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ON THE EXISTENCE AND NONEXISTENCE OF GLOBAL SIGN CHANGING SOLUTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study the global existence and nonexistence of solutions to the following nonlinear reaction-diffusion problem

$$\begin{cases} u_t - \Delta u &= |u|^p \quad \text{in} \quad M^n \times (0,\infty), \\ u(x,0) &= u_0(x) \quad \text{in} \quad M^n, \end{cases}$$

where M^n, $n \geq 3$, is a non-compact complete Riemannian manifold, Δ is the Laplace-Beltrami operator and $p > 1$.We assume that $u_0(x)$ is smooth, bounded function satisfying $\int_{M^n} u_0(x)dx > 0$. We prove that there is an exponent $p^\star$ which is critical in the following sense: when $p \in (1, p^\star)$, the above problem has no global solution and when $p \in (p^\star, \infty)$, the problem has a global solution for some $u_0(x)$.

1. Introduction and statement of the results

In this paper, we study the global existence and nonexistence of solutions to the following semilinear parabolic equation

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$$\begin{cases} u_t - \Delta u = |u|^p & \text{in} \quad M^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad M^n, \end{cases} \tag{1.1}$$

where (M^n, g), $n \geq 3$, is a non-compact complete Riemannian manifold, Δ is the Laplace-Beltrami operator, u_0 are smooth functions and $p > 1$.
The study of existence or nonexistence of global positive solutions to homogeneous semilinear parabolic equations and systems have been studied widely since the result of Fujita [3] in 1966 who proved the following Theorem.

Theorem 1.1. *Consider the Cauchy problem*

$$\begin{cases} u_t - \Delta u = u^p & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $p > 1$, $n \geq 1$ and Δ the standard Laplace operator. The following conclusions hold.

(a) *When $1 < p < 1 + \dfrac{2}{n}$, and $u_0 \geq 0$, problem (1.2) possesses no global solutions.*

(b) *When $p > 1 + \dfrac{2}{n}$ and u_0 smaller then a small Gaussian, then (1.2) has global positive solutions.*

We call $p^\star = 1 + \dfrac{2}{n}$ the critical exponent of Fujita type for the semilinear heat equation (1.2).

These equations are model cases of many equations applied to wide scope and we can find an account of the historic development in the survey papers [2, 6].

After that, Zhang [12] has undertaken the research to prove that Fujita's results on critical exponents can be extended to solutions that may change sign. He considered the problem

$$\begin{cases} u_t - \Delta u = |u|^p & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{1.3}$$

He proved the following result.

Theorem 1.2. *Let $p > 1$, $n \geq 1$ and Δ the standard Laplacien. The following results hold*

(a) *If $1 < p < 1 + \dfrac{2}{n}$, and $\int_{\mathbb{R}^n} u_0(x)\, dx > 0$, then (1.3) has no global solutions.*

(b) *If $p = 1 + \dfrac{2}{n}$ and $\int_{\mathbb{R}^n} u_0 dx > 0$, then (1.3) has no global solutions provided that u_0^- is compactly supported.*

(c) *If* $p > 1 + \dfrac{2}{n}$, *then* (1.3) *has global solutions for some* u_0.

Also, Zhang [11] has undertaken the research on semilinear parabolic operators on Riemannian manifold and obtains a lot of important results in the study of the positive global existence and nonexistence (see [10] and [11]). The method he uses is rather technical, and the main tools are fixed point theorems and many estimates.

As an expansion of [9, 10, 11], we study our problem (1.1) and obtain several meaningful results.

Throughout the paper, for a fixed $x_0 \in M^n$, we make the following assumptions (see [10, 11]).

