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OSCILLATION OF SOLUTIONS OF SECOND ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS

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Abstract. We study the oscillatory behavior of solutions of second order neutral type difference equation. We obtain conditions which ensure that all solutions are oscillatory. Examples are provided to illustrate the results.

1. Introduction

In this article, we study the oscillatory behavior of solutions to the second order nonlinear neutral type difference equation of the form

$$\Delta\left(a_n\psi(x_n)\left(\Delta z_n\right)^{\alpha}\right)+q_nf\left(x_{n-\sigma+1}\right)=0, \tag{1.1}$$

where $n \in N = \{n_0, n_{0+1}, ...\}$, n_0 a nonnegative integer, $z_n = x_n + p_n x_{n-\tau}$ and $\alpha \geq 1$ is a ratio of odd positive integers. Throughout, we assume the following conditions without further mention:

(C1) $\{a_n\}$, $\{p_n\}$, $\{q_n\}$ are real sequences such that $a_n > 0$, $0 \leq p_n < 1$, $q_n \geq 0$ and q_n is not identically zero for infinitely many values of n;

(C2) ψ and f are real-valued continuous function with $\psi > 0$, $uf(u) > 0$ for all $u \neq 0$, and there exist two positive constants K and L such that

$$\frac{f(u)}{u^{\alpha}} \geq K \quad \text{and} \quad \phi(u) \leq L^{-1} \quad \text{for all } u \neq 0;$$

(C3) τ and σ are nonnegative integers;

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(C4) $A_n = \sum\limits_{s=n}^{\infty} \frac{1}{a_s^{1/\alpha}}$ and $A_{n_0} < \infty$.

By a solution of equation (1.1), we mean a real-valued sequence $\{x_n\}$ defined for all $n \geq N_1 \in N$ and satisfies equation (1.1)for all $n \geq N_1$. We consider any solutions satisfying condition $\sup\{|x_n| : n \geq N \geq N_1\} > 0$ and tacitly assume that equation (1.1) possesses such solutions. As usual, a solution of equation (1.1) is called oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

It is well known that various types of neutral difference equations are often appeared in applied problems in science and engineering, see,for example [1]. Recently, there is a great deal of attention in oscillatory properties of neutral type difference equations, see for example [2]-[10].

Next, we briefly review the following related results that motivated our study. Wang and Xu [9] obtained several oscillation criteria for equation (1.1) when $\psi(u) \equiv 1$, one of which we presented below. For the convenience of the reader, in what follows, we use the notations

$$b = k\epsilon^{\alpha}(1-p)^{\alpha}, \quad P_n = \sum_{s=n}^{\infty} \frac{1}{a_s^{1/\alpha}}.$$

Theorem 1.1. ([9], Theorem 2.1) *Assume that $\sigma > \tau$. If $\{p_n\}$ is nondecreasing,*

$$\sum_{n=n_0}^{\infty} q_n = \infty,$$

and

$$\sum_{s=n}^{n-\tau+\sigma} q_s p_s > \frac{1}{b}$$

then every solution equation (1.1) *is oscillatory.*

Note that Theorem 1.1 is not valid for the difference equation

$$\Delta\left(3^n\left(x_n + \frac{1}{2^n}x_{n-2}\right)\right) + 3^n x_{n-1} = 0,$$

since $p_n = 1/2^n$ is decreasing and $\tau > \sigma$. Therefore the main purpose of this paper is to derive new oscillation criteria for equation (1.1) without requiring the condition $\{p_n\}$ is nondecreasing and $\sigma > \tau$. In Section 2, we present new oscillation results for equation (1.1), and in Section 3, we provide two examples to illustrate the main results.

2. Oscillation results

In this section we present some sufficient conditions which ensure that all solutions of equation (1.1) are oscillatory.

