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ON GENERALIZATIONS OF MAJORIZATION INEQUALITY

M. Adil Khan[1], N. Latif[2] and J. Pečarić[3]

[1]Department of Mathematics
University of Peshawar, Peshawar, Pakistan
e-mail: adilswati@gmail.com

[2]Department of Mathematics
Govt. College University, Faisalabad 38000, Pakistan
e-mail: naveed707@gmail.com

[3]Faculty of Textile Technology Zagreb, University of Zagreb
Prilaz Baruna Filipovića 28A, 10000 Zagreb, Croatia
e-mail: pecaric@element.hr

Abstract. In this paper, we give some identities for the difference of majorization inequality by using Abel-Gontscharoff's interpolating polynomials and conditions on Green's functions as well as present the generalizations of majorization theorem for the class of n-convex functions. We obtain the generalizations of classical and weighted majorization theorems. We give bounds for identities related to the generalizations of majorization inequalities by using Čebyšev functionals. We also give Grüss type inequalities and Ostrowski-type inequalities for these functionals. We present mean value theorems and n-exponential convexity which leads to exponential convexity and then log-convexity for these functionals. At the end, we discuss some families of functions which enable us to construct a large families of functions that are exponentially convex and also give Stolarsky type means with their monotonicity.

1. Introduction and preliminaries

Majorization (sub- or supermajorization) introduces a preorder into R^n. For fixed $m \geq 2$, let

$$\mathbf{x} = (x_1, ..., x_m), \;\; \mathbf{y} = (y_1, ..., y_m)$$

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denote two m-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[m]}, \quad y_{[1]} \geq y_{[2]} \geq ... \geq y_{[m]},$$

$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(m)}, \quad y_{(1)} \leq y_{(2)} \leq ... \leq y_{(m)}$$

be their ordered components.

Majorization: ([18, p.319]) $\mathbf{x}$ is said to majorize $\mathbf{y}$ (or $\mathbf{y}$ is said to be majorized by $\mathbf{x}$), in symbol, $\mathbf{x} \succ \mathbf{y}$, if

$$\sum_{i=1}^{l} y_{[i]} \leq \sum_{i=1}^{l} x_{[i]} \tag{1.1}$$

holds for $l = 1, 2, ..., m-1$ and

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i.$$

Note that (1.1) is equivalent to

$$\sum_{i=m-l+1}^{m} y_{(i)} \leq \sum_{i=m-l+1}^{m} x_{(i)}$$

holds for $l = 1, 2, ..., m-1$.

The following theorem is well-known as the majorization theorem and a convenient reference for its proof is given by Marshall and Olkin [15, p.11] (see also [18, p.320]):

Theorem 1.1. *Let* $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ *be two m-tuples such that* x_i*,* $y_i \in [a, b]$ $(i = 1, ..., m)$*. Then*

$$\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i) \tag{1.2}$$

holds for every continuous convex function $\phi : [a, b] \to \mathbb{R}$ *iff* $\mathbf{x} \succ \mathbf{y}$ *holds.*

The following theorem can be regarded as a generalization of Theorem 1.1 known as weighted majorization theorem and is proved by Fuchs in [11] (see also [18, p.323]):

Theorem 1.2. *Let* $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ *be two decreasing real m-tuples with* x_i*,* $y_i \in [a, b]$ $(i = 1, ..., m)$*, let* $\mathbf{w} = (w_1, ..., w_m)$ *be a real m-tuple such that*

$$\sum_{i=1}^{l} w_i\, y_i \leq \sum_{i=1}^{l} w_i\, x_i \;\; \textit{for} \;\; l = 1, ..., m-1; \tag{1.3}$$

and

$$\sum_{i=1}^{m} w_i\, y_i \;=\; \sum_{i=1}^{m} w_i\, x_i. \tag{1.4}$$

Then for every continuous convex function $\phi : [a,b] \to \mathbb{R}$, we have

$$\sum_{i=1}^{m} w_i\, \phi\left(y_i\right) \;\leq\; \sum_{i=1}^{m} w_i\, \phi\left(x_i\right). \tag{1.5}$$

The following theorem is a simple consequence of Theorem 12.14 in [20] (see also [18, p.328]):

Theorem 1.3. *Let $x(t)$, $y(t)$: $[a,b] \to \mathbb{R}$, $x(t)$ and $y(t)$ be decreasing and $w : [a,b] \to \mathbb{R}$ be continuous functions.*
If

$$\int_a^{\nu} w(t)\, y(t)\, dt \;\leq\; \int_a^{\nu} w(t)\, x(t)\, dt \quad \textit{for every} \;\; \nu \in [a,b], \tag{1.6}$$

and

$$\int_a^{b} w(t)\, y(t)\, dt \;=\; \int_a^{b} w(t)\, x(t)\, dt \tag{1.7}$$

hold, then for every continuous convex function ϕ we have

$$\int_a^{b} w(t)\, \phi\left(y(t)\right)\, dt \;\leq\; \int_a^{b} w(t)\, \phi\left(x(t)\right)\, dt. \tag{1.8}$$

For discrete version and generalizations of majorization theorem see [16]. For integral version and generalizations of majorization theorem see [15, p.583], [1, 2, 3, 4, 7, 14, 17].

In this paper, n always denotes a positive integer number. Throughout, in what follows, we shall assume that the function ϕ that is n-times continuously differentiable on the interval $[a,b]$ (i.e., $\phi \in C^n[a,b]$), although this restriction is not necessary.

The Abel-Gontscharoff interpolation problem in the real case was introduced in 1935 by Whittaker [21] and subsequently by Gontscharoff [12] and Davis [9].

The Abel-Gontscharoff interpolating polynomial for two points with integral remainder is given in [5]:

Theorem 1.4. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$ and $\phi \in C^n[a,b]$; then we have*

$$\phi(t) = Q_{n-1}\left(a, b, \phi, t\right) + R\left(\phi, t\right), \tag{1.9}$$

where Q_{n-1} is the Abel-Gontscharoff interpolating polynomial for two-points of degree $n-1$, i.e.,

$$Q_{n-1}(a,b,\phi,t) = \sum_{i=0}^{k} \frac{(t-a)^i}{i!}\phi^{(i)}(a) + \sum_{j=0}^{n-k-2}\left[\sum_{i=0}^{j} \frac{(t-a)^{k+1+i}(a-b)^{j-i}}{(k+1+i)!\,(j-i)!}\right]\phi^{(k+1+j)}(b)$$

and the remainder is given by

$$R(\phi,t) = \int_a^b G_n(t,s)\phi^{(n)}(s)ds,$$

where $G_n(t,s)$ be Green's function [5, p.177]

$$G_n(t,s) = \frac{1}{(n-1)!}\begin{cases} \sum_{i=0}^{k} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1}, & a \le s \le t; \\ -\sum_{i=k+1}^{n-1} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1}, & t \le s \le b. \end{cases} \tag{1.10}$$

Further, for $a \le s, t \le b$ the following inequality hold

$$(-1)^{n-k-1}\frac{\partial^i G_n(t,s)}{\partial t^i} \ge 0, \quad 0 \le i \le k, \tag{1.11}$$

$$(-1)^{n-i}\frac{\partial^i G_n(t,s)}{\partial t^i} \ge 0, \quad k+1 \le i \le n-1. \tag{1.12}$$

We arrange the paper in this manner, in section 2, we give some identities for the difference of majorization inequality by using Abel-Gontscharoff interpolating polynomial for two points and present the generalizations of majorization theorem for the class of n-convex functions. We also obtain the generalizations of classical and weighted majorization theorems. In section 3, we give bounds for identities related to the generalizations of majorization inequalities by using Čebyšev functionals and also give Grüss type inequalities and Ostrowski-type inequalities for these functionals. In section 4, we present Lagrange and Cauchy type mean value theorems related to the functionals which are the differences of the generalizations of majorization inequality and also give n-exponential convexity which leads to exponential convexity and then log-convexity for these defined functionals. At the end, in section 5, we discuss some families of functions which enable us to construct a large families of functions that are exponentially convex and also give Stolarsky type means with their monotonicity.

