbb{R}$ is called a *norm* if the following conditions hold:

$$\|x\|_X > 0 \quad \text{if} \quad x \neq 0, \tag{NQLS 1}$$

$$\|x + y\|_X \leq \|x\|_X + \|y\|_X , \tag{NQLS 2}$$

$$\|\alpha \cdot x\|_X = |\alpha| \, \|x\|_X , \tag{NQLS 3}$$

$$\|x\|_X \leq \|y\|_X \quad \text{if} \quad x \leq y, \tag{NQLS 4}$$

if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that,

$$x \leq y + x_\varepsilon \quad \text{and} \quad \|x_\varepsilon\|_X \leq \varepsilon \quad \text{then} \quad x \leq y. \tag{NQLS 5}$$

A qls X with a norm defined on it is called *normed quasilinear space (briefly, normed qls)*. If any $x \in X$ has inverse element $x' \in X$ then the concept of normed qls coincides with the concept of real normed linear space.

Let X be a normed qls and $x, y \in X$. Hausdorff metric on X is defined by the equality

$$h_X(x, y) = \inf \left\{ r \geq 0 : x \leq y + a_1^r, \; y \leq x + a_2^r, \|a_i^r\| \leq r \right\}.$$

Since $x \leq y + (x - y)$ and $y \leq x + (y - x)$, the function h_X is well-defined and $h_X(x, y) \leq \|x - y\|_X$. It is not hard to see that this function h_X satisfies all of the metric axioms.

Lemma 2.3. ([3]) *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is a continuous function with respect to the Hausdorff metric.*

Lemma 2.4. ([3])

(a) *Suppose that $x_n \to x_0$ and $y_n \to y_0$, and that $x_n \leq y_n$ for any positive integer n. Then $x_0 \leq y_0$.*
(b) *Suppose that $x_n \to x_0$ and $z_n \to x_0$. If $x_n \leq y_n \leq z_n$ for any n, then $y_n \to x_0$.*
(c) *Suppose that $x_n +\, y_n \to x_0$ and $y_n \to \theta$, then $x_n \to x_0$.*

Let X be a real complete normed linear space (a real Banach space). Then X is a complete normed qls with partial order relation given by equality. Conversely, if X is a complete normed qls and any x has inverse element $x^{'}$, then X is a real Banach space and the partial order relation on X is equality. In this case $h_X(x, y) = \|x - y\|_X$.

Let E be a real normed linear space. The norm on $\Omega(E)$ is defined by

$$\|A\|_{\Omega(E)} = \sup \|a\|_E \,.$$

Then $\Omega(E)$ and $\Omega_C(E)$ are normed quasilinear spaces. In this case the Hausdorff metric is defined as usual:

$$h_\Omega(A, B) = \inf\{r \geq 0 : A \subseteq B + S_r(\theta),\ B \subseteq A + S_r(\theta)\},$$

where $S_r(\theta)$ stands for the closed ball centered at θ with radius r.

Let us write $C(S, X)$ to denote the set of all continuous mappings $f : S \to X$. Then $C(S, X)$ is a normed qls with the partial order relation

$$f_1 \leq f_2 \quad \text{if} \quad f_1(s) \leq f_2(s) \quad \text{for any} \quad s \in S$$

and the algebraic sum operation

$$(f_1 + f_2)(s) = f_1(s) + f_2(s)$$

and the operation of multiplication by real numbers

$$(\alpha \cdot f)(s) = \alpha \cdot f(s)$$

as well as the norm

$$\|f\|_C = \max_{s \in S} \|f(s)\|_X \,.$$

3. The main results

In this section we introduce the notions of "quasimodules" and "normed quasimodules" as a generalization of the quasilinear spaces and normed quasilinear spaces given in [3]. Also, we obtain some results concerning with these concepts.

Definition 3.1. A *quasiring* is a nonempty set R equipped with a partial order relation " $\leq$ " on R and two binary operations " $+$ " and " $\cdot$ " called addition and multiplication, respectively, such that

(QR 1) $(R, +, \leq)$ is a commutative ordered monoid with identity element θ :
- (a) $(a+b)+c = a+(b+c)$,
- (b) $\theta + a = a + \theta = a$,
- (c) $a+b = b+a$,
- (d) $a \leq b \Rightarrow a+c \leq b+c$,

(QR 2) $(R, \cdot, \leq)$ is an ordered monoid with identity element 1 :
- (a) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- (b) there exists an element $1 \in R$ such that $1 \cdot a = a \cdot 1 = a$,
- (c) $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ if $a \leq b$,

(QR 3) The multiplication operation sub distributes over the addition operation:
- (a) $a \cdot (b+c) \leq a \cdot b + a \cdot c$,
- (b) $(a+b) \cdot c \leq a \cdot c + b \cdot c$

hold, for all $a, b, c \in R$.

We write $(R, +, \cdot, \leq)$ to denote the quasiring defined in this way. Hence a quasiring is a commutative ordered monoid with respect to " $+$ " and ordered monoid with respect to " $\cdot$ " and satisfies the condition (QR 3).

The following remark is important since it emphasizes that a quasiring is different from ordered semiring.

