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GENERAL FRAMEWORK FOR A SUPER-RELAXED PROXIMAL POINT ALGORITHM AND ITS APPLICATIONS TO BANACH SPACES

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Abstract. Based on the notion of $A-$maximal relaxed accretiveness, first a general framework for a super-relaxed proximal point algorithm is introduced, and then the convergence analysis for the algorithm to the context of approximating solutions to a class of nonlinear inclusion problems is examined along with some auxiliary results on the generalized resolvent operator corresponding to $A-$maximal relaxed accretiveness. The $A-$maximal relaxed accretiveness seems to be applicable generalizing results on the theory of hemivariational inequalities that is a direct generalization to variational inequalities. As a matter of fact, hemivariational inequalities arise from mechanics, engineering sciences, economics relating to nonconvex energy functionals or equivalently relating to nonmonotone possibly multivalued laws, for instance between stresses and strains or reactions and displacements in deformable bodies between heat flux and temperature in thermal problems or between differential and flow intensities in economic network problems.

1. Introduction

Let X be a real Banach space with the norm $\|\cdot\|$ on X and X^*, the dual of X, and the duality pairing $\langle\cdot,\cdot\rangle$ between the elements of X and X^*. We consider the nonlinear inclusion problem: determine a solution to

$$0 \in M(x), \tag{1.1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

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Inspired by the investigations of Xu [18], Agarwal and Verma [2], and Rockafellar [6] on the proximal point algorithm, and of Agarwal and Verma [1] on the super-relaxed proximal point algorithm applied in showing that the sequence for the solutions converges linearly to a solution of (1.1), we introduce a hybrid version of the proximal point algorithm based on the notion of $A-$maximal relaxed accretiveness [4] for solving general inclusion problems. If we look back the scenario when Rockafellar [6] was dealing with the nonexpansiveness of the classical resolvent to the context of achieving linear convergence of the proximal point algorithm, then Rockafellar ([6], Theorem 2) resolved the problem by considering the Lipschitz continuity of the mapping M^{-1} instead with a Lipschitz continuity constant less than one, while linear convergence was not general in nature. Although in the present case, we do have a generalized resolvent that is Lipschitz continuous, but the Lipschitz continuity constant is the quotient of the Lipschitz continuity constant and strong accretiveness constant of a single-valued mapping and this makes the endeavor of achieving a linear convergence just as hard as in the case of the classical resolvent. As a matter of fact, we have skipped the Lipschitz continuity and introduced some new notion instead within the framework of the $A-$maximal $(m)-$relaxed monotonicity, and the selective use of the strong accretiveness of single-valued map A only to showing the generalized resolvent is single-valued, while the strong accretiveness is used in achieving the linear convergence within the proof environment. In a way, the proof technique seems to be unique other than the usual resolvent methods exist in the literature.
Indeed, the notion of the $A-$maximal $(m)-$relaxed monotone mapping generalizes the general class of maximal monotone set-valued mappings, including the $H-$maximal monotone mappings. Recently, Lan, Cho and Verma [4] generalized the notion of $A-$maximal relaxed monotonicity introduced and studied by the Verma [9–11] and $H-$maximal accretiveness introduced by Fang and Huang [3], while approximating the solutions of inclusion problems of the form (1.1). As a result, it unifies a more general class of problems of variational character, including minimization or maximization of functions, variational problems, and minimax problems into the form (1.1). Verma [9–11] examined the notion of $A-$maximal relaxed monotonicity to the context of approximating the solution of an inclusion problem based on the generalized resolvent operator technique. We note that the generalized resolvent operator techniques can also be applied to other problems from a wide spectrum of different fields, such as equilibria problems in economics, global optimization and control theory, operations research, management and decision sciences, and mathematical programming. Furthermore, as it seems most of the investigations on hemivariational inequalities [14] are limited to the classical maximal monotonicity, can be generalized based on the $A-$maximal $(m)-$relaxed

monotonicity as well as $A-$maximal $(m)-$relaxed accretivity to the context of the mathematical theory of hemivariational inequalities and applications. For more literature on the resolvent operator techniques and related materials, we refer the reader [1–22].