(i) There are positive constants $\alpha > 2$, k and c such that:

$$\text{Ricci} \geq -k \quad \text{and} \quad \text{for all } x \in M^n,$$
$$c^{-1} r^\alpha \leq |B_r(x)| \leq c r^\alpha, \quad \text{when r is large.}$$

(ii) $G(x,y,t)$ is the fundamental solution of the linear operator $\frac{\partial}{\partial t} - \Delta$, and satisfies

$$\frac{c}{|B(x, t^{\frac{1}{2}})|} e^{-b\frac{d(x,y)^2}{t}} \geq G(x,y,t) \geq 0, \quad \text{in } M^n \times (0,\infty)$$

and when $t - s \geq d(x,y)^2$, b is a positive constant, the fundamental solution $G(x,y,t-s)$ satisfies

$$G(x,y,t-s) \geq \min\left\{ \frac{c}{|B(x,(t-s)^{\frac{1}{2}})|}, \frac{c}{|B(y,(t-s)^{\frac{1}{2}})|} \right\}.$$

(iii) $\frac{\partial \log g^{\frac{1}{2}}}{\partial r} \leq \frac{c}{r}$, when $r = d(x,x_0)$ is smooth, here $g^{\frac{1}{2}}$ is the volume of density of the manifold.

Since the above assumptions are satisfied, the following lemmas hold.

Lemma 1.3. ([11]) *There exists a constant* $c_1 > 0$*, depending only on* n*,* α *and* $\delta > 0$*, such that*

$$\sup_x \int_{M^n} \frac{1}{d(x,y)^{\alpha-2}[1 + d(y,x_0)^{2+\delta}]} dy \leq c_1.$$

Lemma 1.4. ([11]) *There exists a constant* $c_2 > 0$*, depending only on* n*,* α *and* $\delta > 0$*, such that*

$$\int_{M^n} \frac{1}{d(x,y)^{\alpha-2}[1 + d(y,x_0)^{\alpha+\delta}]} dy \leq \frac{c_2}{1 + d(x,x_0)^{\alpha-2}}.$$

Lemma 1.5. ([11]) *Let* $\Gamma(x, y)$ *the Green's function for the Laplacien, then there exists a constant* $c_3 > 0$ *such that*

$$\int_{M^n} \Gamma(x, y) \frac{1}{1 + d(x, y)^{\alpha+\delta}} dy \leq \frac{c_3}{1 + d(x, x_0)^{\alpha-2}}.$$

Lemma 1.6. ([11]) *Given* $\delta > 0$*, there exists a constant* $c_4 > 0$ *such that*

$$h(x, t) = \int_{M^n} \frac{G(x, y, t)}{1 + d(y, x_0)^{\alpha+\delta}} dy \leq \frac{c_4}{1 + d(x, x_0)^{\alpha}}.$$

Since we are dealing with sign changing solutions on a Riemannian manifold, we take the following definition.

Definition 1.7. $u(x, t) \in L^{\infty}_{loc}(M^n \times (0, \infty), \mathbb{R})$ is called a solution of (1.1) if

$$u(x, t) = \int_{M^n} G(x, y, t) u_0(y) dy + \int_0^t \int_{M^n} G(x, y, t - s)|u(y, s)|^p dy ds.$$

Definition 1.8. On a complete Riemannian manifold, we define Green's function $\Gamma(x, y) = \int_0^{\infty} G(x, y, s) ds$, if the integral on the right hand side converges.

We have

$$\Gamma(x, y) > 0, \quad \triangle\Gamma = -\delta_x y,$$

$$\int_0^t G(x, y, t - s) ds = \int_0^t G(x, y, w) dw \leq \int_0^{\infty} G(x, y, w) dw = \Gamma(x, y). \quad (1.4)$$

Our results are as follows.

Theorem 1.9. *If* $1 < p < 1 + \frac{2}{\alpha}$ *and* $\int_{M^n} u_0(y) dy > 0$*, then* (1.1) *has no global solutions.*

Theorem 1.10. *If* $p > 1 + \frac{2}{\alpha}$ *and* $\int_{M^n} u_0(y) dy > 0$*, then* (1.1) *has global solutions, whenever* $|u_0(x)| < \frac{\epsilon}{1 + d(x, x_0)^{\alpha} + \delta}$ *for some* $\delta > 0$ *and some sufficiently small* $\epsilon > 0$.