2. Main results

We start this section with the identities of generalizations of majorization inequality using Abel-Gontscharoff interpolating polynomial for two points.

Theorem 2.1. *Let* $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ *and* $\mathbf{w} = (w_1, ..., w_m)$ *be* m*-tuples such that* x_r, $y_r \in [a,b]$ *and* $w_r \in \mathbb{R}$ $(r = 1, ..., m)$. *Let also* $\phi \in C^n[a,b]$ *and* G_n *be the Green function defined as in* (1.10), *then*

$$\begin{aligned}
&\sum_{r=1}^{m} w_r\, \phi(x_r) - \sum_{r=1}^{m} w_r\, \phi(y_r) \\
&= \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\sum_{r=1}^{m} w_r (x_r - a)^i - \sum_{r=1}^{m} w_r (y_r - a)^i\right] \\
&\quad + \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\quad \times \left[\sum_{r=1}^{m} w_r (x_r - a)^{k+1+i} - \sum_{r=1}^{m} w_r (y_r - a)^{k+1+i}\right] \\
&\quad + \int_a^b \left(\sum_{r=1}^{m} w_r G_n(x_r, s) - \sum_{r=1}^{m} w_r G_n(y_r, s)\right) \phi^{(n)}(s) ds. \qquad (2.1)
\end{aligned}$$

Proof. Consider the majorization difference

$$\sum_{r=1}^{m} w_r \phi(x_r) - \sum_{r=1}^{m} w_r \phi(y_r). \qquad (2.2)$$

By using Theorem 1.4 we have

$$\begin{aligned}
\phi(t) &= \sum_{i=0}^{k} \frac{(t-a)^i}{i!} \phi^{(i)}(a) \\
&\quad + \sum_{j=0}^{n-k-2} \left[\sum_{i=0}^{j} \frac{(t-a)^{k+1+i} (-1)^{j-i} (b-a)^{j-i}}{(k+1+i)!\,(j-i)!}\right] \phi^{(k+1+j)}(b) \\
&\quad + \int_a^b G_n(t,s)\, \phi^{(n)}(s) ds. \qquad (2.3)
\end{aligned}$$

Substituting this value of ϕ in (2.2) and some arrangements, we get (2.1). $\square$

Integral version of the above theorem can be stated as:

Theorem 2.2. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and $x, y : [\alpha, \beta] \to [a, b]$, $w : [\alpha, \beta] \to \mathbb{R}$ be continuous functions. Let also $\phi \in C^n[a, b]$ and G_n be the Green function defined as in* (1.10)*, then*

$$\begin{aligned}
&\int_\alpha^\beta w(t)\,\phi\left(x(t)\right)dt - \int_\alpha^\beta w(t)\,\phi\left(y(t)\right)dt \\
&= \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^i dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^i dt\right] \\
&\quad + \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\quad \times \left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^{k+1+i} dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^{k+1+i} dt\right] \\
&\quad + \int_a^b \phi^{(n)}(s)\left(\int_\alpha^\beta w(t)G_n\left(x(t), s\right)dt - \int_\alpha^\beta w(t)G_n\left(y(t), s\right)dt\right)ds. \qquad (2.4)
\end{aligned}$$

We give generalizations of majorization inequality for $n-$convex functions.

Theorem 2.3. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ and $\mathbf{w} = (w_1, ..., w_m)$ be m-tuples such that x_r, $y_r \in [a, b]$ and $w_r \in \mathbb{R}$ $(r = 1, ..., m)$ and also G_n be the Green function defined as in* (1.10)*. If for all $s \in [a, b]$*

$$\sum_{r=1}^{m} w_r G_n\left(y_r, s\right) \leq \sum_{r=1}^{m} w_r G_n\left(x_r, s\right), \qquad (2.5)$$

then for every n-convex function $\phi : [a, b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\sum_{r=1}^{m} w_r\,\phi\left(x_r\right) - \sum_{r=1}^{m} w_r\,\phi\left(y_r\right) \\
&\geq \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^i - \sum_{r=1}^{m} w_r\left(y_r-a\right)^i\right] \\
&\quad + \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\quad \times \left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^{k+1+i} - \sum_{r=1}^{m} w_r\left(y_r-a\right)^{k+1+i}\right]. \qquad (2.6)
\end{aligned}$$

If the reverse inequality in (2.5) *holds, then also the reverse inequality in* (2.6) *holds.*

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)}(x) \geq 0$, for all $x \in [a,b]$. Hence we can apply Theorem 2.1 to get (2.6). □

Integral version of the above theorem can be stated as:

Theorem 2.4. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and $x, y : [\alpha, \beta] \to [a,b]$, $w : [\alpha, \beta] \to \mathbb{R}$ be continuous functions and also G_n be the Green function defined as in* (1.10). *If for all $s \in [a,b]$*

$$\int_{\alpha}^{\beta} w(t)\, G_n\left(y(t), s\right) dt \leq \int_{\alpha}^{\beta} w(t)\, G_n\left(x(t), s\right) dt, \tag{2.7}$$

then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds

$$\begin{aligned}
&\int_{\alpha}^{\beta} w(t)\, \phi\left(x(t)\right) dt - \int_{\alpha}^{\beta} w(t)\, \phi\left(y(t)\right) dt \\
&\geq \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\int_{\alpha}^{\beta} w(t)\left(x(t)-a\right)^{i} dt - \int_{\alpha}^{\beta} w(t)\left(y(t)-a\right)^{i} dt \right] \\
&\quad + \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\quad \times \left[\int_{\alpha}^{\beta} w(t)\left(x(t)-a\right)^{k+1+i} dt - \int_{\alpha}^{\beta} w(t)\left(y(t)-a\right)^{k+1+i} dt \right].
\end{aligned} \tag{2.8}$$

If the reverse inequality in (2.7) *holds, then also the reverse inequality in* (2.8) *holds.*

The following theorem is the generalization of classical majorization theorem:

Theorem 2.5. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ be m-tuples such that x_r, $y_r \in [a,b]$ and $\mathbf{x} \succ \mathbf{y}$ and also G_n be the Green function defined as in* (1.10).