2. Preliminaries and $A-$Maximal Relaxed Accretiveness

In this section we state some auxiliary results based on basic properties of $A-$maximal relaxed accretiveness [4] and its variant forms. Let X be a real Banach space, and let $\|\cdot\|$ denote the norm on X and X^*, the dual of X. Let $\langle .,. \rangle$ denote the duality pairing between the elements of X and X^*. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, that is, the set $\{(x, y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x, y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x, y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$D(M) = \{x \in X : \exists\, y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

D(M)=X, shall denote the full domain of M, and the range of M is defined by

$$R(M) = \{y \in X : \exists\, x \in X : (x, y) \in M\}.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{x, \rho y) : (x, y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M\}.$$

Now we define the generalized duality mapping $J_q : X \to 2^{X^*}$ by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\| \cdot \|f^*\|, \|f^*\| = \|x\|^{q-1}\}\, \forall\, x \in X.$$

As special cases, for $q = 2$, J_q reduces to the normalized duality mapping. Also, we noticed that $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$, and J_q is single-valued if X^* is strictly convex. The modulus of smoothness is characterized as

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

Based on the modulus of smoothness, a Banach space is uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0,$$

and X is $q-$uniformly smooth if there is a positive constant c such that

$$\rho_X(t) \leq ct^q \quad \text{whenever} \quad q > 1.$$

Next we mention the lemma [17] on $q-$uniformly smooth Banach spaces.

Lemma 2.1. *Let X be a real uniformly smooth Banach space. Then X is $q-$uniformly smooth if and only if there exists a positive constant c_q such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 2.2. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) $(r)-$ strongly accretive if there exists a positive constant r such that

$$\langle u^* - v^*, J_q(u-v)\rangle \geq r\|u-v\|^q \,\forall\, (u,u^*),(v,v^*) \in M.$$

(ii) $(m)-$relaxed accretive if there exists a positive constant m such that

$$\langle u^* - v^*, J_q(u-v)\rangle \geq (-m)\|u-v\|^q \,\forall\, (u,u^*),(v,v^*) \in M.$$

(iii) $(c)-$cocoercive if there exists a positive constant c such that

$$\langle u^* - v^*, J_q(u-v)\rangle \geq c\|u^*-v^*\|^q \,\forall\, (u,u^*),(v,v^*) \in M.$$

Definition 2.3. ([4]) Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be $A-$maximal $(m)-$relaxed accretive if

(i) M is $(m)-$ relaxed accretive.
(ii) $R(A+\rho M) = X$ whenever $\rho > 0$.

Proposition 2.4. *Let $A : X \to X$ be an $(r)-$strongly accretive mapping, and let $M : X \to 2^X$ be an $A-$maximal $(m)-$relaxed accretive mapping. Then the operator $(A+\rho M)^{-1}$ is single-valued for $r - \rho m > 0$.*

Proof. For some $z \in X$, assume $x, y \in (A+\rho M)^{-1}(z)$. Then we have

$$-A(x) + z \in \rho M(x) \text{ and } -A(y) + z \in \rho M(y).$$

Since M is $A-$maximal $(m)-$relaxed accretive and A is $(r)-$strongly accretive, it follows that

$$\begin{aligned}
&-\rho m\|x-y\|^q \leq -\langle A(x) - A(y), J_q(x-y)\rangle \leq -r\|x-y\|^q \\
\Longrightarrow\quad & (r-\rho m)\|x-y\|^q \leq 0 \\
\Longrightarrow\quad & x = y \quad \text{whenever} \quad r - \rho m > 0.
\end{aligned}$$

□

Definition 2.5. Let $A : X \to X$ be an $(r)-$strongly accretive mapping and let $M : X \to 2^X$ be an $A-$maximal $(m)-$relaxed accretive mapping. Then the generalized resolvent operator $J^M_{\rho,A} : X \to X$ is defined by

$$J^M_{\rho,A}(u) = (A+\rho M)^{-1}(u),$$

where $r - \rho\, m > 0$.

Definition 2.6. ([3]) Let $H : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be $H-$maximal accretive if

(i) M is accretive.
(ii) $R(H + \rho M) = X$ whenever $\rho > 0$.

Proposition 2.7. *Let $H : X \to X$ be an $(r)-$strongly accretive mapping, and let $M : X \to 2^X$ be an $H-$ maximal accretive mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued for $r > 0$.*

Proof. For some $z \in X$, assume $x, y \in (H + \rho M)^{-1}(z)$. Then we have

$$-H(x) + z \in \rho M(x) \ and \ -H(y) + z \in \rho M(y).$$

Since M is $H-$maximal accretive and H is $(r)-$strongly accretive, it follows that

$$\begin{aligned} & 0 \le -\langle H(x) - H(y), J_q(x - y)\rangle \le -r\|x - y\|^q \\ \Longrightarrow\ & r\|x - y\|^q \le 0 \\ \Longrightarrow\ & x = y \quad \text{whenever} \quad r > 0. \end{aligned}$$

□

Definition 2.8. Let $H : X \to X$ be an $(r)-$strongly accretive mapping and let $M : X \to 2^X$ be an $H-$maximal accretive mapping. Then the generalized resolvent operator $J^M_{\rho,H} : X \to X$ is defined by

$$J^M_{\rho,H}(u) = (H + \rho M)^{-1}(u),$$

where $r > 0$.