Theorem 2.1. *Assume that there is a constant M such that $\psi(x) \geq M > 0$. Suppose that there exist two positive real sequences $\{\rho_n\}$ and $\{m_n\}$ such that $\{\rho_n\}$ is nondecreasing and*

$$\frac{m_n}{(LM)^{1/\alpha} a_n^{1/\alpha} A_n} + \Delta m_n \leq 0, \quad 1 - p_n \frac{m_{n-\tau}}{m_n} > 0, \tag{2.1}$$

$$\sum_{n=n_0}^{\infty} \left[p_n Q_N - \frac{1}{LK(\alpha+1)^{\alpha+1}} \frac{(\Delta \rho_n)^{\alpha+1} a_{n-\sigma}}{p_n^{\alpha+1}} \right] = \infty, \tag{2.2}$$

$$\sum_{n=n_0}^{\infty} \left[K q_n A_{n+1}^{\alpha} \left(1 - p_{n-\sigma+1} \frac{m_{n-\sigma-\tau+1}}{m_{n-\sigma+1}} \right)^{\alpha} - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{A_n^{\alpha^2-1}}{L A_{n+1}^{\alpha^2} a_n^{1/\alpha}} \right] = \infty. \tag{2.3}$$

Then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). The proofs for eventually positive and for eventually negative solutions are similar. If $\{y_n\}$ is a negative solution, then $x_n = -y_n$ may not be a solution of equation (1.1), but x_n satisfies key estimates such as (2.1) with $\psi(-x_n)$ existead of $\psi(x_n)$. Then we can use that $\psi(x_n)$ and $\psi(-x_n)$ have same bounds, $M \leq \psi(.) \leq 1/L$.

We assume that there exists an integer $n_1 \in N$ such may $x_n > 0$, $x_{\sigma(n)} > 0$, and $x_{\tau(n)} > 0$ for all $n \geq n_1$. Then $z_n > 0$. From equation (1.1) it follows that for all $n \geq n_1$,

$$\Delta \left(a_n \psi(x_n) (\Delta z_n)^{\alpha} \right) \leq -K q_n x_{n-\sigma+1}^{\alpha} \leq 0. \tag{2.4}$$

Hence, there is an integer $n_2 \geq n_1$ such that either $\Delta z_n > 0$ or $\Delta z_n < 0$ for all $n \geq n_2$. We consider these two cases separately.

Case 1. Let $\Delta z_n > 0$ for all $n \geq n_2$. As in the proof of [9], we obtain a contradiction to (2.2).

Case 2. Let $\Delta z_n < 0$ for all $n \geq n_2$. For $n \geq n_2$, we define

$$w_n = \frac{a_n \psi(x_n) (\Delta z_n)^{\alpha}}{z_n^{\alpha}}. \tag{2.5}$$

Then $w_n < 0$ for all $n \geq n_2$. Since $\Delta \left(a_n \psi(x_n) (\Delta (z_n)^{\alpha}) \right) \leq 0$, $a_n \psi(x_n) (\Delta z_n)^{\alpha}$ is nonincreasing. Thus, for all $s \geq n \geq n_2$

$$(a_s \psi(x_s))^{1/\alpha} \Delta z_s \leq (a_n \psi(x_n))^{1/\alpha} \Delta z_n.$$

Dividing the last inequality by $(a_s\psi(x_s))^{1/\alpha}$ and summing the resulting inequality from n to l, for all $l \ge n \ge n_2$, we have

$$z_l \le z_n + (a_n\psi(x_n))^{1/\alpha}\,\Delta z_n \sum_{s=n}^{l} \frac{1}{(a_s\psi(x_s))^{1/\alpha}}.$$

Since $\Delta z_n < 0$ and $\psi \le 1/L$, we conclude that, for all $l \ge n \ge n_2$,

$$z_l \le z_n + (La_n\psi(x_n))^{1/\alpha}\,\Delta z_n \sum_{s=n}^{l} \frac{1}{a_s^{1/\alpha}}.$$