Remark 3.2. The concept of quasiring is different from the notion of ordered semiring. Because *an ordered semiring* is a semiring equipped with a partial order relation " $\leq$ " such that

$$a + c \leq b + c \quad \text{and} \quad c + a \leq c + b \quad \text{if} \quad a \leq b,$$
$$\theta \leq a \cdot b \quad \text{if} \quad \theta \leq a \quad \text{and} \quad \theta \leq b,$$

for all $a, b, c \in R$. Whereas, a quasiring is a mathematical structure satisfying the conditions (QR 1)-(QR 3).

Now we give an example account for this case:

The semiring $(\mathbb{R}, +, \cdot)$ is an ordered semiring with the partial order relation " $\leq$ ", but $(\mathbb{R}, +, \cdot)$ is not a quasiring with the same partial order relation " $\leq$ " since the condition (QR 2)-(c) can not be satisfied. To illustrate this, we consider the following fact

$$-3 \leq -1 \quad \text{but} \quad -3 \cdot (-2) \nleq -1 \cdot (-2).$$

On the other hand $(\mathbb{R}, +, \cdot)$ is an ordered semiring and it is a quasiring when the relation " $=$ " is considered as partial ordered relation instead of the relation " $\leq$ ".

An ordered semiring $(R, +, \cdot)$ is a quasiring when

$$x \leq y \quad \text{if and only if} \quad x = y$$

holds.

Example 3.3. Every ring and every semiring is a quasiring with partial order relation " $=$ ".

Now, we will give an example showing that there exists a quasiring which is not a ring. Since it shows that the concept of quasiring is a generalization of the notion of ring, the following example is considerable.

Example 3.4. Consider the set $\Omega_C(\mathbb{R})$ with the operations " $+$ " and " $\cdot$ " defined by

$$\begin{aligned}[a,b] + [c,d] &= \{u + v : u \in [a,b], \ v \in [c,d]\} \\ &= [a + c, b + d],\end{aligned}$$

and

$$\begin{aligned}[a,b] \cdot [c,d] &= \{u \cdot v : u \in [a,b], \ v \in [c,d]\} \\ &= [\min(a \cdot c, a \cdot d, b \cdot c, b \cdot d), \max(a \cdot c, a \cdot d, b \cdot c, b \cdot d)]\end{aligned}$$

for all $[a,b]$, $[c,d] \in \Omega_C(\mathbb{R})$. It can be easily seen that the inclusion relation defined by

$$[a,b] \leq [c,d] \text{ if and only if } [a,b] \subseteq [c,d]$$

is a partial order relation on $\Omega_C(\mathbb{R})$.

Since it is easy to show that the conditions (QR 1)-(a),(b),(c) hold, we omit its. Where, we only show that the condition (QR 1)-(d) holds:

$$\begin{aligned}&[a,b] \subseteq [c,d] \\ &\Rightarrow (c \leq a) \ \wedge \ (b \leq d) \\ &\Rightarrow (c + e \leq a + e) \ \wedge \ (b + f \leq d + f) \\ &\Rightarrow [a + e, b + f] \subseteq [c + e, d + f] \\ &\Rightarrow [a,b] + [e,f] \subseteq [c,d] + [e,f]\end{aligned}$$

for all $[a,b]$, $[c,d] \in \Omega_C(\mathbb{R})$. Hence $(\Omega_C(\mathbb{R}), +, \subseteq)$ is a commutative ordered monoid. Moreover $(\Omega_C(\mathbb{R}), \cdot, \subseteq)$ is an ordered monoid (see [1]). The singletons

$\{0\}$ and $\{1\}$ are identity elements of addition and mutiplication operations, respectively. Furthermore one can easily see that the includings

$$[a,b]\cdot([c,d]+[e,f])\subseteq[a,b]\cdot[c,d]+[a,b]\cdot[e,f]$$

and

$$([a,b]+[c,d])\cdot[e,f]\subseteq[a,b]\cdot[e,f]+[c,d]\cdot[e,f]$$

hold. So $(\Omega_C(\mathbb{R}),+,\cdot,\subseteq)$ is a quasiring.

Whereas $(\Omega_C(\mathbb{R}),+,\cdot)$ is not a ring since there exist some elements which have not inverse in $\Omega_C(\mathbb{R})$. For example the singular elements $[a,b]$ (with the condition $a\neq b$) have not inverse with respect to the operation " + ".

Example 3.5. Consider the set $\Omega(\mathbb{R})$ with the operations Minkowskii addition and set multiplication defined by

$$A+B=\{a+b: a\in A \text{ and } b\in B\},$$

and

$$A\cdot B=\{a\cdot b: a\in A \text{ and } b\in B\},$$

respectively. Then $(\Omega(\mathbb{R}),+,\subseteq)$ is a commutative ordered monoid and $(\Omega(\mathbb{R}),\cdot,\subseteq)$ is an ordered monoid (see [1]). So $(\Omega(\mathbb{R}),+,\cdot,\subseteq)$ is a quasiring.

Definition 3.6. Let $(R,+,\cdot,\leq)$ is a quasiring and $R'\subseteq R$. Then we say that $(R',+,\cdot,\leq)$ is a *subquasiring* of $(R,+,\cdot,\leq)$ if R' is a quasiring with the algebraic operations and partial order relation on R.

By the way, $\Omega_C(\mathbb{R})$ is a subquasiring of $\Omega(\mathbb{R})$ with algebraic operations and partial order relation defined before.