Remark 1.11. By Theorems (1.9) $-$ (1.10), it is easy to see that $p^{\star}=1+\dfrac{2}{\alpha}$ is the critical exponent of Fujita type for the semilinear parabolic equation (1.1).

2. Nonexistence of global solutions

In this section, we prove the Theorem 1.9 and c is always a constant that may be change for line to line.

Proof of Theorem 1.9.

Step 1. Let

$$\omega(x,t)=\int_{M^n}G(x,y,t)u_0(y)dy, \tag{2.1}$$

where G is the fundamental solution to the heat equation $\Delta u-u_t$, we will prove in this step that

$$\omega(x,t)\geq\frac{c_0}{t^{\frac{\alpha}{2}}} \tag{2.2}$$

when $(x,t)\in Q_R=B_R(0)\times[4R^2,8R^2]$ for a sufficiently large R. Here c_0 can be chosen independently of R.

Choosing $R_0>0$ to be determined later, we have

$$\omega(x,t)=\int_{B_{R_0}(0)}G(x,y,t)u_0(y)dy+\int_{B^C_{R_0}(0)}G(x,y,t)u_0(y)dy$$

and hence

$$\begin{aligned}\omega(x,t)\geq&\int_{B_{R_0}(0)}G(x,y,t)u_0^+(y)dy-\int_{B_{R_0}(0)}G(x,y,t)u_0^-(y)dy\\&-\int_{B^C_{R_0}(0)}G(x,y,t)u_0^-(y)dy.\end{aligned}$$

Suppose $(x,t)\in Q_R$, by assumption (i) and (ii) we obtain for $R>R_0$,

$$\begin{aligned}\omega(x,t)\geq&\frac{c}{t^{\frac{\alpha}{2}}}\int_{B_{R_0}(0)}u_0^+(y)dy-\frac{c}{t^{\frac{\alpha}{2}}}\int_{B_{R_0}(0)}e^{-b\frac{d(x,y)^2}{t}}u_0^-(y)dy\\&-\frac{c}{t^{\frac{\alpha}{2}}}\int_{B^C_{R_0}(0)}e^{-b\frac{d(x,y)^2}{t}}u_0^-(y)dy.\end{aligned}$$

Since $b>0$, we have

$$\omega(x,t)\geq\frac{c}{t^{\frac{\alpha}{2}}}\int_{B_{R_0}(0)}u_0(y)dy-\frac{c}{t^{\frac{\alpha}{2}}}\int_{B^C_{R_0}(0)}u_0^-(y)dy.$$

By our assumption on u_0, there exists a $\delta > 0$ such that

$$\int_{B_{R_0}(0)} u_0(y)dy \geq \delta,$$

when R_0 is sufficiently large. For $R > R_0$, we have

$$\omega(x,t) \geq \frac{c\delta}{t^{\frac{\alpha}{2}}} - \frac{c}{t^{\frac{\alpha}{2}}} \int_{B^C_{R_0}(0)} u_0^-(y)dy. \tag{2.3}$$

Since $u_0 \in L^1(M^n)$ and the constants in the inequality (2.3) are independent of R_0, we can take R_0 sufficiently large so that for $(x,t) \in Q_R$,

$$\omega(x,t) \geq \frac{c_0\delta}{t^{\frac{\alpha}{2}}} \cdot \tag{2.4}$$

Step 2. We will use the methode of contradiction. Suppose that u is a global solution to (1.1). By Definition 1.7 and the inequality (2.4), we have

$$u(x,t) \geq \omega(x,t) > 0 \quad \text{for } (x,t) \in Q_R. \tag{2.5}$$

Let ϕ, $\eta \in C^\infty$ be tow functions satisfying

$$\begin{cases} \phi(r) \in [0,1], & \text{if } r \in [0,+\infty), \\ \phi(r) = 1, & \text{if } r \in \left[0, \frac{1}{2}\right], \\ \phi(r) = 0, & \text{if } r \in [1,+\infty), \\ \eta(t) \in [0,1], & \text{if } t \in [0,+\infty), \\ \eta(t) = 1, & \text{if } t \in \left[0, \frac{1}{4}\right], \\ \eta(t) = 0, & \text{if } t \in [1,+\infty), \\ -C \leq \phi'(r) \leq 0; & |\phi''(r)| \leq C; |\eta'(t)| \leq C. \end{cases}$$