Letting $l \to \infty$, and using $z_n > 0$, we see that for all $n \ge n_2$,

$$0 \le z_n + (La_n\psi(x_n))^{1/\alpha}\,\Delta z_n A_n,$$

that is, for all $n \ge n_2$,

$$(a_n\psi(x_n))^{1/\alpha} A_n \frac{\Delta z_n}{z_n} \ge -\frac{1}{L^{1/\alpha}}. \tag{2.6}$$

Hence, by (2.5), we conclude that, for all $n \ge n_2$,

$$-\frac{1}{L} \le w_n A_n^{\alpha} \le 0. \tag{2.7}$$

From (2.6) and $M \le \psi$, we obtain

$$\frac{\Delta z_n}{z_n} \ge -\frac{1}{L^{1/\alpha}(a_n\psi(x_n))^{1/\alpha} A_n} \ge -\frac{1}{(LM)^{1/\alpha}a_n^{1/\alpha}A_n}.$$

Thus, we have

$$\begin{aligned}\Delta\left(\frac{z_n}{m_n}\right) &= \frac{m_n\Delta z_n - z_n\Delta m_n}{m_n m_{n+1}} \\ &\ge -\frac{z_n}{m_n m_{n+1}}\left[\frac{m_n}{(LM)^{1/\alpha}a_n^{1/\alpha}A_n} + \Delta m_n\right] \ge 0.\end{aligned}$$

Hence, the sequence $\{z_n/m_n\}$ is nondecreasing, and so

$$x_n \ge z_n - p_n\frac{m_{n-\tau}}{m_n}z_n = \left(1 - p_n\frac{m_{n-\tau}}{m_n}\right)z_n.$$

From (2.5), we have

$$\begin{aligned}\Delta w_n &= \frac{\Delta\left(a_n\psi(x_n)(\Delta z_n)^{\alpha}\right)}{z_n^{\alpha}} - \frac{a_{n+1}\psi(x_{n+1})(\Delta z_{n+1})^{\alpha}\Delta z_n^{\alpha}}{z_{n+1}^{\alpha}z_n^{\alpha}} \\ &\le -Kq_n\left(1 - p_{n-\sigma+1}\frac{m_{n-\sigma-\tau+1}}{m_{n-\sigma+1}}\right)\frac{z_{n-\sigma+1}^{\alpha}}{z_n^{\alpha}} - w_n\frac{\Delta z_n^{\alpha}}{z_{n+1}^{\alpha}}.\end{aligned} \tag{2.8}$$

By Mean value theorem, we have

$$\Delta z_n^\alpha = \alpha t^{\alpha-1}\Delta z_n,$$

where $z_{n+1} < t < z_n$. Thus,

$$\Delta z_n^\alpha \le \alpha z_{n+1}^{\alpha-1}\Delta z_n.$$

From (2.8), and the last inequality we obtain

$$\Delta w_n \le -Kq_n\left(1 - p_{n-\sigma+1}\frac{m_{n-\sigma-\tau+1}}{m_{n-\sigma+1}}\right) - \alpha w_n\frac{\Delta z_n}{z_n}, \tag{2.9}$$

where we have used $z_n > 0$ and nonincreasing. From (2.9) and (2.5) we have

$$\Delta w_n + Kq_n\left(1 - p_{n-\sigma+1}\frac{m_{n-\sigma-\tau+1}}{m_{n-\sigma+1}}\right) + \frac{\alpha L^{1/\alpha}}{a_n^{1/\alpha}}(-w_n)^{\frac{\alpha+1}{\alpha}}\frac{\Delta z_n}{z_n} \le 0. \tag{2.10}$$