Proposition 3.7. *θ_R is a minimal element of the quasiring* $(R,+,\cdot,\leq)$.

Proof. Suppose that $x\leq\theta_R$ for $x\in R$. Then we can write

$$(-1_R)\cdot x\leq(-1_R)\cdot x,$$

where (-1_R) is inverse of 1_R with respect to " + ". From (QR 1)-(d), we can write

$$x+(-1_R)\cdot x\leq\theta_R+(-1_R)\cdot x=(-1_R)\cdot x. \tag{3.1}$$

Also, by using the conditions (QR 1)-(b), (QR 3)-(b) and the relation (3.1), we get

$$\theta_R=(1_R+(-1_R))\cdot x\leq x+(-1_R)\cdot x\leq(-1_R)\cdot x.$$

On the other hand, from (QR 2)-(c)

$$(-1_R)\cdot\theta_R\leq(-1_R)\cdot(-1_R)\cdot x=x \tag{3.2}$$

is obtained and we know that

$$(-1_R) \cdot \theta_R = \theta_R. \tag{3.3}$$

Thus, (3.2) and (3.3) give us that

$$\theta_R \leq x.$$

Hence $x = \theta_R$. This completes the proof. □

The following lemma gives the relation between quasiring and field:

Lemma 3.8. *If every element of a quasiring R (except θ_R) has an inverse with respect to " $+$ " and " $\cdot$ ", the partial order relation " $\leq$ " turns to the relation " $=$ " and this quasiring becomes a field.*

Proof. Assume that $x \leq y$. Then, from (QR 1)-(d), we get

$$x + y' \leq y + y' = \theta_R$$

where y' is inverse of y with respect to "$+$". The minimality of the element θ_R implies that $x + y' = \theta_R$. Since inverse element is unique, then $x = y$. In this case, the relation " $\leq$ " turns to the relation " $=$ " and (QR 1)-(d), (QR 2)-(c) are automatically satisfied. Also (QR 3)-(a) and (QR 3)-(b) turn into the property of distributivity. Consequently this quasiring becomes a field. □

Any field is a quasiring if and only if the relation " $=$ " is considered as partial order relation.

Definition 3.9. An element of a quasiring R which has inverse with respect to " $+$ " is called *additive regular element.* Similarly an element of a quasiring R is called *multiplicative regular element* if it has inverse with respect to " $\cdot$ " It can be easily seen that if inverse of an element exists, then it is unique. R_{ar} and R_{mr} stand for the sets of all additive regular and multiplicative regular elements in R, respectively. Note that the family of all additive and multiplicative regular elements of a quasiring R is subquasiring of R.

For example, a singleton in the quasiring $\Omega_C(\mathbb{R})$ is an additive regular and a multiplicative regular element. But, an interval has not inverse with respect to addition and multiplication operation.

Proposition 3.10. *In a quasiring $(R, +, \cdot, \leq)$, every additive regular element is minimal.*

Proof. We must show that $y \leq x$ implies $y = x$ for each $x \in R_{ar}$. Assume that $y \leq x$. Then, from (QR 1)-(d), we obtain

$$y + x^{-1} \leq x + x^{-1} = \theta_R,$$

where x^{-1} is inverse of x with respect to "$+$". The minimality of the element θ_R implies that $y + x^{-1} = \theta_R$. Since inverse element is unique then $y = x$. □

Proposition 3.11. *In a quasiring $(R, +, \cdot, \leq)$, every multiplicative regular element is minimal.*

Proof. Let $x \in R_{mr}$ and $y \leq x$. Then, using (QR 2)-(c) we get

$$y \cdot x' \leq x \cdot x' = 1_R,$$

where x' is inverse of x with respect to "$\cdot$". The minimality of the additive regular element 1_R implies that $y \cdot x' = 1_R$. Since inverse element is unique then $y = x$. □

Proposition 3.12. *Let $(R, +, \cdot, \leq)$ be a quasiring. Then*

$$a \leq b + (a - b)$$

and

$$b \leq a + (b - a)$$

hold for every $a, b \in R$.

Proof. Since $\theta_R = (1_R + (-1_R)) \cdot a \leq 1_R \cdot a + (-1_R) \cdot a = a - a$, taking into account (QR 1)-(d) we have $\theta_R + b \leq (a - a) + b$, where $-a$ denotes $(-1_R) \cdot a$. On the other hand, we write $b \leq a + (-a + b)$ from the conditions (QR 1)-(a) and (QR 1)-(c). Thus

$$\begin{aligned} b &\leq a + (b + (-1_R) \cdot a) \\ &= a + (b - a) \end{aligned}$$

holds. Similarly it can be obtained that $a \leq b + (a - b)$. □

Now let us now give another important definition.