(i) *If k is odd and n is even or k is even and n is odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds*

$$\begin{aligned} &\sum_{r=1}^{m} \phi(x_r) - \sum_{r=1}^{m} \phi(y_r) \\ &\geq \sum_{i=2}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\sum_{r=1}^{m} (x_r - a)^i - \sum_{r=1}^{m} (y_r - a)^i \right] \\ &\quad + \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\ &\quad \times \left[\sum_{r=1}^{m} (x_r - a)^{k+1+i} - \sum_{r=1}^{m} (y_r - a)^{k+1+i} \right]. \end{aligned} \tag{2.9}$$

Moreover if $\phi^{(i)}(a) \geq 0$ for $i = 0, ..., k$ and $\phi^{(k+1+j)}(b) \geq 0$ if $j - i$ is even and $\phi^{(k+1+j)}(b) \leq 0$ if $j - i$ is odd for $i = 0, ..., j$ and $j = 0, ..., n-k-2$, then the right hand side of (2.9) *will be non-negative that is* (1.2) *holds.*

(ii) *If k and n both are even or odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, the reverse inequality in* (2.9) *holds. Moreover, if $\phi^{(i)}(a) \leq 0$ for $i = 0, ..., k$ and $\phi^{(k+1+j)}(b) \leq 0$ if $j - i$ is even, and $\phi^{(k+1+j)}(b) \geq 0$ if $j-i$ is odd for $i = 0, ..., j$ and $j = 0, ..., n-k-2$, then the right hand side of reverse inequality in* (2.9) *will be non-positive that is the reverse inequality in* (1.2) *holds.*

Proof. By using (1.11), for $a \leq s, t \leq b$ the following inequality hold

$$(-1)^{n-k-1} \frac{\partial^2 G_n(t,s)}{\partial t^2} \geq 0,$$

we conclude easily that if k is odd and n is even or k is even and n is odd then $\partial^2 G_n(t,s)/\partial t^2 \geq 0$ and also if k and n both are even or odd then $\partial^2 G_n(t,s)/\partial t^2 \leq 0$. So k is odd and n is even or k is even and n is odd, G_n is convex with respect to first variable therefore by using Theorem 1.1 we have

$$\sum_{r=1}^{m} G_n(y_r, s) \leq \sum_{r=1}^{m} G_n(x_r, s).$$

Hence by Theorem 2.3 for $w_r = 1$, $(r = 1, ..., m)$ we get (2.9).
By using the other conditions the non-negativity of the right-hand side of (2.9) is obvious that is (1.2) holds. Similarly we can prove for part (ii). □

The following theorem is the generalization of weighted majorization theorem:

Theorem 2.6. *Let* $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ *be decreasing and* $\mathbf{w} = (w_1, ..., w_m)$ *be any* m*-tuples such that* x_r, $y_r \in [a, b]$ *and* $w_r \in \mathbb{R}$ $(r = 1, ..., m)$ *satisfies* (1.3) *and* (1.4) *and also* G_n *be the Green function defined as in* (1.10)*.*

(i) *If* k *is odd and* n *is even or* k *is even and* n *is odd, then for every* n*-convex function* $\phi : [a, b] \to \mathbb{R}$*, it holds*

$$\begin{aligned}
&\sum_{r=1}^{m} w_r \, \phi(x_r) - \sum_{r=1}^{m} w_r \, \phi(y_r) \\
&\geq \sum_{i=2}^{k} \frac{\phi^{(i)}(a)}{i!} \left[\sum_{r=1}^{m} w_r (x_r - a)^i - \sum_{r=1}^{m} w_r (y_r - a)^i \right] \\
&\quad + \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i}}{(k+1+i)!(j-i)!} \phi^{(k+1+j)}(b) \\
&\quad \times \left[\sum_{r=1}^{m} w_r (x_r - a)^{k+1+i} - \sum_{r=1}^{m} w_r (y_r - a)^{k+1+i} \right]. \qquad (2.10)
\end{aligned}$$

Moreover, if $\phi^{(i)}(a) \geq 0$ *for* $i = 0, ..., k$ *and* $\phi^{(k+1+j)}(b) \geq 0$ *if* $j - i$ *is even and* $\phi^{(k+1+j)}(b) \leq 0$ *if* $j - i$ *is odd for* $i = 0, ..., j$ *and* $j = 0, ..., n-k-2$*, then the right hand side of* (2.10) *will be non-negative that is* (1.5) *holds.*

(ii) *If* k *and* n *both are even or odd, then for every* n*-convex function* $\phi : [a, b] \to \mathbb{R}$*, the reverse inequality in* (2.10) *holds. Moreover, if* $\phi^{(i)}(a) \leq 0$ *for* $i = 0, ..., k$ *and* $\phi^{(k+1+j)}(b) \leq 0$ *if* $j - i$ *is even, and* $\phi^{(k+1+j)}(b) \geq 0$ *if* $j - i$ *is odd for* $i = 0, ..., j$ *and* $j = 0, ..., n-k-2$*, then the right hand side of the reverse inequality in* (2.10) *will be non-positive that is the reverse inequality in* (1.5) *holds.*

Proof. The proof is similar to the proof of Theorem 2.5 but use Theorem 1.2 instead of Theorem 1.1. □

The following theorem is weighted majorization theorem for n-convex function in integral case:

Theorem 2.7. *Let* $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $x, y : [\alpha, \beta] \to [a, b]$ *be increasing and* $w : [\alpha, \beta] \to \mathbb{R}$ *be continuous functions satisfying* (1.6) *and* (1.7) *and also* G_n *be the Green function defined as in* (1.10)*.*

(i) *If k is odd and n is even or k is even and n is odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, it holds*

$$\begin{aligned}
&\int_\alpha^\beta w(t)\,\phi\left(x(t)\right)dt - \int_\alpha^\beta w(t)\,\phi\left(y(t)\right)dt \\
&\geq \sum_{i=2}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^i dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^i dt\right] \\
&\quad + \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\quad \times \left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^{k+1+i} dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^{k+1+i} dt\right]. \qquad (2.11)
\end{aligned}$$

Moreover, if $\phi^{(i)}(a) \geq 0$ for $i = 0, ..., k$ and $\phi^{(k+1+j)}(b) \geq 0$ if $j-i$ is even and $\phi^{(k+1+j)}(b) \leq 0$ if $j-i$ is odd for $i = 0, ..., j$ and $j = 0, ..., n-k-2$, then the right hand side of (2.11) *will be non-negative that is* (1.8) *holds.*

(ii) *If k and n both are even or odd, then for every n-convex function $\phi : [a,b] \to \mathbb{R}$, then the reverse inequality holds in* (2.11). *Moreover, if $\phi^{(i)}(a) \leq 0$ for $i = 0, ..., k$ and $\phi^{(k+1+j)}(b) \leq 0$ if $j-i$ is even, and $\phi^{(k+1+j)}(b) \geq 0$ if $j-i$ is odd for $i = 0, ..., j$ and $j = 0, ..., n-k-2$, then right hand side of the reverse inequality in* (2.11) *will be non-positive that is the reverse inequality in* (1.8) *holds.*

3. Bounds for identities related to generalizations of majorization inequality

For two Lebesgue integrable functions $f, h : [a,b] \to \mathbb{R}$ we consider the Čebyšev functional

$$\Omega(f,h) = \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \,.\, \frac{1}{b-a}\int_a^b h(t)dt. \qquad (3.1)$$

In [8], the authors proved the following theorems:

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $(.-a)(b-.)[h']^2 \in L[a,b]$. Then we have the inequality*

$$\mid \Omega(f,h) \mid \leq \frac{1}{\sqrt{2}}\left[\Omega(f,f)\right]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)\left[h'(x)\right]^2 dx\right)^{\frac{1}{2}}. \qquad (3.2)$$

The constant $\frac{1}{\sqrt{2}}$ *in* (3.2) *is the best possible.*

Theorem 3.2. *Assume that* $h : [a, b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a, b]$ *and* $f : [a, b] \to \mathbb{R}$ *is absolutely continuous with* $f' \in L_\infty[a, b]$*. Then we have the inequality*

$$| \Omega(f, h) | \leq \frac{1}{2(b-a)} \parallel f' \parallel_\infty \int_a^b (x-a)(b-x) dh(x). \tag{3.3}$$

The constant $\frac{1}{2}$ *in* (3.3) *is the best possible.*

In the sequel we use the above theorems to obtain generalizations of the results proved in the previous section.
For m-tuples $\mathbf{w} = (w_1, ..., w_m)$, $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ with x_r, $y_r \in [a, b], w_r \in R$ $(r = 1, ..., m)$ and the function G_n as defined above, denote

$$\Upsilon(t) = \sum_{r=1}^{m} w_r G_n(x_r, s) - \sum_{r=1}^{m} w_r G_n(y_r, s), \quad s \in [a, b], \tag{3.4}$$

similarly for $x, y : [\alpha, \beta] \to [a, b]$ and $w : [\alpha, \beta] \to \mathbb{R}$ be continuous functions and for all $s \in [a, b]$, denote

$$\widetilde{\Upsilon}(s) = \int_\alpha^\beta w(t) G_n(x(t), s)\, dt - \int_\alpha^\beta w(t) G_n(y(t), s)\, dt. \tag{3.5}$$

Consider the *Č*eby*š*ev functionals defined as:

$$\Omega(\Upsilon, \Upsilon) = \frac{1}{b-a} \int_a^b \Upsilon^2(s) ds - \left(\frac{1}{b-a} \int_a^b \Upsilon(s) ds \right)^2, \tag{3.6}$$

$$\Omega(\widetilde{\Upsilon}, \widetilde{\Upsilon}) = \frac{1}{b-a} \int_a^b \widetilde{\Upsilon}^2(s) ds - \left(\frac{1}{b-a} \int_a^b \widetilde{\Upsilon}(s) ds \right)^2. \tag{3.7}$$

Theorem 3.3. *Let* $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\phi : [a, b] \to \mathbb{R}$ *be such that* $\phi \in C^n[a, b]$ *with* $(. - a)(b - .) \left[\phi^{(n+1)}\right]^2 \in L[a, b]$, *and* $\boldsymbol{x} = (x_1, ..., x_m)$, $\boldsymbol{y} = (y_1, ..., y_m)$ *and* $\mathbf{w} = (w_1, ..., w_m)$ *be* m*-tuples such that* x_r, $y_r \in [a, b]$ *and* $w_r \in \mathbb{R}$ $(r = 1, ..., m)$*. Let the functions* G_n, Υ *and* Ω *be defined in* (1.10), (3.4) *and* (3.6) *respectively. Then*

$$\begin{aligned}
&\sum_{r=1}^{m} w_r\,\phi\left(x_r\right) - \sum_{r=1}^{m} w_r\,\phi\left(y_r\right) \\
&= \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^i - \sum_{r=1}^{m} w_r\left(y_r-a\right)^i\right] \\
&\quad + \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\quad \times \left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^{k+1+i} - \sum_{r=1}^{m} w_r\left(y_r-a\right)^{k+1+i}\right] \\
&\quad + \frac{\phi^{(n-1)}(b)-\phi^{(n-1)}(a)}{b-a}\int_a^b \Upsilon(t)dt + H_n^1(\phi;a,b), \qquad (3.8)
\end{aligned}$$

where the remainder $H_n^1(\phi;a,b)$ *satisfies the estimation*

$$\left|H_n^1(\phi;a,b)\right| \le \sqrt{\frac{b-a}{2}}\,[\Omega(\Upsilon,\Upsilon)]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}. \qquad (3.9)$$

Proof. If we apply Theorem 3.1 for $f \to \Upsilon$ and $h \to \phi^{(n)}$ we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \Upsilon(t)\phi^{(n)}(t)dt - \frac{1}{b-a}\int_a^b \Upsilon(t)dt.\frac{1}{b-a}\int_a^b \phi^{(n)}(t)dt\right| \\
&\le \frac{1}{\sqrt{2}}\,[\Omega(\Upsilon,\Upsilon)]^{\frac{1}{2}}\,\frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)\left[\phi^{(n+1)}(t)\right]^2 dt\right|^{\frac{1}{2}}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\int_a^b \Upsilon(t)\phi^{(n)}(t)dt \\
&= \frac{\phi^{(n-1)}(b)-\phi^{(n-1)}(a)}{b-a}\int_a^b \Upsilon(t)dt + H_n^1(\phi;a,b)
\end{aligned}$$

where the remainder $H_n^1(\phi;a,b)$ satisfies the estimation (3.9). Now from the identity (2.1) we obtain (3.8). □

Integral case of the above theorem can be given:

Theorem 3.4. *Let* $n,k \in \mathbb{N}$, $n \ge 2$, $0 \le k \le n-1$, $\phi : [a,b] \to \mathbb{R}$ *be such that* $\phi \in C^n[a,b]$ *with* $(.-a)(b-.)\left[\phi^{(n+1)}\right]^2 \in L[a,b]$, *and* $x,y : [\alpha,\beta] \to [a,b]$,

$w:[\alpha,\beta]\to\mathbb{R}$ be continuous functions and also let the functions $G_n, \widetilde{\Upsilon}$ and Ω be defined in (1.10)*,* (3.5) *and* (3.7) *respectively. Then*

$$\begin{aligned}
&\int_\alpha^\beta w(t)\,\phi\left(x(t)\right)dt-\int_\alpha^\beta w(t)\,\phi\left(y(t)\right)dt\\
&=\sum_{i=0}^{k}\frac{\phi^{(i)}(a)}{i!}\left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^i dt-\int_\alpha^\beta w(t)\left(y(t)-a\right)^i dt\right]\\
&\quad+\sum_{j=0}^{n-k-2}\sum_{i=0}^{j}\frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\\
&\quad\times\left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^{k+1+i}dt-\int_\alpha^\beta w(t)\left(y(t)-a\right)^{k+1+i}dt\right]\\
&\quad+\frac{\phi^{(n-1)}(b)-\phi^{(n-1)}(a)}{b-a}\int_a^b\widetilde{\Upsilon}(s)ds+\widetilde{H}_n^1(\phi;a,b),
\end{aligned}\tag{3.10}$$

where the remainder $\widetilde{H}_n^1(\phi;a,b)$ satisfies the estimation

$$\left|\widetilde{H}_n^1(\phi;a,b)\right|\le\sqrt{\frac{b-a}{2}}\left[\Omega(\widetilde{\Upsilon},\widetilde{\Upsilon})\right]^{\frac{1}{2}}\left|\int_a^b(t-a)(b-t)\left[\phi^{(n+1)}(t)\right]^2dt\right|^{\frac{1}{2}}.$$

Using Theorem 3.2 we obtain the following Grüss type inequalities.