Multiplying (2.10) by A_α^{n+1} and summing the resulting inequality from n_3 to $n-1$, we deduce that

$$\begin{aligned}
&A_n^\alpha w_n - A_{n_3}^\alpha w_{n_3} + \sum_{s=n_3}^{n-1}\alpha w_s\frac{A_s^{\alpha-1}}{a_s^{1/\alpha}}\\
&+K\sum_{s=n_3}^{n-1}A_{s+1}^\alpha q_s\left(1 - p_{n-\sigma+1}\frac{m_{n-\sigma-\tau+1}}{m_{n-\sigma+1}}\right)^\alpha\\
&+\alpha L^{1/\alpha}\sum_{s=n_3}^{n-1}\frac{A_{s+1}^\alpha}{a_s^{1/\alpha}}(-w_s)^{\frac{\alpha+1}{\alpha}} \le 0.
\end{aligned} \tag{2.11}$$

Let

$$\begin{aligned}
p &= \frac{\alpha+1}{\alpha}, \qquad q = \alpha+1,\\
a &= L^{1/\alpha+1}\,(\alpha+1)^{\alpha/\alpha+1}\,A_{n+1}^{\alpha^2/\alpha+1},\\
b &= L^{-1/\alpha+1}\,\frac{\alpha}{(\alpha+1)^{\alpha/\alpha+1}}\,\frac{A_n^{\alpha-1}}{A_{n+}^{\alpha^2/\alpha+1}}.
\end{aligned}$$

Using Young's inequality

$$|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \quad \text{where } a, b \in R,\ p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$-\alpha\, A_n^{\alpha-1}\, w_n \le \alpha\, L^{1/\alpha}\, A_{n+1}^\alpha\,(-w_n)^{\alpha+1/\alpha} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\,\frac{A_n^{\alpha^2-1}}{LA_{n+1}^{\alpha^2}}$$

and hence

$$\frac{-\alpha A_n^{\alpha-1}w_n}{a_n^{1/\alpha}} \le \frac{\alpha L^{1/\alpha}A_{n+1}^{\alpha}(-w_n)^{\alpha+1/\alpha}}{a_n^{1/\alpha}} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{A_n^{\alpha^2-1}}{LA_{n+1}^{\alpha^2}a_n^{1/\alpha}}.$$

Therefore, it follows from (2.7) and (2.11) that

$$\sum_{s=n_3}^{n-1}\left[Kq_sA_{s+1}^{\alpha}\left(1-p_{n-\sigma+1}\frac{m_{n-\sigma-\tau+1}}{m_{n-\sigma+1}}\right)^{\alpha} - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{A_s^{\alpha^2-1}}{LA_{s+1}^{\alpha^2}a_s^{1/\alpha}}\right]$$
$$\le A_{n_3}^{\alpha}w_{n_3}^{\alpha} - A_n^{\alpha}w_n \le \frac{1}{L} + A_{n_3}^{\alpha}w_{n_3}^{\alpha},$$

which contradicts (2.3). The proof is now completed. □

Remark 2.2. The sequence $\{m_n\}$ in Theorem 2.1 can be obtained by setting $m_n = A_n$ in the case $LM \ge 1$.

If the restriction $\psi(x) \ge M > 0$ is not satisfied then Theorem 2.1 cannot be applicable. For example when

$$\psi(u) = \frac{1}{u^2+1}$$

the following result proves to be useful.

Theorem 2.3. *Assume that condition* (1.1) *holds. Let* $\psi(x)$ *be nonincreasing for all* $x > 0$, *and nondecreasing for all* $x < 0$. *Suppose there exist two positive real sequences* $\{\rho_n\}$ *and* $\{m_n\}$ *such that, for any fixed constant* $l > 0$,

$$\frac{m_n}{(L\psi(l))^{1/\alpha}\ a_n^{1/\alpha}\ A_n} + \Delta m_n \le 0, \quad 1 - p_n\frac{m_{n-\tau}}{m_n} > 0 \tag{2.12}$$

and such that conditions (2.2) *and* (2.3) *are satisfied. Then every solution of equation* (1.1) *is oscillatory.*