Definition 3.13. Let X be a nonempty set and $(R, +, \cdot, \leq)$ is a quasiring. Then X is called *a quasimodule* on the quasiring R if the following conditions hold:

$$\alpha \odot (\beta \odot x) = (\alpha \cdot \beta) \odot x, \tag{QM 1}$$

$$\alpha \odot (x \oplus y) = (\alpha \odot x) \oplus (\alpha \odot y), \tag{QM 2}$$

$$1_R \odot x = x, \tag{QM 3}$$

$$0_R \odot x = \theta, \tag{QM 4}$$

$$(\alpha + \beta) \odot x \precsim (\alpha \odot x) \oplus (\beta \odot x), \tag{QM 5}$$

$$x \precsim y \text{ and } z \precsim v \Rightarrow x \oplus z \precsim y \oplus v, \tag{QM 6}$$

$$\alpha \leq \beta \text{ and } x \precsim y \Rightarrow \alpha \odot x \precsim \beta \odot y \tag{QM 7}$$

for all $x, y, z, v \in X$ and $\alpha, \beta \in R$. Where " $\oplus$ " and " $\odot$ " are two operations defined by

$$\begin{aligned} \oplus : X \times X &\to X \\ (x, y) &\to x \oplus y \end{aligned}$$

and

$$\begin{aligned} \odot : R \times X &\to X \\ (\alpha, x) &\to \alpha \odot x \end{aligned}$$

such that X is a commutative monoid with " $\oplus$ ".

Note that, since real numbers set $\mathbb{R}$,which has property of field is subquasiring of $\Omega(\mathbb{R})$ and $\Omega_C(\mathbb{R})$, then quasimodules on this field coincide with the concept of qls. Hence quasimodules is a generalization of quasilinear spaces.

The principal aim of this paper is to give foundation of mathematical structures of quasiring and quasimodules which constitutes the basics of investigations concerning with quasimodules especially on the quasiring $\Omega_C(\mathbb{R})$. We think that investigations on quasimodules on a quasiring may provide some important contributions to improvement of some branches of quasilinear functional analysis such as the duality theory of quasilinear spaces. Further the notion of quasimodule is more suitable backdrop as regards theory of quasilinear spaces in examination of quasilinear functional.

Proposition 3.14. *Every quasiring is a quasimodule on itself.*

Proof. Assume that $(Q, +, \cdot, \leq)$ is a quasiring. Consider $X = Q$, $R = Q$ and $\precsim = \leq$. If the operation " $\oplus$ " and " $\odot$ " are considered as the addition and multiplication operations on Q, respectively, then $(R, +, \leq)$ is a commutative ordered monoid and the following conditions hold:

$$\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x,$$

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y,$$

$$1_R \cdot x = x,$$

$$0_R \cdot x = \theta,$$

$$(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x.$$

If $x \leq y$ and $z \leq v$ then by using (QR 1)-(d) we write $x+z \leq y+z$, $y+z \leq y+v$ and since " $\leq$ " is a partial order relation we have $x + z \leq y + v$. If $\alpha \leq \beta$ and $x \leq y$ then by using (QR 2)-(c) we get $\alpha \cdot x \leq \alpha \cdot y$, $\alpha \cdot y \leq \beta \cdot y$ and since " $\leq$ " is a partial order relation we have $\alpha \cdot x \leq \beta \cdot y$. Hence the quasiring R is a quasimodule on itself. □

Definition 3.15. Let X be a quasimodule and $Y \subseteq X$. Then we called that Y is a subquasimodules of X if Y is a quasimodule with the operations and relation on X.

In a quasimodule, an element which has inverse is called *regular element.* Since the operation of multiplication of an element with an interval is an external operation, the notion of inverse of an element with respect to multiplication operation is meaningless in a quasimodule in contrast to field. Hence it is not need that concept of inverse of an element with respect to multiplication and so inverse of an element means the inverse of this element respect to addition operation, in a quasi module.

Definition 3.16. Let X be a quasimodule on a quasiring R. Then the set of all regular (singular) elements is a quasimodule with the operations and relation on X and this set is called regular (singular) subquasimodule of X and denoted by X_r and X_s, respectively. An element $x \in X$ is called symmetric element if $[-1,-1] \odot x = x$, and X_d denotes the set of all symmetric elements. Note that this set is a quasimodule with the operations and relation on X.

Proposition 3.17. *$\Omega(\mathbb{R})$ is a quasimodule on the quasiring $\Omega_C(\mathbb{R})$.*

Proof. Consider the operations

$$\begin{aligned} \oplus : \Omega(\mathbb{R}) \times \Omega(\mathbb{R}) \to \Omega(\mathbb{R}) \\ (A,B) \to A \oplus B = \{a+b : a \in A,\ b \in B\} \end{aligned}$$

and

$$\begin{aligned} \odot : \Omega_C(\mathbb{R}) \times \Omega(\mathbb{R}) \to \Omega(\mathbb{R}) \\ (C,A) \to C \odot A = \{c \cdot a : c \in C,\ a \in A\} \end{aligned}$$

and the partial order relation " $\subseteq$ " on $\Omega(\mathbb{R})$.