For $R > 0$, we define a cut-off function

$$\Psi_R(x,t) = \phi_R(x)\eta_R(t),$$

where $\phi_R(x) = \phi(\frac{|x|}{R})$ and $\eta_R(t) = \eta(\frac{t-4R^2}{4R^2})$. Clearly ϕ_R is radial function and we have

$$-\frac{C}{R} \leq \frac{\partial \phi_R}{\partial r} \leq 0, \quad |\frac{\partial^2 \phi_R}{\partial r^2}| \leq \frac{C}{R^2}, \quad |\eta'_R(t)| \leq \frac{C}{R^2}. \tag{2.6}$$

For $R > 0$, we set

$$I_R = \int_{Q_R} |u|^p(x,t)\Psi_R^q(x,t)dxdt, \tag{2.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since u is a solution of (1.1), we have

$$I_R = \int_{Q_R} \left[u_t(x,t) - \Delta u(x,t)\right] \Psi_R^q(x,t)dxdt, \tag{2.8}$$

which implies, via integration by parts (Stokes Formula)

$$
\begin{aligned}
I_R \leq & \int_{B_R(0)} u(x,.)\Psi_R^q(x,.)|_{4R^2}^{8R^2}dx - \int_{Q_R} u(x,t)\phi_R^q(x)q\eta_R^{q-1}(t)\eta_R'(t)dxdt \\
& + \int_{4R^2}^{8R^2}\int_{\partial B_R(0)} u(x,t)\frac{\partial\phi_R^q(x)}{\partial n}\eta_R^q(t)\ dS_x dt \\
& - \int_{4R^2}^{8R^2}\int_{\partial B_R(0)} \Psi_R^q(x,t)\frac{\partial u}{\partial n}(x,t)dS_x dt \\
& - \int_{Q_R} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)dxdt.
\end{aligned} \tag{2.9}
$$

Here and later $\frac{\partial}{\partial n}$ denotes the outward normal derivative with respect to the relevant bounded domain.

Using the boundary conditions on the cut-off function and (2.9), and since

$$
\frac{\partial\phi_R^q}{\partial n} = q\phi_R^{q-1}\phi_R'(\frac{\partial r}{\partial n}) \leq 0 \quad \text{on} \quad \partial B_{R_0}(0),
$$

we obtain

$$
\begin{aligned}
I_R \leq & -q\int_{Q_R} u(x,t)\phi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt \\
& - \int_{Q_R} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)dxdt.
\end{aligned} \tag{2.10}
$$

Because

$$
\Delta\phi_R^q(x) = q\phi_R^{q-1}(x)\Delta\phi_R(x) + q(q-1)\phi_R^{q-2}(x)|\nabla\phi_R(x)|^2,
$$

the inequality (2.10) yields

$$
\begin{aligned}
I_R \leq & -q\int_{Q_R} u(x,t)\phi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt \\
& - q\int_{Q_R} u(x,t)\phi_R^{q-1}(x)\Delta\phi_R(x)\eta_R^q(t)dxdt.
\end{aligned} \tag{2.11}
$$