Theorem 3.5. *Let $n,k\in\mathbb{N}$, $n\ge 2$, $0\le k\le n-1$, $\phi:[a,b]\to\mathbb{R}$ be such that $\phi\in C^n[a,b]$ and $\phi^{(n+1)}\ge 0$ on $[a,b]$ and let the function Υ and Ω be defined by* (3.4) *and* (3.6) *respectively. Then we have the representation* (3.8) *and the remainder $H_n^1(\phi;a,b)$ satisfies the bound*

$$\left|H_n^1(\phi;a,b)\right|\le\left\|\Upsilon'\right\|_\infty\left\{\frac{\phi^{(n-1)}(b)+\phi^{(n-1)}(a)}{2}-\frac{\phi^{(n-2)}(b)-\phi^{(n-2)}(a)}{b-a}\right\}.\tag{3.11}$$

Proof. Applying Theorem 3.2 for $f\to\Upsilon$ and $h\to\phi^{(n)}$ we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b\Upsilon(t)\phi^{(n)}(t)dt-\frac{1}{b-a}\int_a^b\Upsilon(t)dt.\frac{1}{b-a}\int_a^b\phi^{(n)}(t)dt\right|\\
&\le\frac{1}{2(b-a)}\left\|\Upsilon'\right\|_\infty\int_a^b(t-a)(b-t)\phi^{(n+1)}(t)dt.
\end{aligned}\tag{3.12}$$

Since

$$\begin{aligned}
&\int_a^b(t-a)(b-t)\phi^{(n+1)}(t)dt=\int_a^b\left[2t-(a+b)\right]\phi^{(n)}(t)dt\\
&=(b-a)\left[\phi^{(n-1)}(b)+\phi^{(n-1)}(a)\right]-2\left(\phi^{(n-2)}(b)-\phi^{(n-2)}(a)\right),
\end{aligned}$$

using the identities (2.1) and (3.12) we deduce (3.11). □

Integral version of the above theorem can be given as:

Theorem 3.6. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\phi : [a,b] \to \mathbb{R}$ be such that $\phi \in C^n[a,b]$ and $\phi^{(n+1)} \geq 0$ on $[a,b]$ and also let the functions $\widetilde{\Upsilon}$ and Ω be defined by* (3.5) *and* (3.7) *respectively. Then we have the representation* (3.10) *and the remainder $\widetilde{H}_n^1(\phi; a, b)$ satisfies the bound*

$$\left|\widetilde{H}_n^1(\phi; a,b)\right| \leq \left\|\widetilde{\Upsilon}'\right\|_\infty \left\{\frac{\phi^{(n-1)}(b) + \phi^{(n-1)}(a)}{2} - \frac{\phi^{(n-2)}(b) - \phi^{(n-2)}(a)}{b-a}\right\}.$$

We give the Ostrowski-type inequalities related to the generalizations of majorization inequality.

Theorem 3.7. *Suppose that all the assumptions of Theorem* 2.1 *hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $\left|\phi^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
&\Bigg|\sum_{r=1}^m w_r\,\phi\left(x_r\right) - \sum_{r=1}^m w_r\,\phi\left(y_r\right) - \sum_{i=0}^k \frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^m w_r\left(x_r - a\right)^i - \sum_{r=1}^m w_r\left(y_r - a\right)^i\right] \\
&\quad - \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\quad \times \left[\sum_{r=1}^m w_r\left(x_r - a\right)^{k+1+i} - \sum_{r=1}^m w_r\left(y_r - a\right)^{k+1+i}\right]\Bigg| \\
&\leq \left\|\phi^{(n)}\right\|_p \left(\int_a^b \left|\sum_{r=1}^m w_r G_n\left(x_r, s\right) - \sum_{r=1}^m w_r G_n\left(y_r, s\right)\right|^q dt\right)^{\frac{1}{q}}. \qquad (3.13)
\end{aligned}$$

The constant on the right-hand side of (3.13) *is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.*

Proof. As we have

$$\Upsilon(s) = \sum_{r=1}^m w_r G_n\left(x_r, s\right) - \sum_{r=1}^m w_r G_n\left(y_r, s\right), \quad s \in [a,b].$$

Using the identity (2.1) and applying Hölder's inequality we obtain

$$\begin{aligned}&\Bigg|\sum_{r=1}^{m} w_r\,\phi\left(x_r\right)-\sum_{r=1}^{m} w_r\,\phi\left(y_r\right)\\&\quad-\sum_{i=0}^{k}\frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^i-\sum_{r=1}^{m} w_r\left(y_r-a\right)^i\right]\\&\quad-\sum_{j=0}^{n-k-2}\sum_{i=0}^{j}\frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\\&\quad\times\left[\sum_{r=1}^{m} w_r\left(x_r-a\right)^{k+1+i}-\sum_{r=1}^{m} w_r\left(y_r-a\right)^{k+1+i}\right]\Bigg|\\&=\left|\int_a^b \Upsilon(t)\phi^{(n)}(t)dt\right|\le\left\|\phi^{(n)}\right\|_p\left(\int_a^b\left|\Upsilon(t)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_a^b\left|\Upsilon(t)\right|^q dt\right)^{\frac{1}{q}}$ let us find a function ϕ for which the equality in (3.13) is obtained.
For $1<p<\infty$ take ϕ to be such that

$$\phi^{(n)}(t)=sgn\Upsilon(t)\left|\Upsilon(t)\right|^{\frac{1}{p-1}}.$$

For $p=\infty$ take $\phi^{(n)}(t)=sgn\Upsilon(t)$.
For $p=1$ we prove that

$$\left|\int_a^b \Upsilon(t)\phi^{(n)}(t)\right|\le\max_{t\in[a,b]}\left|\Upsilon(t)\right|\left(\int_a^b\left|\phi^{(n)}(t)\right|dt\right) \tag{3.14}$$

is the best possible inequality. Suppose that $|\Upsilon(t)|$ attains its maximum at $t_0\in[a,b]$. First we assume that $\Upsilon(t_0)>0$. For ϵ small enough we define $\phi_\epsilon(t)$ by

$$\phi_\epsilon(t):=\begin{cases}0, & a\le t\le t_0,\\ \frac{1}{\epsilon\, n!}\left(t-t_0\right)^n, & t_0\le t\le t_0+\epsilon,\\ \frac{1}{n!}\left(t-t_0\right)^{n-1}, & t_0+\epsilon\le t\le b.\end{cases}$$

Then for ϵ small enough

$$\left|\int_a^b \Upsilon(t)\phi^{(n)}(t)\right|=\left|\int_{t_0}^{t_0+\epsilon}\Upsilon(t)\frac{1}{\epsilon}dt\right|=\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}\Upsilon(t)dt.$$