For every $C, C' \in \Omega_C(\mathbb{R})$ and $A, B, D, E \in \Omega(\mathbb{R})$, the following equalities hold:

$$\begin{gathered} C \odot \big(C' \odot A\big) = (C \cdot C') \odot A, \\ C \odot (A \oplus B) = (C \odot A) \oplus (C \odot B), \\ 1_R \odot A = \{1 \cdot a : a \in A\} = \{a : a \in A\} = A, \\ \theta_{\Omega_C(\mathbb{R})} \odot A = \{\theta \cdot a : a \in A\} = \{\theta\colon \theta \in \Omega(\mathbb{R})\} = \theta, \\ (C + C') \odot A = (C \odot A) \oplus (C' \odot A), \\ \text{if } C \subseteq C' \quad \text{and} \quad A \subseteq B \quad \text{then} \quad C \odot A \subseteq C' \odot B, \end{gathered}$$

and

$$\text{if } A \subseteq B \quad \text{and} \quad D \subseteq E \quad \text{then} \quad A \oplus D \subseteq B \oplus E,$$

where $1_{\Omega_C(\mathbb{R})} = \{1\}$, $1 \in \mathbb{R}$ and $\theta_{\Omega_C(\mathbb{R})} = \{0\}$, $0 \in \mathbb{R}$. Consequently, $\Omega(\mathbb{R})$ is a quasimodule on the quasiring $\Omega_C(\mathbb{R})$. □

Remark 3.18. The set $\Omega_C(\mathbb{R})$ is not a quasimodule on the quasiring $\Omega(\mathbb{R})$ although $\Omega(\mathbb{R})$ is a quasimodule on the quasiring $\Omega_C(\mathbb{R})$. Because the operation

$$\begin{aligned} \odot : \Omega(\mathbb{R}) \times \Omega_C(\mathbb{R}) &\to \Omega_C(\mathbb{R}) \\ (A, B) &\to A \odot B = \{a \cdot b : \ a \in A, \ b \in B\} \end{aligned}$$

is not well defined. Indeed, producting of a compact set with an interval may not be interval. For example, for elements $[3, 4] \cup [9, 10] \in \Omega(\mathbb{R})$ and $[1, 2] \in \Omega_C(\mathbb{R})$,

$$([3, 4] \cup [9, 10]) \odot [1, 2] = [3, 8] \cup [9, 20]$$

but $[3, 8] \cup [9, 20] \notin \Omega_C(\mathbb{R})$.

We denote by $\Omega_C^n(\mathbb{R})$ the family of all $n-$tuples intervals which constitute an important part of interval analysis.

$$\Omega_C^n(\mathbb{R}) = \{X = (X_1, X_2, ..., X_n) : X_i \in \Omega_C(\mathbb{R}) \text{ for } 1 \le i \le n\} .$$

We emphasize that $\Omega_C^n(\mathbb{R})$ is different from $\Omega_C(\mathbb{R}^n)$ which is the family of all closed, bounded and convex subsets of $\mathbb{R}^n$, [9].

Example 3.19. $\Omega_C^n(\mathbb{R})$ is a quasimodule on $\Omega_C(\mathbb{R})$ with the operations $\oplus$, $\odot$ and partial order relation $\ll$ defined by

$$\begin{aligned} \oplus : \Omega_C^n(\mathbb{R}) \times \Omega_C^n(\mathbb{R}) &\to \Omega_C^n(\mathbb{R}), \\ F \oplus G &= (F_1 + G_1, F_2 + G_2, ..., F_n + G_n) \end{aligned}$$

and

$$\begin{aligned} \odot : \Omega_C(\mathbb{R}) \times \Omega_C^n(\mathbb{R}) &\to \Omega_C^n(\mathbb{R}), \\ A \odot F &= (A \cdot F_1, A \cdot F_2, ..., A \cdot F_n) \end{aligned}$$

and

$$F \ll G \quad \Leftrightarrow \quad F_i \subseteq G_i \quad \text{for every} \quad i \in \{1, 2, ..., n\}$$

for F, $G \in \Omega_C^n(\mathbb{R})$ and $A \in \Omega_C(\mathbb{R})$.

We note that $\Omega_C^n(\mathbb{R})$ is a commutative monoid with "$\oplus$". Since it is easy to show that the conditions (QM 1)-(QM 4) hold, we only prove that the conditions (QM 5)-(QM 7) hold for all $F, G, H, I \in \Omega_C^n(\mathbb{R})$ and $A, B \in \Omega_C(\mathbb{R})$:

(QM 5)

$$\begin{aligned}(A+B)\odot F &= (A+B)\odot(F_1,F_2,...,F_n)\\ &= ((A+B)\cdot F_1,(A+B)\cdot F_2,...,(A+B)\cdot F_n)\\ &\ll (A\cdot F_1+B\cdot F_1, A\cdot F_2+B\cdot F_2,...,A\cdot F_n+B\cdot F_n)\\ &= (A\cdot F_1,A\cdot F_2,...,A\cdot F_n)\oplus(B\cdot F_1,B\cdot F_2,...,B\cdot F_n)\\ &= (A\odot(F_1,F_2,...,F_n))\oplus(B\odot(F_1,F_2,...,F_n))\\ &= (A\odot F)\oplus(B\odot F),\end{aligned}$$

(QM 6) Let $F \ll G$ and $H \ll I$. Then $F_i \subseteq G_i$ and $H_i \subseteq I_i$ for all $i \in \{1,2,...,n\}$.

$$\begin{aligned}F\oplus H &= (F_1,F_2,...,F_n)\oplus(H_1,H_2,...,H_n)\\ &= (F_1+H_1,F_2+H_2,...,F_n+H_n)\\ &\ll (G_1+I_1,G_2+I_2,...,G_n+I_n)\\ &= (G_1,G_2,...,G_n)\oplus(I_1,I_2,...,I_n)=G\oplus I,\end{aligned}$$

(QM 7) Let $A \subseteq B$ and $F \ll G$. Then $F_i \subseteq G_i$ for all $i \in \{1,2,...,n\}$.

$$\begin{aligned}A\odot F &= A\odot(F_1,F_2,...,F_n)=(A\cdot F_1,A\cdot F_2,...,A\cdot F_n)\\ &\ll (B\cdot G_1,B\cdot G_2,...,B\cdot G_n)=B\odot(G_1,G_2,...,G_n)\\ &= B\odot G.\end{aligned}$$

Hence $\Omega_C^n(\mathbb{R})$ is a quasimodule on $\Omega_C(\mathbb{R})$.