Recalling of the supports of $\phi_R(x)$ and $\eta_R(t)$ that is

$$
\begin{cases}
\eta_R(t) = 1, \eta_R'(t) = 0, & \text{if } t \in [4R^2, 7R^2], \\
\phi_R(x) = 1, \Delta\phi_R(x) = 0, & \text{if } r = |x| \in [0, \frac{R}{2}],
\end{cases} \tag{2.12}
$$

we can reduce (2.11) to

$$
\begin{aligned}
I_R \leq & -q \int_{7R^2}^{8R^2} \int_{B_R(0)} u(x,t)\phi_R^q(x,t)\eta_R^{q-1}(t)\eta_R'(t)dtdx \\
& - q \int_{4R^2}^{8R^2} \int_{B_R(0)\setminus B_{\frac{R}{2}}(0)} u(x,t)\phi_R^{q-1}(x,t)\Delta\phi_R(x)\eta_R^q(t)dxdt.
\end{aligned} \tag{2.13}
$$

Since ϕ_R is radial, we have

$$
\Delta\phi_R = \phi_R'' + \left[\frac{n-1}{r} + \frac{\partial \log g^{\frac{1}{2}}}{\partial r}\right]\phi_R'.
$$

Taking R sufficiently large, by assumption (iii), that is

$$
\frac{\partial \log g^{\frac{1}{2}}}{\partial r} \leq \frac{C}{R},
$$

we obtain, when $x \in B_R(0) \setminus B_{\frac{R}{2}}(0)$

$$
\Delta\phi_R(x) \geq -\frac{c}{R^2}. \tag{2.14}
$$

Merging (2.13), (2.14) and (2.6), we know

$$
\begin{aligned}
I_R \leq & \frac{c}{R^2} \int_{7R^2}^{8R^2} \int_{B_R(0)} u(x,t)\phi_R^q(x,t)\eta_R^{q-1}(t)dxdt \\
& + \frac{c}{R^2} \int_{4R^2}^{8R^2} \int_{B_R(0)\setminus B_{\frac{R}{2}}(0)} u(x,t)\phi_R^{q-1}(x,t)\eta_R^q(t)dxdt.
\end{aligned} \tag{2.15}
$$

Therefore, as $\phi_R \leq 1$ and $\eta_R \leq 1$,

$$
\begin{aligned}
I_R \leq & \frac{c}{R^2} \int_{7R^2}^{8R^2} \int_{B_R(0)} u(x,t)\Psi_R^{q-1}(x,t)dxdt \\
& + \frac{c}{R^2} \int_{4R^2}^{8R^2} \int_{B_R(0)\setminus B_{\frac{R}{2}}(0)} u(x,t)\Psi_R^{q-1}(x,t)dxdt.
\end{aligned} \tag{2.16}
$$

Using Hölder inequality, we can deduce from (2.16) that

$$I_R \leq \frac{c}{R^2}\left[\int_{7R^2}^{8R^2}\int_{B_R(0)} u^p(x,t)\Psi_R^{p(q-1)}(x,t)dxdt\right]^{\frac{1}{p}}\left[\int_{7R^2}^{8R^2}\int_{B_R(0)} dxdt\right]^{\frac{1}{q}}$$
$$+\frac{c}{R^2}\left[\int_{4R^2}^{8R^2}\int_{B_R(0)\backslash B_{\frac{R}{2}}(0)} u^p(x,t)\Psi_R^{p(q-1)}(x,t)dxdt\right]^{\frac{1}{p}}$$
$$\times\left[\int_{4R^2}^{8R^2}\int_{B_R(0)\backslash B_{\frac{R}{2}}(0)} dxdt\right]^{\frac{1}{q}}.$$

Therefore, as $\frac{1}{p}+\frac{1}{q}=1$, we have

$$I_R \leq c\left[\int_{7R^2}^{8R^2}\int_{B_R(0)} u^p(x,t)\Psi_R^{q}(x,t)dxdt\right]^{\frac{1}{p}} R^{\frac{2+\alpha}{q}-2}$$
$$+\, c\left[\int_{4R^2}^{8R^2}\int_{B_R(0)\backslash B_{\frac{R}{2}}(0)} u^p(x,t)\Psi_R^{q}(x,t)dxdt\right]^{\frac{1}{p}} R^{\frac{2+\alpha}{q}-2},$$