Now from the inequality (3.14) we have

$$\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}\Upsilon(t)dt \le \Upsilon(t_0)\int_{t_0}^{t_0+\epsilon}\frac{1}{\epsilon}dt = \Upsilon(t_0).$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon}\Upsilon(t)dt = \Upsilon(t_0)$$

the statement follows. In the case $\Upsilon(t_0) < 0$, we define $\phi_\epsilon(t)$ by

$$\phi_\epsilon(t) := \begin{cases} \frac{1}{n!}\left(t-t_0-\epsilon\right)^{n-1}, & a \le t \le t_0, \\ -\frac{1}{\epsilon\, n!}\left(t-t_0-\epsilon\right)^{n}, & t_0 \le t \le t_0+\epsilon, \\ 0, & t_0+\epsilon \le t \le b, \end{cases}$$

and the rest of the proof is the same as above. □

Integral version of the above theorem can be stated as:

Theorem 3.8. *Suppose that all the assumptions of Theorem* 2.2 *hold. Assume* (p, q) *is a pair of conjugate exponents, that is* $1 \le p, q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\left|\phi^{(n)}\right|^p : [a, b] \to \mathbb{R}$ *be an R-integrable function for some* $n \in \mathbb{N}$. *Then we have*

$$\begin{aligned}
&\Bigg|\int_\alpha^\beta w(t)\,\phi\left(x(t)\right)dt - \int_\alpha^\beta w(t)\,\phi\left(y(t)\right)dt \\
&\quad - \sum_{i=0}^{k}\frac{\phi^{(i)}(a)}{i!}\left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^i dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^i dt\right] \\
&\quad - \sum_{j=0}^{n-k-2}\sum_{i=0}^{j}\frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b) \\
&\quad \times\left[\int_\alpha^\beta w(t)\left(x(t)-a\right)^{k+1+i}dt - \int_\alpha^\beta w(t)\left(y(t)-a\right)^{k+1+i}dt\right]\Bigg| \\
&\le \left\|\phi^{(n)}\right\|_p\left(\int_a^b\left|\int_\alpha^\beta w(t)G_n\left(x(t),s\right)dt - \int_\alpha^\beta w(t)G_n\left(y(t),s\right)dt\right|^q ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.15}$$

The constant on the right-hand side of (3.15) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

4. n-exponential convexity and exponential convexity

Motivated by the inequality (2.6) and (2.8), we define functional $\Theta_1(\phi)$ and $\Theta_2(\phi)$ by

$$\begin{aligned}\Theta_1(\phi) &= \sum_{r=1}^{m} w_r\,\phi\,(x_r) - \sum_{r=1}^{m} w_r\,\phi\,(y_r)\\ &\quad - \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\sum_{r=1}^{m} w_r\,(x_r-a)^i - \sum_{r=1}^{m} w_r\,(y_r-a)^i\right]\\ &\quad - \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\\ &\quad \times\left[\sum_{r=1}^{m} w_r\,(x_r-a)^{k+1+i} - \sum_{r=1}^{m} w_r\,(y_r-a)^{k+1+i}\right], \end{aligned} \tag{4.1}$$

$$\begin{aligned}\Theta_2(\phi) &= \int_{\alpha}^{\beta} w(t)\,\phi\,(x(t))\,dt - \int_{\alpha}^{\beta} w(t)\,\phi\,(y(t))\,dt\\ &\quad - \sum_{i=0}^{k} \frac{\phi^{(i)}(a)}{i!}\left[\int_{\alpha}^{\beta} w(t)\,(x(t)-a)^i\,dt - \int_{\alpha}^{\beta} w(t)\,(y(t)-a)^i\,dt\right]\\ &\quad - \sum_{j=0}^{n-k-2}\sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}}{(k+1+i)!(j-i)!}\phi^{(k+1+j)}(b)\\ &\quad \times\left[\int_{\alpha}^{\beta} w(t)\,(x(t)-a)^{k+1+i}\,dt - \int_{\alpha}^{\beta} w(t)\,(y(t)-a)^{k+1+i}\,dt\right]. \end{aligned} \tag{4.2}$$

Remark 4.1. Under the assumptions of Theorem 2.3 and Theorem 2.4, it holds $\Theta_i(\phi) \geq 0$, $i = 1, 2$ for all n-convex functions ϕ.

Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems:

Theorem 4.2. *Let $\phi : [a, b] \to R$ be such that $\phi \in C^n[a, b]$. If the inequalities in (2.5) $(i = 1)$, (2.7) $(i = 2)$ hold, then there exist $\xi_i \in [a, b]$ such that*

$$\Theta_i(\phi) = \phi^{(n)}(\xi_i)\Theta_i(\eta), \quad i = 1, 2, \tag{4.3}$$

where $\eta(x) = \frac{x^n}{n!}$ and Θ_1, Θ_2 are defined in (4.1) and (4.2).

Proof. Similar to the proof of Theorem 7 in [6]. □

Theorem 4.3. *Let $\phi, \psi : [a, b] \to R$ be such that $\phi, \psi \in C^n[a, b]$. If the inequalities in* (2.5) $(i = 1)$, (2.7) $(i = 2)$ *hold, then there exist $\xi_i \in [a, b]$ such that*

$$\frac{\Theta_i(\phi)}{\Theta_i(\varphi)} = \frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)}, \quad i = 1, 2, \tag{4.4}$$

provided that the denominators are non-zero and Θ_1, Θ_2 are defined in (4.1) *and* (4.2).

Proof. Similar to the proof of Corollary 12 in [6]. □

Definition 4.4. ([18, p. 2]) A function $\phi : I \to \mathbb{R}$ is convex on an interval I if

$$\phi(x_1)(x_3 - x_2) + \phi(x_2)(x_1 - x_3) + \phi(x_3)(x_2 - x_1) \geq 0, \tag{4.5}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Now, let us recall some definitions and facts about exponentially convex functions (see [13]):

Definition 4.5. A function $\phi : I \to \mathbb{R}$ is *n-exponentially convex* in the Jensen sense on I if

$$\sum_{k,l=1}^{n} \alpha_k \alpha_l \phi\left(\frac{x_k + x_l}{2}\right) \geq 0$$

holds for $\alpha_k \in \mathbb{R}$ and $x_k \in I$, $k = 1, 2, ..., n$.

Definition 4.6. A function $\phi : I \to \mathbb{R}$ is *n-exponentially convex* on I if it is *n-exponentially convex* in the Jensen sense and continuous on I.

Remark 4.7. From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}, m \leq n$.

Proposition 4.8. *If $\phi : I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}, m \leq n$. Particularly,*

$$\det\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{m} \geq 0,$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

Definition 4.9. A function $\phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

Definition 4.10. A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 4.11. It is easy to show that $\phi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \geq 0$$

holds for every $\alpha, \beta \in R$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

Corollary 4.12. *If $\phi : I \to (0, \infty)$ is an exponentially convex function, then ϕ is a* log-*convex function that is*

$$\phi(\lambda x + (1-\lambda)y) \leq \phi^{\lambda}(x)\phi^{1-\lambda}(y), \;\; for \; all \;\; x, y \in I, \;\; \lambda \in [0, 1].$$

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

Definition 4.13. Let ϕ be a real-valued function defined on the segment $[a, b]$. The divided difference of order n of the function ϕ at distinct points $x_0, ..., x_n \in [a, b]$ is defined recursively (see [5], [18]) by

$$\phi[x_i] = \phi(x_i), \quad (i = 0, ..., n)$$

and

$$\phi[x_0, ..., x_n] = \frac{\phi[x_1, ..., x_n] - \phi[x_0, ..., x_{n-1}]}{x_n - x_0}.$$

The value $\phi[x_0, ..., x_n]$ is independent of the order of the points $x_0, ..., x_n$.