3. Super Relaxed Proximal Point Algorithm

This section deals with a hybrid version of the proximal point algorithm, which generalizes the proximal point algorithm studied in [2], and its application to approximation solvability of the inclusion problem (1.1) based on the A-maximal relaxed accretiveness. Although, the generalized resolvent corresponding to M within the framework $A-$maximal $(m)-$relaxed accretiveness is Lipschitz continuous, it seems quite unlikely to achieve a linear convergence with that setting. That is why, we consider the $(\gamma)-$cocoerciveness of $A\text{o}J^M_{\rho_k,A}$ and skip completely the Lipschitz continuity of A with a limited use of the strong accretiveness of A, especially during the final phases of the proof. Note that $(m)-$ relaxed accretiveness of M and $(r)-$strong accretiveness of A are significantly crucial to showing that the generalized resolvent is single-valued. However, we observe that the present framework for linear convergence breaks down when $A = I$ (identity mapping), so in a way the introduction of the map

A is vitally important to avoid the repeat of similar frameworks as that of the classical resolvents.

Theorem 3.1. *Let X be a real Banach space, let $A : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed accretive. Then the following statements are mutually equivalent:*

(i) *An element $u \in X$ is a solution to* (1.1).

(ii) *For an $u \in X$, we have*

$$u = J^M_{\rho_k,A}(A(u)) \;\forall\, k \geq 0,$$

where

$$J^M_{\rho_k,A}(u) = (A + \rho_k M)^{-1}(u),$$

$r - \rho_k\, m > 0$, m is a positive constant, and $\{\rho_k\}$ is a sequence of all positive real numbers.

Proof. Although the proof is straightforward, we include it for the sake of the completeness. To show, (i) $\Rightarrow$ (ii), we start with $(\rho_k > 0)$

$$\begin{aligned} & 0 \in \rho_k M(u) \\ \Rightarrow\;\; & A(u) \in (A + \rho_k M)(u) \\ \Rightarrow\;\; & J^M_{\rho_k,A}(A(u)) = u. \end{aligned}$$

Next, to show, (ii)$\Rightarrow$ (i), we have

$$\begin{aligned} & J^M_{\rho_k,A}(A(u)) = u \\ \Rightarrow\;\; & A(u) \in (A + \rho_k M)(u) \\ \Rightarrow\;\; & 0 \in M(u). \end{aligned}$$

□

Theorem 3.2. ([3]) *Let X be a real Banach space, let $H : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $H-$maximal accretive. Then the following statements are mutually equivalent:*

(i) *An element $u \in X$ is a solution to* (1.1).

(ii) *For an $u \in X$, we have*

$$u = J^M_{\rho_k,H}(H(u)),$$

where

$$J^M_{\rho_k,H}(u) = (H + \rho_k M)^{-1}(u) \text{ whenever } r > 0.$$

Next, we apply the hybrid proximal point algorithm to approximate the solution of (1.1), which results showing in linear convergence.

Theorem 3.3. *Let X be a real $q-$uniformly smooth Banach space, let $A : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed accretive. Furthermore, we suppose that $AoJ^M_{\rho_k,A}$ is $(\gamma)-$cocoercive for a positive constant γ. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by an iterative procedure*

$$A(x^{k+1}) = (1-\alpha_k)A(x^k) + \alpha_k y^k \,\forall\, k \geq 0, \tag{3.1}$$

and y^k satisfies

$$\|y^k - A(J^M_{\rho_k,A}(A(x^k)))\| \leq \delta_k \|y^k - A(x^k)\|,$$

where $\delta_k \to 0$, $J^M_{\rho_k,A} = (A+\rho_k M)^{-1}$, $r - \rho_k m > 0$, and

$$\{\alpha_k\},\ \{\delta_k\},\{\rho_k\} \subseteq [0,\infty)$$

are scalar sequences such that

$$\alpha = \limsup_{k\to\infty} \alpha_k, \quad \rho_k \uparrow \rho \leq \infty, \quad \alpha_k \geq 1, \quad \text{and} \quad \sum_{k=1}^{\infty} \delta_k < \infty.$$

Then the sequence $\{x^k\}$ converges linearly to a solution of (1.1) with convergence rate

$$\sqrt[q]{(1-\alpha_k)^q + \Big(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\Big)\frac{1}{\gamma^{\frac{q}{q-1}}}} < 1, \tag{3.2}$$

for $\big(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\big)\frac{1}{\gamma^{\frac{q}{q-1}}} > 0$, where $c_q > 0$ and $\gamma > 1$.