Set valued maps provide a useful framework for control theory, optimization theory, game theory, robotics, chemical engineering and mathematical economics (See [4], [8], [9]). For this reason $C([a,b],\Omega_C(\mathbb{R}))$, which is a class of interval valued maps, has an important place in set valued analysis. Now we claim that $C([a,b],\Omega_C(\mathbb{R}))$ is a quasimodule on the quasiring $\Omega_C(\mathbb{R})$.

Example 3.20. $C([a,b],\Omega_C(\mathbb{R}))$ is a quasimodule on $\Omega_C(\mathbb{R})$ with the operations $\oplus$, $\odot$ and partial order relation $\precsim$ defined by

$$\begin{aligned}&\oplus : C([a,b],\Omega_C(\mathbb{R}))\times C([a,b],\Omega_C(\mathbb{R}))\to C([a,b],\Omega_C(\mathbb{R})),\\ &(f\oplus g)(t)=\{f(t)+g(t) : t\in[a,b],\ f(t),g(t)\in\Omega_C(\mathbb{R})\}\end{aligned}$$

and

$$\begin{aligned}&\odot : \Omega_C(\mathbb{R})\times C([a,b],\Omega_C(\mathbb{R}))\to C([a,b],\Omega_C(\mathbb{R})),\\ &([c,d]\odot f)(t)=\{s\cdot f(t) : s\in[c,d],\ t\in[a,b]\}\end{aligned}$$

and

$$f \precsim g \quad\Leftrightarrow\quad f(t)\subseteq g(t), \text{for all } t\in[a,b]$$

for $f, g \in C([a,b], \Omega_C(\mathbb{R}))$ and $[c,d] \in \Omega_C(\mathbb{R})$. Since it can be easily seen that $C([a,b], \Omega_C(\mathbb{R}))$ is a commutative monoid with "$\oplus$" and the conditions (QM 1)-(QM 4) hold, we give only the proof of the conditions (QM 5)-(QM 7):

(QM 5)

$$\begin{aligned}((A+B)\odot f)(t) &= \{(a+b)v\colon\ a \in A\ ,\ b \in B\ ,\ v \in f(t)\} \\ &\subseteq \{(av+bv\colon\ a \in A\ ,\ b \in B\ ,\ v \in f(t)\} \\ &= ((A \odot f) \oplus (B \odot f))(t).\end{aligned}$$

Hence we have $(A+B)\odot f \precsim A \odot f \oplus B \odot f$.

(QM 6) If $f \precsim g$ and $h \precsim i$ then respectively $f(t) \subseteq g(t)$ and $h(t) \subseteq i(t)$ for all $t \in [a,b]$. Thus

$$(f \oplus h)(t) = f(t) + h(t) \subseteq g(t) + i(t) = (g \oplus i)(t)$$

So we obtain $f \oplus h \precsim g \oplus i$.

(QM 7) Let $A \subseteq B$ and $f \precsim g$. Then $f(t) \subseteq g(t)$, for all $t \in [a,b]$.

$$\begin{aligned}(A \odot f)(t) &= \{av :\ a \in A\ ,\ v \in f(t)\} \\ &\subseteq \{av :\ a \in B\ ,\ v \in g(t)\} = (B \odot g)(t)\end{aligned}$$

Thus we get $A \odot f \precsim B \odot g$. Hence $C([a,b], \Omega_C(\mathbb{R}))$ is a quasimodule on $\Omega_C(\mathbb{R})$.

Now, we introduce the notion of normed quasimodule which constitutes another important part of this section. Note that we have inspired from Aseev [3] for giving the following definition.

Definition 3.21. Let X be a quasimodule on the quasiring $\Omega_C(\mathbb{R})$. A real function $\|\cdot\| : X \to \mathbb{R}$ is called *a norm* if the following conditions hold:

$$\|x\| = 0 \Leftrightarrow x = \theta, \tag{NQM 1}$$

$$\|\alpha \odot x\| = |\alpha|\,\|x\|\,,\ \alpha = [\underline{\alpha}, \overline{\alpha}],\ \ |\alpha| = \max\{|\underline{\alpha}|\,, |\overline{\alpha}|\}, \tag{NQM 2}$$

$$\|x \oplus y\| \le \|x\| + \|y\|\,, \tag{NQM 3}$$

$$\text{if } x \ll y \text{ then } \ \|x\| \le \|y\|\,, \tag{NQM 4}$$

$$\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \tag{NQM 5}$$

$$x \ll y \oplus x_\varepsilon \text{ and } \ \|x_\varepsilon\| \le \varepsilon \text{ then } x \ll y.$$

A quasimodule X on which a norm is defined is called *normed quasimodule*.

It is known that every field is a quasiring. When $\mathbb{R}$ is considered as quasiring, the notion of normed qls defined in [3] is obtained. This implies that normed quasimodules are a generalization of normed quasilinear spaces.