Proof. As in the proof of Theorem 2.1, we only need to prove the case where $\Delta z_n < 0$. In this case, there exists a constant $l > 0$ such that $0 < x_n \le z_n \le l$. Using the monotonicity of ψ, we deduce that $\psi(x) \ge \psi(l)$. Along the same lines as in Theorem 2.1, we conclude that

$$\frac{\Delta z_n}{z_n} \ge \frac{1}{L^{1/\alpha}\ (a_n\psi(x_n))^{1/\alpha}\ A_n} \ge -\frac{1}{(L\psi(l))^{1/\alpha}\ a_n^{1/\alpha}\ A_n}.$$

Hence, we have

$$\Delta\left(\frac{z_n}{m_n}\right) = \frac{m_n\Delta z_n - z_n\Delta m_n}{m_nm_{n+1}}$$
$$\ge -\frac{z_n}{m_nm_{n+1}}\left[\frac{m_n}{(L\psi(l))^{1/\alpha}a_n^{1/\alpha}A_n} + \Delta m_n\right] \ge 0.$$

Thus, the sequence $\{z_n/m_n\}$ is nondecreasig. The remaining part is similar to that of Theorem 2.1 and hence is omitted. $\square$

3. Examples

In this section we present two examples to illustrate the theoretical results obtained in the previous section.

Example 3.1. Consider the second order neutral delay difference equation

$$\Delta\left(n(n+1)\frac{x_n^2+2}{x_n^2+1}\Delta\left(x_n+\frac{1}{4}x_{n-2}\right)\right)+\frac{15}{2}(n-1)^2x_{n-4}=0, \quad n\geq 4. \tag{3.1}$$

Here, $a_n = n(n+1)$, $\psi(x) = \frac{x^2+2}{x^2+1}$, $p_n = 1/4$, $f(x) = x$ and $q_n = \frac{15}{2}(n+1)^2$. Then $1 \leq \psi(x) \leq 2$, and we can fix $M = 1$, $K = 1$ and $L = 1/2$. Let $m_n = \frac{1}{n(n+2)}$ and $\rho_n = 1$. Since all conditions of Theorem 2.1 are satisfied and therefore every solution of equation (3.1) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (3.1).

Example 3.2. Consider the second order neutral difference equation of the form

$$\Delta\left(\frac{n(n+1)}{x_n^2+1}\Delta\left(x_n+\frac{1}{n}x_{n-1}\right)\right)+nx_{n-1}=0, \quad n\geq 2. \tag{3.2}$$

Here, $a_n = n(n+1)$, $\psi(x) = \frac{1}{x^2+1}$, $p_n = 1/n$, $f(x) = x$, $\sigma(n) = \tau(n) = n-1$ and $q_n = n$. Then $\psi(x) \leq 1$, and we can fix $K = 1$ and $L = 1$. Let $m_n = n^{-1-l^2}$ and $\rho_n = 1$. It is easy to verify that all conditions of Theorem 2.3 are satisfied and therefore every solution of equation (3.2) is oscillatory.

We conclude this paper with the following remark.

Remark 3.3. In this paper, using Riccati transformation and Young's inequality, we have established new oscillation criteria for the neutral difference equation (1.1) assuming condition (2.1) is satisfied. Note that in the study of oscillation of solution of equation (1.1), the case $\Delta z_n < 0$ brings additional difficulties. One of the important difficulties one encounters lies in the fact that if $\{x_n\}$ is an eventually positive solution of equation (1.1), then the inequality

$$x_n \geq (1-p_n)z_n$$

does not hold when $\Delta z_n < 0$ is satisfied. However in this paper, we obtain similar inequality by using condition (2.1) or (2.12). Thus, we have presented new criteria for the oscillation of all solutions of equation (1.1). It would be interesting to study the oscillatory properties of equation (1.1) without using the condition (2.1) or (2.12), and it remains an open problem at the moment.

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