Proof. Suppose that x^* is a zero of M. Then from Theorem 3.1, it follows that any solution to (1.1) is a fixed point of $J^M_{\rho_k,A}oA$. For all $k \geq 0$, we express

$$A(z^{k+1}) = (1-\alpha_k)A(x^k) + \alpha_k A(J^M_{\rho_k,A}(A(x^k))).$$

Next, since $AoJ^M_{\rho_k,A}$ is $(\gamma)-$cocoercive, we have

$$\begin{aligned} & \langle A(J^M_{\rho,A}(A(x^k))) - A(J^M_{\rho,A}(A(x^*))), J_q(A(x^k) - A(x^*))\rangle \\ \geq\ & \gamma\|A(J^M_{\rho,A}(A(x^k))) - A(J^M_{\rho,A}(A(x^*)))\|^q. \end{aligned}$$

It follows that

$$\|A(J^M_{\rho,A}(A(x^k))) - A(J^M_{\rho,A}(A(x^*)))\| \leq \frac{1}{\gamma^{\frac{1}{q-1}}}\|A(x^k) - A(x^*)\|. \tag{3.3}$$

Now we find the estimate on applying (3.3) that

$$\begin{aligned}
&\|A(z^{k+1})-A(x^*)\|^q\\
&=\|(1-\alpha_k)A(x^k)+\alpha_k A(J^M_{\rho_k,A}(A(x^k)))\\
&\qquad-[(1-\alpha_k)A(x^*)+\alpha_k A(J^M_{\rho_k,A}(A(x^*)))]\|^q\\
&=\|(1-\alpha_k)(A(x^k)-A(x^*))\\
&\qquad+\alpha_k\Big(A(J^M_{\rho_k,A}(A(x^k)))-A(J^M_{\rho_k,A}(A(x^*)))\Big)\|^q\\
&=(1-\alpha_k)^q\|A(x^k)-A(x^*)\|^q\\
&\qquad+q\alpha_k(1-\alpha_k)\\
&\qquad\cdot\langle A(J^M_{\rho_k,A}(A(x^k)))-A(J^M_{\rho_k,A}(A(x^*))),J_q(A(x^k)-A(x^*))\rangle\\
&\qquad+c_q\alpha_k^q\|A(J^M_{\rho_k,A}(A(x^k)))-A(J^M_{\rho_k,A}(A(x^*)))\|^q\\
&\le(1-\alpha_k)^q\|A(x^k)-A(x^*)\|^q\\
&\qquad+q\alpha_k(1-\alpha_k)\gamma\|A(J^M_{\rho_k,A}(A(x^k)))-A(J^M_{\rho_k,A}(A(x^*)))\|^q\\
&\qquad+c_q\alpha_k^q\|A(J^M_{\rho_k,A}(A(x^k)))-A(J^M_{\rho_k,A}(A(x^*)))\|^q\\
&=(1-\alpha_k)^q\|A(x^k)-A(x^*)\|^q\\
&\qquad+[c_q\alpha_k^q+q\alpha_k(1-\alpha_k)\gamma]\|A(J^M_{\rho_k,A}(A(x^k)))-A(J^M_{\rho_k,A}(A(x^*)))\|^q\\
&\le(1-\alpha_k)^q\|A(x^k)-A(x^*)\|^q\\
&\qquad+[c_q\alpha_k^q+q\alpha_k(1-\alpha_k)\gamma]\frac{1}{\gamma^{\frac{q}{q-1}}}\|A(x^k-A(x^*)\|^q\\
&=\left[(1-\alpha_k)^q+\Big(c_q\alpha_k^q+q\alpha_k(1-\alpha_k)\gamma\Big)\frac{1}{\gamma^{\frac{q}{q-1}}}\right]\|A(x^k-A(x^*)\|^q,
\end{aligned}$$

where $\left[c_q\alpha_k^q+q\alpha_k(1-\alpha_k)\gamma\right]\frac{1}{\gamma^{\frac{q}{q-1}}}>0.$