The definition may be extended to include the case that some (or all) the points coincide. Assuming that $\phi^{(j-1)}(x)$ exists, we define

$$\phi\,\underbrace{[x, ..., x]}_{j-times} = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \tag{4.6}$$

We use an idea from [13] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [19]):

Theorem 4.14. *Let $\Phi = \{\phi_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a, b]$ in $\mathbb{R}$ such that the function $s \mapsto \phi_s[x_0, ..., x_l]$ is an n-exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $x_0, ..., x_l \in [a, b]$. Let $\Theta_i(\phi_s)$, $i = 1, 2$ be linear functionals defined as in (4.1) and (4.2). Then the following statements hold:*

(i) *The function $s \mapsto \Theta_i(\phi_s)$ is an n-exponentially convex function in the Jensen sense on J and the matrix $\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}$, $m \le n$, $s_1, ..., s_m \in J$. Particularly*

$$\det\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^{m} \ge 0 \quad \text{for all } m \in \mathbb{N},\ m = 1, ..., n.$$

(ii) *If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is n-exponentially convex function on J.*

Proof. (i) For $\vartheta_i \in \mathbb{R}$ and $s_i \in J$, $i = 1, ..., n$ we define the function

$$\delta(x) = \sum_{i,j=1}^{n} \vartheta_i \vartheta_j \phi_{\frac{s_i+s_j}{2}}(x).$$

Using the assumption that the function $s \mapsto \phi_s[x_0, ..., x_l]$ is l-exponentially convex in the Jensen sense, we have

$$\delta[x_0, ..., x_l] = \sum_{i,j=1}^{n} \vartheta_i \vartheta_j \phi_{\frac{s_i+s_j}{2}}[x_0, ..., x_l] \ge 0,$$

which in turn implies that δ is a l-convex function on J, so it is $\Theta_i(\delta) \ge 0$, $i = 1, 2$ hence

$$\sum_{i,j=1}^{n} \vartheta_i \vartheta_j \Theta_i(\phi_{\frac{s_i+s_j}{2}}) \ge 0.$$

We conclude that the function $s \mapsto \Theta_i(\phi_s)$ is n-exponentially convex function in the Jensen sense on J. The remaining part follows from Proposition 1.
(ii) If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then $s \mapsto \Theta_i(\phi_s)$ is n-exponentially convex by definition. □

The following corollaries are an immediate consequences of the above theorem:

Corollary 4.15. *Let $\Phi = \{\phi_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a, b]$ in $\mathbb{R}$ such that the function $s \mapsto \phi_s [x_0, ..., x_l]$ is an exponentially convex in the Jensen sense on J for every $(l + 1)$ mutually different points $x_0, ..., x_l \in [a, b]$. Let $\Theta_i(\phi)$, $i = 1, 2$ be linear functionals defined as in (4.1) and (4.2). Then the following statements hold:*

(i) *The function $s \mapsto \Theta_i(\phi_s)$ is an exponentially convex function in the Jensen sense on J and the matrix $\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}$, $m \leq n$, $s_1, ..., s_m \in J$. Particularly*

$$\det\left[\Theta_i\left(\phi_{\frac{s_i+s_j}{2}}\right)\right]_{i,j=1}^{m} \geq 0 \quad \text{for all } m \in \mathbb{N},\ m = 1, ..., n.$$

(ii) *If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is exponentially convex function on J.*

Corollary 4.16. *Let $\Phi = \{\phi_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a, b]$ in $\mathbb{R}$, such that the function $s \mapsto \phi_s [x_0, ..., x_l]$ is an 2-exponentially convex in the Jensen sense on J for every $(l + 1)$ mutually different points $x_0, ..., x_l \in [a, b]$. Let $\Theta_i(\phi)$, $i = 1, 2$ be linear functionals defined as in (4.1) and (4.2). Then the following statements hold:*

(i) *If the function $s \mapsto \Theta_i(\phi_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Theta_i(\phi_s)$ is additionally strictly positive, then it is* log-*convex on J. Furthermore, the Lypunov's inequality holds true:*

$$[\Theta_i(\phi_s)]^{t-r} \leq [\Theta_i(\phi_r)]^{t-s} [\Theta_i(\phi_t)]^{s-r}, \quad i = 1, 2, \tag{4.7}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $s \mapsto \Theta_i(\phi_s)$ is strictly positive and differentiable on J, then for every $s, q, u, v \in J$, such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Theta_i, \Phi) \leq \mu_{u,v}(\Theta_i, \Phi), \tag{4.8}$$

where

$$\mu_{s,q}(\Theta_i, \Phi) = \begin{cases} \left(\frac{\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Theta_i(\phi_s)}{\Theta_i(\phi_q)}\right), & s = q, \end{cases} \tag{4.9}$$

for $\phi_s, \phi_q \in \Phi$.

Proof. (i) This is an immediate consequence of Theorem 4.14 and Remark 4.11.
(ii) Since $s \mapsto \Theta_i(\phi_s)$ is positive and continuous, by (i) we have that the function $s \mapsto \Theta_i(\phi_s)$ is log-convex on J. So, we get

$$\frac{\log \Theta_i(\phi_s) - \log \Theta_i(\phi_q)}{s-q} \leq \frac{\log \Theta_i(\phi_u) - \log \Theta_i(\phi_v)}{u-v} \tag{4.10}$$

for $s \leq u$ and $q \leq v$, $s \neq q$, $u \neq v$, so we conclude that

$$\mu_{s,q}(\Theta_i, \Phi) \leq \mu_{u,v}(\Theta_i, \Phi).$$

Cases $s = q$ and $u = v$ follows from (4.10) as limiting cases. □

Remark 4.17. Note that the results from Theorem 4.14, Corollary 4.15 and Corollary 4.16 still hold when two of the points $x_0, ..., x_l \in [a, b]$ coincide, say $x_1 = x_0$, for a family of differentiable functions ϕ_s such that the function $s \mapsto \phi_s[x_0, ..., x_l]$ is an n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(l+1)$ points coincide for a family of l differentiable functions with the same property. The proofs are obtained by (4.6) and suitable characterization of convexity.

5. Applications to Stolarsky type means

In this section, we present several families of functions which fulfill the conditions of Theorem 4.14, Corollary 4.15, Corollary 4.16 and Remark 4.17. This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [10].

Example 5.1. Let

$$\Lambda_1 = \{\psi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

be a family of functions defined by

$$\psi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & \text{t} \neq 0; \\ \frac{x^n}{n!}, & \text{t} = 0. \end{cases}$$

We have $\frac{d^n \psi_t}{dx^n}(x) = e^{tx} > 0$ which shows that ψ_t is n-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^n \psi_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.14 we also have that $t \mapsto \psi_t[x_0, ..., x_n]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 4.15 we conclude that $t \mapsto \Theta_i(\psi_t)$, $i = 1, 2$ are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $t \mapsto \psi_t$ is not continuous for $t = 0$), so it is exponentially

convex.
For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_1)$, $i = 1, 2$ from (4.9), becomes

$$\mu_{t,q}(\Theta_i, \Lambda_1) = \begin{cases} \left(\frac{\Theta_i(\psi_t)}{\Theta_i(\psi_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(\frac{\Theta_i(id.\psi_t)}{\Theta_i(\psi_t)} - \frac{n}{t}\right), & t = q \neq 0; \\ exp\left(\frac{1}{n+1}\frac{\Theta_i(id.\psi_0)}{\Theta_i(\psi_0)}\right), & t = q = 0, \end{cases}$$

where id is the identity function. Now, using (4.8) it is monotone function in parameters t and q.