We note that when quasimodules on the quasiring $\Omega_C(\mathbb{R})$ is considered, it should be recalled that $\Omega_C(\mathbb{R})$ is normed qls. So $\Omega_C(\mathbb{R})$ satisfies the conditions (NQLS 1)-(NQLS 5), that is $\Omega_C(\mathbb{R})$ already satisfies the following conditions

$$\|A\| \leq \|B\|, \text{ if } A \subseteq B$$

and

$$\text{if for any } \varepsilon > 0 \text{ there exists an element } A_\varepsilon \in \Omega_C(\mathbb{R}) \text{ such that}$$
$$A \subseteq B + A_\varepsilon \text{ and } \|A_\varepsilon\| \leq \varepsilon \text{ then } A \subseteq B.$$

On the other hand, conditions about accordance between norm on quasimodule and partial order relation will present extra in Definition 3.21 (see (NQM 4) and (NQM 5)).

Let X be a quasimodule on the quasiring $\Omega_C(\mathbb{R})$. Taking into account the conditions of quasimodule, since $\Omega_C(\mathbb{R})$ is a quasimodule, we get $A \odot x \ll B \odot x$ for $A, B \in \Omega_C(\mathbb{R})$ and $x \in X$, when $A \subseteq B$. Then $\|A \odot x\| \leq \|B \odot x\|$ holds by (NQM 4).

Example 3.22. $\Omega_C(\mathbb{R})$ is a normed quasimodule with the following norm

$$\|A\| = \sup_{a \in A} |a|.$$

Now let us define a norm function on the quasimodules $\Omega_C^2(\mathbb{R})$.

Example 3.23. $\Omega_C^2(\mathbb{R})$ is a normed quasimodule on the quasiring $\Omega_C(\mathbb{R})$ with the following norm

$$\|X\| = \|(X_1, X_2)\| = |X_1| + |X_2|$$

for $X_1 = [\underline{X_1}, \overline{X_1}]$, $X_2 = [\underline{X_2}, \overline{X_2}] \in \Omega_C(\mathbb{R})$, where $|X_1| = \max\{|\underline{X_1}|, |\overline{X_1}|\}$, $|X_2| = \max\{|\underline{X_2}|, |\overline{X_2}|\}$.

Since it is easy to see that the conditions (NQM 1)-(NQM 4) hold we omit its, we only show that the condition (NQM 5) holds.

Let $X = (X_1, X_2), Y = (Y_1, Y_2) \in \Omega_C^2(\mathbb{R})$. Suppose that, for any $\epsilon > 0$, there exists an element $X_\epsilon = (X_{\epsilon,1}, X_{\epsilon,2}) \in \Omega_C^2(\mathbb{R})$ such that

$$X \ll Y \oplus X_\epsilon \text{ and } \|X_\epsilon\| \leq \epsilon.$$

Assume that $X \not\ll Y$. Then we have $X_1 \subsetneq Y_1$ or $X_2 \subsetneq Y_2$. If $X_1 \subsetneq Y_1$ then there exists an element $x_1 \in X_1$ such that $x_1 \notin Y_1$. Since Y_1 is closed, the distance between the element x_1 and the set Y_1 is

$$d(x_1, Y_1) = \inf_{y \in Y_1} \|x_1 - y\| \neq 0.$$

By the hypotesis, for

$$\epsilon = \frac{d(x_1, Y_1)}{2}$$

there exists an $X_\epsilon \in \Omega_C^2(\mathbb{R})$ such that $X \ll Y \oplus X_\epsilon$ and $\|X_\epsilon\| = |X_{\epsilon,1}| + |X_{\epsilon,2}| < \epsilon$.

We note that $X \ll Y \oplus X_\epsilon$ implies that $X_1 \subseteq Y_1 \oplus X_{\epsilon,1}$ and $X_2 \subseteq Y_2 \oplus X_{\epsilon,2}$. Thus $x_1 \in X_1$ and $x_1 \in Y_1 \oplus X_{\epsilon,1}$. Then we have

$$x_1 = y + x_{\epsilon,1}$$

for $y \in Y_1$ and $x_{\epsilon,1} \in X_{\epsilon,1}$. So

$$\begin{aligned} 0 &= \|x_1 - (y + x_{\epsilon,1})\| \geq |\|x_1 - y\| - \|x_{\epsilon,1}\|| \\ &\geq |d(x_1, Y_1) - \|x_{\epsilon,1}\|| \\ &\geq \left| d(x_1, Y_1) - \frac{d(x_1, Y_1)}{2} \right| \\ &= \frac{d(x_1, Y_1)}{2} = \epsilon \end{aligned}$$

This is a contradiction. Thus $x_1 \in Y_1$ and $X_1 \subseteq Y_1$. Analogously we obtain $X_2 \subseteq Y_2$. Hence we get $X \ll Y$.