It follows that

$$\|A(z^{k+1})-A(x^*)\|\le\theta_k\|A(x^k)-A(x^*)\|,$$

where

$$\theta_k=\sqrt[q]{(1-\alpha_k)^q+\Big(c_q\alpha_k^q+q\alpha_k(1-\alpha_k)\gamma\Big)\frac{1}{\gamma^{\frac{q}{q-1}}}}<1$$

for $\alpha_k\ge 1$. Since $A(x^{k+1})=(1-\alpha_k)A(x^k)+\alpha_k y^k$, we have

$$A(x^{k+1})-A(x^k)=\alpha_k(y^k-A(x^k)).$$

It further follows that

$$\begin{aligned}
&\|A(x^{k+1}) - A(z^{k+1})\| \\
&= \|(1-\alpha_k)A(x^k) + \alpha_k y^k - [(1-\alpha_k)A(x^k) \\
&\quad + \alpha_k A(J^M_{\rho_k,A}(A(x^k)))]\| \\
&= \|\alpha_k(y^k - A(J^M_{\rho,A}(A(x^k))))\| \\
&\leq \alpha_k \delta_k \|y^k - A(x^k)\|.
\end{aligned}$$

Next, we find the estimate

$$\begin{aligned}
&\|A(x^{k+1}) - A(x^*)\| \\
&= \|A(z^{k+1}) - A(x^*) + A(x^{k+1}) - A(z^{k+1})\| \\
&\leq \|A(z^{k+1}) - A(x^*)\| + \|A(x^{k+1}) - A(z^{k+1})\| \\
&\leq \|A(z^{k+1}) - A(x^*)\| + \alpha_k\delta_k\|y^k - A(x^k)\| \\
&= \|A(z^{k+1}) - A(x^*)\| + \delta_k\|A(x^{k+1}) - A(x^k)\| \\
&\leq \|A(z^{k+1}) - A(x^*)\| + \delta_k\|A(x^{k+1}) - A(x^*)\| \\
&\quad +\delta_k\|A(x^k) - A(x^*)\| \\
&\leq \theta_k\|A(x^k) - A(x^*)\| + \delta_k\|A(x^{k+1}) - A(x^*)\| \\
&\quad +\delta_k\|A(x^k) - A(x^*)\|. \qquad (3.4)
\end{aligned}$$

This implies that

$$\|A(x^{k+1}) - A(x^*)\| \leq \frac{\theta_k + \delta_k}{1-\delta_k}\|A(x^k) - A(x^*)\|, \qquad (3.5)$$

where

$$\begin{aligned}
&\limsup \frac{\theta_k + \delta_k}{1-\delta_k} = \limsup \theta_k \\
&= \sqrt[q]{(1-\alpha_k)^q + \Big(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\Big)\frac{1}{\gamma^{\frac{q}{q-1}}}} < 1,
\end{aligned}$$

for $\left[c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\right]\frac{1}{\gamma^{\frac{q}{q-1}}} > 0.$

Based on (3.5), we infer that the sequence $\{A(x^k)\}$ converges to $A(x^*)$, while A is $(r)-$strongly accretive (and hence $\|A(x^k)-A(x^*)\| \geq r\|x^k - x^*\|$). Hence, the sequence $\{x^k\}$ converges linearly to x^*. □

Corollary 3.4. *Let X be a real $q-$uniformly smooth Banach space, let $H : X \to X$ be $(r)-$strongly accretive, and let $M : X \to 2^X$ be $H-$maximal accretive. Furthermore, we suppose that $H o J^M_{\rho_k,H}$ is $(\gamma)-$cocoercive for a positive* *constant* γ. *For an arbitrarily chosen initial point* x^0, *suppose that the sequence* $\{x^k\}$ *is generated by an iterative procedure*

$$H(x^{k+1}) = (1-\alpha_k)H(x^k) + \alpha_k y^k \,\forall\, k \geq 0, \tag{3.6}$$

and y^k *satisfies*

$$\|y^k - H(J^M_{\rho_k,H}(H(x^k)))\| \leq \delta_k \|y^k - H(x^k)\|,$$

where $\delta_k \to 0$, $J^M_{\rho_k,A} = (H + \rho_k M)^{-1}$, *and*

$$\{\alpha_k\},\ \{\delta_k\}\,,\{\rho_k\} \subseteq [0,\infty)$$

are scalar sequences such that

$$\alpha = \limsup_{k\to\infty} \alpha_k, \quad \rho_k \uparrow \rho \leq \infty, \quad \alpha_k \geq 1, \quad \text{and} \quad \sum_{k=1}^{\infty} \delta_k < \infty,$$

$$\sqrt[q]{(1-\alpha_k)^q + \Big(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\Big)\frac{1}{\gamma^{\frac{q}{q-1}}}} < 1, \tag{3.7}$$

for $\left(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\right)\frac{1}{\gamma^{\frac{q}{q-1}}} \geq 0$, $c_q > 0$ *and* $\gamma > 1$.