We observe here that $\left(\frac{\frac{d^n\psi_t}{dx^n}}{\frac{d^n\psi_q}{dx^n}}\right)^{\frac{1}{t-q}}(lnx) = x$ so using Theorem 4.3 it follows that

$$M_{t,q}(\Theta_i, \Lambda_1) = ln\mu_{t,q}(\Theta_i, \Lambda_1), \quad i = 1, 2$$

satisfies

$$a \leq M_{t,q}(\Theta_i, \Lambda_1) \leq b, \quad i = 1, 2.$$

This shows that $M_{t,q}(\Theta_i, \Lambda_1)$ is mean for $i = 1, 2$. Because of the above inequality (4.8), this mean is also monotonic.

Example 5.2. Let

$$\Lambda_2 = \{\lambda_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

be a family of functions defined by

$$\lambda_t(x) = \begin{cases} \frac{x^t}{t(t-1)...(t-n+1)}, & \text{t} \notin \{0,1,..., \text{n-1}\}; \\ \frac{x^j lnx}{(-1)^{n-1-j} j!(n-1-j)!}, & \text{t} = \text{j} \in \{0,1,..., \text{n-1}\}. \end{cases}$$

Here, $\frac{d^n\lambda_t}{dx^n}(x) = x^{t-n} = e^{(t-n)lnx} > 0$ which shows that λ_t is n-convex on $(0, \infty)$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^n\psi_t}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 5.1 we get the mappings $t \mapsto \Theta_i(\lambda_t)$, $i = 1, 2$ are exponentially convex. In this case we assume that $[a, b] \in \mathbb{R}^+$.

For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_1)$, $i = 1, 2$ from (4.9), becomes

$$\mu_{t,q}(\Theta_i, \Lambda_2) = \begin{cases} \left(\frac{\Theta_i(\lambda_t)}{\Theta_i(\lambda_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left((-1)^{n-1}(n-1)!\frac{\Theta_i(\lambda_0\lambda_t)}{\Theta_i(\lambda_t)} + \sum_{k=0}^{n-1}\frac{1}{k-t}\right), & \text{t=q} \notin \{0,1,..., \text{n-1}\}; \\ exp\left((-1)^{n-1}(n-1)!\frac{\Theta_i(\lambda_0\lambda_t)}{2\Theta_i(\lambda_t)} + \sum_{k=0,k\neq t}^{n-1}\frac{1}{k-t}\right), & \text{t=q} \in \{0,1,..., \text{n-1}\}. \end{cases}$$

We observe that $\left(\frac{\frac{d^n\lambda_t}{dx^n}}{\frac{d^n\lambda_q}{dx^n}}\right)^{\frac{1}{t-q}}(x) = x$, so if Θ_i $(i = 1, 2)$ are positive, then Theorem 4.3 yield that there exists some $\xi_i \in [a, b]$, $i = 1, 2$ such that

$$\xi_i^{t-q} = \frac{\Theta_i(\lambda_t)}{\Theta_i(\lambda_q)}, \quad i = 1, 2.$$

Since the function $\xi \to \xi^{t-q}$ is invertible for $t \neq q$, we then have

$$a \leq \left(\frac{\Theta_i(\lambda_t)}{\Theta_i(\lambda_q)}\right)^{\frac{1}{t-q}} \leq b, \quad i = 1, 2.$$

This shows that $\mu_{t,q}(\Theta_i, \Lambda_2)$ is mean for $i = 1, 2$. Because of the above inequality (4.8), this mean is also monotonic.

Example 5.3. Let

$$\Lambda_3 = \{\zeta_t : (0, \infty) \to (0, \infty) : t \in (0, \infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(lnt)^n}, & \text{t} \neq 1; \\ \frac{x^n}{n!}, & \text{t} = 1. \end{cases}$$

Since $\frac{d^n\zeta_t}{dx^n}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [22]) it is exponentially convex. Obviously ζ_t are n-convex functions for every $t > 0$.

For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_3)$, $i = 1, 2$, in this case for $[a, b] \in \mathbb{R}^+$, from (4.9) becomes

$$\mu_{t,q}(\Theta_i, \Lambda_3) = \begin{cases} \left(\frac{\Theta_i(\zeta_t)}{\Theta_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Theta_i(id.\zeta_t)}{t\Theta_i(\zeta_t)} - \frac{n}{t\,lnt}\right), & t = q \neq 1; \\ exp\left(-\frac{1}{n+1}\frac{\Theta_i(id.\zeta_1)}{\Theta_i(\zeta_1)}\right), & t = q = 1. \end{cases}$$

This is monotonous function in parameters t and q by (4.8). Using Theorem 4.3 it follows that

$$M_{t,q}(\Theta_i, \Lambda_3) = -L(t,q) ln\mu_{t,q}(\Theta_i, \Lambda_3), \quad i = 1, 2.$$

satisfy

$$a \leq M_{t,q}(\Theta_i, \Lambda_3) \leq b, \quad i = 1, 2.$$

This shows that $M_{t,q}(\Theta_i, \Lambda_3)$ is mean for $i = 1, 2$. Because of the above inequality (4.8), this mean is also monotonic. $L(t,q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t - \log q}, & \text{t} \neq \text{q}; \\ t, & \text{t} = \text{q}. \end{cases}$$

Example 5.4. Let

$$\Lambda_4 = \{\gamma_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{t^n}.$$

Since $\frac{d^n\gamma_t}{dx^n}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [22]) it is exponentially convex. Obviously γ_t are n-convex function for every $t > 0$.
For this family of functions, $\mu_{t,q}(\Theta_i, \Lambda_4)$, $i = 1, 2$, in this case for $[a,b] \in \mathbb{R}^+$, from (4.9) becomes

$$\mu_{t,q}(\Theta_i, \Lambda_4) = \begin{cases} \left(\frac{\Theta_i(\gamma_t)}{\Theta_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ exp\left(-\frac{\Theta_i(id.\gamma_t)}{2\sqrt{t}\Theta_i(\gamma_t)} - \frac{n}{t}\right), & t = q. \end{cases}$$

This is monotonous function in parameters t and q by (4.8). Using Theorem 4.3 it follows that

$$M_{t,q}\left(\Theta_i, \Lambda_4\right) = -\left(\sqrt{t} + \sqrt{q}\right) ln\mu_{t,q}\left(\Theta_i, \Lambda_4\right), \quad i = 1, 2$$

satisfy

$$a \leq M_{t,q}\left(\Theta_i, \Lambda_4\right) \leq b, \quad i = 1, 2.$$

This shows that $M_{t,q}\left(\Theta_i, \Lambda_4\right)$ is mean for $i = 1, 2$. Because of the above inequality (4.8), this mean is also monotonic.

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