which yields

$$I_R \leq c\, I_R^{\frac{1}{p}}\, R^{\frac{2+\alpha}{q}-2}. \tag{2.17}$$

Step 3. (Conclusion) From (2.17), we obtain

$$I_R^{\frac{1}{q}} \leq c\, R^{\frac{2+\alpha}{q}-2}$$

and so

$$I_R^{\frac{1}{q}} \leq c\, R^{(2+\alpha)-2q} \equiv c\, R^{k_1}. \tag{2.18}$$

On the other hand, from (2.4), (2.5), in Step 1 and (2.12), we have for sufficiently large R,

$$I_R \geq \int_{7R^2}^{8R^2}\int_{B_R(0)} |u(x,t)|^p dxdt \geq \int_{7R^2}^{8R^2}\int_{B_R(0)} \omega(x,t)^p dxdt \geq c\, R^{\alpha+2-\alpha p}$$

then

$$I_R \geq c\, R^{k_2}. \tag{2.19}$$

Since $p<1+\frac{2}{\alpha}$, so $k_2>0$. Also $p<1+\frac{2}{\alpha}$, then $q>\frac{\alpha+2}{2}$. Therefore $k_1<0$.

If the solution u is global then (2.18) and (2.19) lead to a contradiction, when R is large. So u cannot be global. Theorem 1.9 is proved. □

3. Existence of global solutions

In this section, we prove the Theorem 1.10.

Proof of Theorem 1.10. As in [7], we define the Green's function

$$\Gamma(x,y)=\int_0^{+\infty}G(x,y,s)ds.$$

We have $\Gamma(x,y)>0$, $\Delta\Gamma=-\delta_x(y)$ and

$$\int_0^t G(x,y,t-s)ds=\int_0^t G(x,y,\omega)d\omega<\int_0^\infty G(x,y,\omega)d\omega=\Gamma(x,y). \quad (3.1)$$

Let Γ the integral operator defined by

$$\Gamma u(x,t)=\int_{M^n}G(x,y,t)u_0(y)dy+\int_0^t\int_{M^n}G(x,y,t-s)|u|^p(y,s)dyds, \quad (3.2)$$

for $u(x,t)\in L^\infty_{loc}(M^n\times[0,+\infty),\mathbb{R})$. For $N\in(0,1)$, we denote

$$\mathcal{H}_N=\left\{u(x,t)\in C(M^n\times(0,+\infty),\mathbb{R})/|u(x,t)|\leq\frac{N}{1+d(x,x_0)^\alpha}\right\}. \quad (3.3)$$

Next, we show that the operator Γ has a fixed point in $\mathcal{H}_N$, for some N. For $\varepsilon>0>$ and $\delta>0$ to be chosen later, select u_0 satisfying

$$|u_0(x)|\leq\frac{N}{1+d(x,x_0)^{\alpha+\delta}}. \quad (3.4)$$

By Lemma 1.6

$$\left|\int_{M^n}G(x,y,t)u_0(y)dy\right|\leq\varepsilon\int_{M^n}\frac{G(x,y,t)}{1+d(x,x_0)^{\alpha+\delta}}dy\leq\frac{\varepsilon c_4}{1+d(x,x_0)^\alpha}. \quad (3.5)$$

By assumption (3.3), we have for $u(x,t)\in\mathcal{H}_N$,

$$\begin{aligned}&\left|\int_0^t\int_{M^n}G(x,y,t-s)|u(y,s)|^p dyds\right|\\ &\leq N^p\int_0^t\int_{M^n}\frac{G(x,y,t-s)}{(1+d(y,x_0)^\alpha)^p}dyds.\end{aligned} \quad (3.6)$$

Since $p>1+\frac{2}{\alpha}$, we can find $c_5>0$ and $\delta>0$ such that

$$\frac{1}{(1+d(y,x_0)^\alpha)^p}\leq N^p\frac{c_5}{1+d(y,x_0)^{\alpha+2+\delta}}. \quad (3.7)$$