Proof. The proof is similar to that of Theorem 3.3, but we include for the sake of the completeness. Suppose that x^* is a zero of M. Then from Theorem 3.2, it follows that any solution to (1.1) is a fixed point of $J^M_{\rho_k,H}oH$. For all $k \geq 0$, we express

$$H(z^{k+1}) = (1-\alpha_k)H(x^k) + \alpha_k H(J^M_{\rho_k,H}(H(x^k))).$$

Next, since $HoJ^M_{\rho_k,H}$ is $(\gamma)-$cocoercive, we have

$$\begin{aligned} &\langle H(J^M_{\rho,H}(H(x^k))) - H(J^M_{\rho,H}(H(x^*))), J_q(H(x^k) - H(x^*))\rangle \\ \geq\ &\gamma\|H(J^M_{\rho,H}(H(x^k))) - H(J^M_{\rho,H}(H(x^*)))\|^q. \end{aligned}$$

It follows that

$$\begin{aligned} &\|H(J^M_{\rho,H}(H(x^k))) - H(J^M_{\rho,H}(H(x^*)))\| \\ \leq\ &\frac{1}{\gamma^{\frac{1}{q-1}}}\|H(x^k) - H(x^*)\|. \end{aligned} \tag{3.8}$$

Now we find the estimate, on applying the above inequality and (3.8) that

$$
\begin{aligned}
& \|H(z^{k+1}) - H(x^*)\|^q \\
= & \|(1-\alpha_k)H(x^k) + \alpha_k H(J^M_{\rho_k,H}(H(x^k))) \\
& \quad -[(1-\alpha_k)H(x^*) + \alpha_k H(J^M_{\rho_k,H}(H(x^*)))]\|^q \\
= & \|(1-\alpha_k)(H(x^k) - H(x^*)) \\
& \quad +\alpha_k\Big(H(J^M_{\rho_k,H}(H(x^k))) - H(J^M_{\rho_k,H}(H(x^*)))\Big)\|^q \\
= & (1-\alpha_k)^q\|H(x^k) - H(x^*)\|^q \\
& \quad +q\alpha_k(1-\alpha_k) \\
& \quad \cdot\langle H(J^M_{\rho_k,H}(H(x^k))) - H(J^M_{\rho_k,H}(H(x^*))), J_q(H(x^k) - A(x^*))\rangle \\
& \quad +c_q\alpha_k^q\|H(J^M_{\rho_k,H}(H(x^k))) - H(J^M_{\rho_k,H}(H(x^*)))\|^q \\
\leq & (1-\alpha_k)^q\|H(x^k) - H(x^*)\|^q \\
& \quad +q\alpha_k(1-\alpha_k)\gamma\|H(J^M_{\rho_k,H}(H(x^k))) - H(J^M_{\rho_k,H}(H(x^*)))\|^q \\
& \quad +c_q\alpha_k^q\|H(J^M_{\rho_k,H}(H(x^k))) - H(J^M_{\rho_k,H}(H(x^*)))\|^q \\
= & (1-\alpha_k)^q\|H(x^k) - H(x^*)\|^q \\
& \quad +[c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma]\|H(J^M_{\rho_k,H}(H(x^k))) - H(J^M_{\rho_k,H}(H(x^*)))\|^q \\
\leq & (1-\alpha_k)^q\|H(x^k) - H(x^*)\|^q \\
& \quad +\left[c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\right]\frac{1}{\gamma^{\frac{q}{q-1}}}\|H(x^k - H(x^*)\|^q \\
= & \left[(1-\alpha_k)^q + \big(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\big)\frac{1}{\gamma^{\frac{q}{q-1}}}\right]\|H(x^k - H(x^*)\|^q,
\end{aligned}
$$

where $[c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma]\frac{1}{\gamma^{\frac{q}{q-1}}} > 0$.

It follows that

$$\|H(z^{k+1}) - A(x^*)\| \leq \theta_k\|H(x^k) - H(x^*)\|,$$

where

$$\theta_k = \sqrt[q]{(1-\alpha_k)^q + \Big(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\Big)\frac{1}{\gamma^{\frac{q}{q-1}}}} < 1$$

for $\alpha_k \geq 1$.