Example 3.24. $C([a,b], \Omega_C(\mathbb{R}))$ is a normed quasimodule on the quasiring $\Omega_C(\mathbb{R})$ with the following norm

$$\|f\| = \max_{t \in [a,b]} \left\{ \|f(t)\|_{\Omega_C(\mathbb{R})} \right\} = \max_{t \in [a,b]} \left\{ \sup_{v \in f(t)} |v| \right\}.$$

Since it can be easly seen that the conditions (NQM 1)-(NQM 3) hold, we only show that (NQM 4) and (NQM 5) are satisfied:

Let $f \precsim g$. Then $f(t) \subseteq g(t)$ for all $t \in [a,b]$. Since $f(t), g(t) \in \Omega_C(\mathbb{R})$ and $\Omega_C(\mathbb{R})$ is a normed qls, we have $\|f(t)\|_{\Omega_C(\mathbb{R})} \leq \|g(t)\|_{\Omega_C(\mathbb{R})}$. So

$$\max_{t \in [a,b]} \left\{ \|f(t)\|_{\Omega_C(\mathbb{R})} \right\} \leq \max_{t \in [a,b]} \left\{ \|g(t)\|_{\Omega_C(\mathbb{R})} \right\}$$

holds, that is $\|f\| \leq \|g\|$.

Suppose that, for any $\epsilon > 0$, there exists an element $f_\epsilon \in C([a,b], \Omega_C(\mathbb{R}))$ such that

$$f \precsim g \oplus f_\epsilon \text{ and } \|f_\epsilon\| \leq \epsilon.$$

By the hypotesis, we have

$$f(t) \subseteq (g \oplus f_\epsilon)(t) = g(t) + f_\epsilon(t) \text{ and } \|f_\epsilon(t)\|_{\Omega_C(\mathbb{R})} \leq \epsilon$$

for all $t \in [a,b]$. Because of the fact that $f(t), g(t), f_\epsilon(t) \in \Omega_C(\mathbb{R})$ and $\Omega_C(\mathbb{R})$ is a normed qls, we obtain $f(t) \subseteq g(t)$ for all $t \in [a,b]$. Hence $f \precsim g$.

Definition 3.25. Let $(X, \|\cdot\|)$ be a normed quasimodule. *Hausdorff metric* on X is defined by the equality

$$h_X(x,y) = \inf\{r \geq 0 : x \ll y \oplus a_1^r, y \ll x \oplus a_2^r, \\ \|a_i^r\|_X \leq r,\ a_i^r \in X,\ i = 1,2\}. \tag{3.4}$$

Since $x \ll y \oplus (x - y)$ and $y \ll x \oplus (y - x)$, h_X is well defined for any elements $x, y \in X$.

We note that the equality $h_X(x,y) = \|x - y\|_X$ may not be satisfied for every $x, y \in X$. But, the inequality

$$h_X(x,y) \leq \|x - y\|_X \tag{3.5}$$

is always true. Therefore, when we deal with topological properties of normed quasimodules, to analyze according to the metric derived from this norm is more convenient instead of using the norm. Because, the equality

$$d(x,y) = \|x - y\| \tag{3.6}$$

doesn't define a metric function. If X is a normed linear space then we know that $h_X(x,y) = d(x,y)$.

It is not hard to see that h_X satisfies all of the metric axioms. Further the following conditions hold:

$$h_X(A \odot x, A \odot y) = \|A\|_{\Omega_C(\mathbb{R})} \cdot h_X(x,y),$$

$$h_X(x \oplus y\ ,\ z \oplus v) \leq h_X(x,z) + h_X(y,v).$$

Remark 3.26. We note that the metric d induced by a norm on a normed quasimodule X is not translation invariant. But this metric satisfies the inequality

$$d(x \oplus a, y \oplus a) \leq d(x,y)$$

since

$$d(x \oplus a, y \oplus a) \leq d(x,y) + d(a,a) = d(x,y)$$

holds.

Definition 3.27. If a normed quasimodule X is complete according to the Hausdorff metric on X then normed quasimodule is called *complete normed quasimodule.*

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References

[1] G. Alefeld and G. Mayer, *Interval Analysis: theory and applications*, J. Comput. Appl. Math., **121** (2000), 421–464.

[2] M. Artin, *Algebra*, Prentice Hall, India (1991).

[3] S.M. Aseev, *Quasilinear operators and their application in the theory of multivalued mappings*, Proc. Steklov Inst. Math., **2** (1986), 23–52.

[4] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston (1980).

[5] G. Birkhoff and R. Pierce, *Lattice-ordered rings*, An. Acad. Brasil. Cienc., **28** (1956), 41–69.

[6] H. Bozkurt, S. Çakan and Y. Yılmaz, *Quasilinear Inner Product Spaces and Hilbert Quasilinear Spaces*, Inter. J. of Anal., (2014), Article ID 258389, 7 pages.

[7] S. Çakan and Y. Yılmaz, *Lower and Upper Semi Basis in Quasilinear Spaces*, Erciyes University Journal of the Institute of Science and Technology, **31(2)** (2015), 97–104.

[8] V. Lakshmikantham, T. Gnana Bhaskar and J. Vasundhara Devi, *Theory of set differential equations in metric spaces*, Cambridge Scientific Publ., Florida (2006).

[9] R.E. Moore, R.B. Kearfott and M.J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia (2009).

[10] Ö. Talo and F. Başar, *Quasilinearity of the classical sets of sequences of fuzzy numbers and some related results*, Taiwanese J. Math., **14**(5) (2010), 1799–1819.

[11] Y. Yılmaz, S. Çakan and Ş. Aytekin, *Topological Quasilinear Spaces*, Abstr. Appl. Anal., (2012), Article ID 951374, 10 pages.