Substituting (3.7) on the right hand side of (3.6) and by Lemma 1.5

$$\begin{aligned}
&\left|\int_0^t \int_{M^n} G(x,y,t-s)|u(y,s)|^p dyds\right| \\
&\le N^p c_5 \int_{M^n} \Gamma(x,y)\frac{1}{1+d(y,x_0)^{\alpha+2+\delta}}dy \qquad (3.8)\\
&\le N^p c_3 c_5 \frac{1}{d(x,y)^{\alpha}},
\end{aligned}$$

when ε and N are sufficiently small, we have

$$\Gamma u(x,y) \le \frac{N}{1+d(x,x_0)^{\alpha}}. \qquad (3.9)$$

This shows that $\Gamma\mathcal{H}_N \subset \mathcal{H}_N$. To obtain the global existence of solution to (1.1), we check that Γ is contractive operator. Let $u_i(x,t) \in \mathcal{H}_N, i = 1,2,$ then noticing that $||a|-|b|| \le |a-b|$ and

$$|u_1^p(x,t)-u_2^p(x,t)| \le p|u_1(x,t)-u_2(x,t)| \max\left\{u_1^{p-1}(x,t), u_2^{p-1}(x,t)\right\}. \qquad (3.10)$$

We have

$$\begin{aligned}
&|\Gamma u_1(x,t) - \Gamma u_2(x,t)| \\
&\le \int_0^t \int_{M^n} G(x,y,t-s)|u_1^p(y,s)-u_2^p(y,s)|dyds \qquad (3.11)\\
&\le p\int_0^t \int_{M^n} G(x,y,t-s)\left(\frac{N}{1+d(y,x_0)^{\alpha}}\right)^{p-1}|u_1^p(y,s)-u_2^p(y,s)|dyds.
\end{aligned}$$

Denoting $\|.\| = \max\limits_{x\in M^n, t>0} |.|$, we have

$$\begin{aligned}
&|\Gamma u_1(x,t) - \Gamma u_2(x,t)| \\
&\le pN^{p-1}\|u_1-u_2\| \int_0^t \int_{M^n} G(x,y,t-s)\frac{1}{[1+d(y,x_0)^{\alpha}]^{p-1}}dyds. \qquad (3.12)
\end{aligned}$$

Since $p > 1+\frac{2}{\alpha}$, we can select $\delta > 0$ such that $(p-1)\alpha \ge 2+\delta$ and then, there exist a constant $c_6 > 0$ such that

$$\frac{1}{[1+d(y,x_0)^{\alpha}])^{p-1}} \le \frac{c_6}{1+d(y,x_0)^{2+\delta}}. \qquad (3.13)$$

From (3.12), (3.13) and (3.1), we obtain

$$\begin{aligned}
&|\Gamma u_1(x,t) - \Gamma u_2(x,t)| \\
&\le c_6 pN^{p-1}\|u_1-u_2\| \int_{M^n} \Gamma(x,y)\frac{1}{1+d(y,x_0)^{2+\delta}}dy. \qquad (3.14)
\end{aligned}$$

By Li and Yau in [7], there is a positive constant $\lambda > 0$ such that $\Gamma(x,y) \sim \frac{1}{1+d(x,y)^{\alpha-2}}$, when $d(x,y) > \lambda$, then we have for some constant $c_7 > 0$

$$\begin{aligned} &|\Gamma u_1(x,t) - \Gamma u_2(x,t)| \\ &\leq pc_6c_7N^{p-1}\|u_1 - u_2\| \int_{M^n} \frac{1}{d(x,y)^{\alpha-2}[1 + d(y,x_0)^{2+\delta}]} dy \\ &\leq pc_6c_7c_1N^{p-1}\|u_1 - u_2\|, \end{aligned}$$

by Lemma 1.3.

If N is small enough so that $pc_1c_6c_7N^{p-1} < 1$ and δ good chosen, then Γ is contractive. Hence problem (1.1) has a global solution and the proof of Theorem1.10 is completed. □

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