Since $H(x^{k+1}) = (1-\alpha_k)H(x^k) + \alpha_k y^k$, we have

$$H(x^{k+1}) - H(x^k) = \alpha_k(y^k - H(x^k)).$$

It follows that

$$\begin{aligned}
&\|H(x^{k+1}) - H(z^{k+1})\| \\
= \; & \|(1-\alpha_k)H(x^k) + \alpha_k y^k - [(1-\alpha_k)H(x^k) \\
& \quad +\alpha_k H(J^M_{\rho_k,H}(H(x^k)))]\| \\
= \; & \|\alpha_k(y^k - H(J^M_{\rho,H}(H(x^k))))\| \\
\leq \; & \alpha_k\delta_k\|y^k - H(x^k)\|.
\end{aligned}$$

Next, we find the estimate

$$\begin{aligned}
&\|H(x^{k+1}) - H(x^*)\| \\
&= \|H(z^{k+1}) - H(x^*) + H(x^{k+1}) - H(z^{k+1})\| \\
&\leq \|H(z^{k+1}) - H(x^*)\| + \|H(x^{k+1}) - H(z^{k+1})\| \\
&\leq \|H(z^{k+1}) - H(x^*)\| + \alpha_k\delta_k\|y^k - H(x^k)\| \\
&= \|H(z^{k+1}) - H(x^*)\| \\
&\qquad +\delta_k\|H(x^{k+1}) - H(x^k)\| \\
&\leq \|H(z^{k+1}) - H(x^*)\| \\
&\qquad +\delta_k\|H(x^{k+1}) - H(x^*)\| + \delta_k\|H(x^k) - H(x^*)\| \\
&\leq \theta_k\|H(x^k) - H(x^*)\| + \delta_k\|H(x^{k+1}) - H(x^*)\| \\
&\qquad +\delta_k\|H(x^k) - H(x^*)\|. \qquad (3.9)
\end{aligned}$$

This implies that

$$\|H(x^{k+1}) - H(x^*)\| \leq \frac{\theta_k + \delta_k}{1-\delta_k}\|H(x^k) - H(x^*)\|, \qquad (3.10)$$

where

$$\begin{aligned}
&\limsup \frac{\theta_k + \delta_k}{1-\delta_k} = \limsup\, \theta_k \\
&= \sqrt[q]{(1-\alpha_k)^q + \Big(c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\Big)\frac{1}{\gamma^{\frac{q}{q-1}}}} \; < 1,
\end{aligned}$$

for $\left[c_q\alpha_k^q + q\alpha_k(1-\alpha_k)\gamma\right]\frac{1}{\gamma^{\frac{q}{q-1}}} > 0$.

Based on (3.10), we infer that the sequence $\{H(x^k)\}$ converges to $H(x^*)$, while H is $(r)-$strongly accretive (and hence $\|H(x^k)-H(x^*)\| \geq r\|x^k - x^*\|$). Hence, the sequence $\{x^k\}$ converges linearly to x^*. □

Remark 3.5. We observe that there is not much difference in the proof approach of Corollary 3.4 than that of Theorem 3.3, but if we compare Propositions 2.4 and 2.7, there is a marked difference between the corresponding constants $r - \rho m$ and r leading to generalized resolvents based on single-valuedness property that is crucial to achieving a linear convergence.

4. Concluding Remarks

We consider the evolution equation

$$u'(t) + Mu(t) - \omega u(t) \ni b(t),\ u(0) = u_0 \tag{4.1}$$

for almost $t \in (0, T)$, where T is fixed with $0 < T < \infty$, and $M : H \to 2^H$ is maximal monotone on a real Hilbert space H. Based on [13] and ([22], Theorem 55A), inclusion problem (4.1) is described as follows:

Theorem 4.1. *If $u_0 \in D(M)$, $b \in W_2^1(0, T; H)$, and $\omega \in \mathbb{R}$, are given fixed quantities, inclusion problem (4.1) has exactly one solution $u \in W_2^1(0, T; H)$ provided $M : H \to 2^H$ is maximal monotone mapping on a real separable Hilbert space H.*
Moreover, the solution u is Lipschitz continuous and $u'(t)$ exists for almost all $t \in (0, T)$ in the sense that the classical derivative is the limit of the difference quotient, i.e., $u' \in L_\infty(0, T; H)$.

We state the next theorem on the solvability of the evolution inclusion

$$u'(t) + Mu(t) \ni f(t), \quad u(0) = u_0, \quad 0 \le t \le T, \tag{4.2}$$

where $u(t)$ belongs to a real Banach space X and $M : D(M) \subseteq X \to 2^X$ is $m-$accretive.

Based on ([22], Theorem 57A), inclusion problem (4.2) is described as follows:

Theorem 4.2. *If $M : D(M) \subseteq X \to 2^X$ is $m-$accretive, $f \in L_1(0, T; X)$, is given and fixed for fixed T such that $0 < T < \infty$ and $u_0 \in \overline{D(M)}$ is fixed. Then (4.2) has exactly one integral solution, each continuous solution $u : [0, T] \to X$ of (4.2) that has a generalized derivative $u' \in L_1(0, T; X)$ is also an integral solution to (4.2). Furthermore, if*

$$f_n \to f \in L_1(0, T; X) \text{ and } x_0^n \to u_0 \in X \text{ as } n \to \infty,$$

then $\{x^n\}$ converges uniformly on $[0, T]$ to the integral solution u of (4.2). Moreover, if v is an arbitrary integral solution to (4.2), then

$$\|v(t) - u(t)\| \le \|v(s) - u(s)\|$$

holds for all $s, t, 0 \le s \le t \le T$.

We observe that Theorem 4.1 and Theorem 4.2, respectively, can be generalized to the case of $A-$maximal $(m)-$relaxed monotone mappings in a Hilbert space setting and to the case of $A-$maximal $(m)-$relaxed accretive mappings in a real Banach space setting. Moreover, Theorem 4.2 shows that the concept of an integral solution provides a suitable solution concept of (4.2). It seems proposed generalizations of Theorems 4.1 and 4.2 seem to be consistent with the recent work [13] on first-order evolution equations based on $H-$maximal monotonicity assumptions.

References

[1] R. P. Agarwal and R. U. Verma, *Super-relaxed $\eta-$proximal point algorithms, relaxed $\eta-$proximal point algorithms, linear convergence analysis, and nonlinear variational inclusions*, Fixed Point Theory and Applications **2009** (2009), Article ID 957407, 47 pages, 2009.

[2] R. P. Agarwal and R. U. Verma, *Inexact $A-$proximal point algorithm and applications to nonlinear variational inclusion problems*, Journal of Optimization Theory and Applications **144(3)** (2010), 431–444.

[3] Y. P. Fang and N. J. Huang, *$H-$accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces*, Applied Mathematics Letters **17** (2004), 647–653.

[4] H. Y. Lan, Y. J. Cho and R. U. Verma, *Nonlinear relaxed cocoercive variational inclusions involving $(A, \eta)-$accretive mappings in Banach spaces*, Computers & Mathematics with Applications **51** (2006), 1529–1538.

[5] S. M. Robinson, *Composition duality and maximal monotonicity*, Mathematical Programming **85** (1999), 1–13.

[6] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal of Control and Optimization **14** (1976), 877–898.

[7] R. T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Mathematics of Operations Research **1** (1976), 97–116.

[8] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM Journal of Control and Optimization **38** (2000), 431–446.

[9] R. U. Verma, *$A-$monotonicity and its role in nonlinear variational inclusions*, Journal of Optimization Theory and Applications **129(3)** (2006), 457–467.

[10] R. U. Verma, *Sensitivity analysis for generalized strongly monotone variational inclusions based on the $(A, \eta)-$resolvent operator technique*, Applied Mathematics Letters **19** (2006), 1409–1413.

[11] R. U. Verma, *$A-$monotone nonlinear relaxed cocoercive variational inclusions*, Central European Journal of Mathematics **5(2)** (2007), 1–11.

[12] R. U. Verma, *General implicit variational inclusion problems involving $A-$maximal relaxed accretive mappings in Banach spaces*, Archivum Mathematicum (Brno) 45 (2009), 171–177.

[13] R. U. Verma, Generalized first-order nonlinear evolution equations and generalized Yosida approximations based on $H-$maximal monotonicity frameworks, Electron. Journal of Differential Equations **2009** (85) (2009), 1–16.

[14] R. U. Verma, *Nonlinear variational and constrained hemivariational inequalities involving relaxed operators*, ZAMM: Z. Angew. Math. Mech. **77(5)** (1977), 387–391.

[15] R. U. Verma, *Relatively relaxed proximal point algorithms for generalized maximal monotone mappings and Douglas-Rachford splitting methods*, WSEAS Transactions on Mathematics **10(8)** (2011), 259– 269.

[16] R. U. Verma, *New approach based on relatively inexact proximal point algorithms to linear convergence analysis and applications*, Journal of Advanced Modelling and Optimization **13(2)** (2011), 233–243.

[17] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Analysis **16** (1991), 1127–138.

[18] H. K. Xu, *Iterative algorithms for nonlinear operators*, Journal of the London Mathematical Society **66(2)** (2002), 240–256.

[19] E. Zeidler, *Nonlinear Functional Analysis and its Applications* I, Springer- Verlag, New York, 1986.

[20] E. Zeidler, *Nonlinear Functional Analysis and its Applications* II/A, Springer- Verlag, New York, 1990.

[21] E. Zeidler, *Nonlinear Functional Analysis and its Applications* II/B, Springer- Verlag, New York, 1990.

[22] E. Zeidler, *Nonlinear Functional Analysis and its Applications* III, Springer- Verlag, New York, 1985.