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TRIPLED COMMON FIXED POINT THEOREMS IN PARTIALLY ORDERED $b-$METRIC SPACES AND ITS APPLICATION TO INTEGRAL EQUATIONS

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Abstract. In this paper, we prove triple common fixed point theorems in partially ordered b-metric spaces depended on another function. The presented results generalize the theorem of Aydi, Karapinar and Mustafa [9], Berinde and Borcut [16], Borcut and Berinde [19] and Borcut [20]. Our results extend and improve several known results from the context of ordered metric spaces to the setting of ordered b-metric spaces. As an application, we prove the existence of a unique solution to a class of nonlinear integral equations.

1. Introduction

Fixed points theorems in partially ordered metric spaces were firstly obtained in 2004 by Ran and Reurings [35], and then by Nieto and Lopez [32]. In this direction several authors obtained further results under weak contractive conditions (see [1], [8], [11], [22], [25], [26]). Berinde initiated in [12] the concept of almost contractions and obtained several interesting fixed point theorems. This has been a subject of intense study since then, see [13, 14, 15, 34, 39]. Some authors used related notions as 'condition (B)' (Babu et al. [10]) and 'almost generalized contractive condition' for two maps (Ćirić et al. [21]), and for four maps (Aghajani et al. [4]). See also a note by Pacurar [34]. On the other hand, the concept of b-metric space was introduced by Czerwik in [24]. After that, several interesting results of the existence of fixed point for single-valued and multivalued operators in b-metric spaces have been

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obtained (see [3, 5, 6, 9, 16, 17, 18, 19, 31, 37, 38]). Pacurar [33] proved some results on sequences of almost contractions and fixed points in b-metric spaces. Recently, Hussain and Shah [27] obtained results on KKM mappings in cone b-metric spaces. Using the concepts of partially ordered metric spaces, almost generalized contractive condition, and b-metric spaces, we define a new concept of almost generalized $(\psi, \varphi, L)-$ contractive condition. We determine in this paper some triple common fixed point theorems for nonlinear contractions in the framework of partially ordered generalized b-metric spaces and obtain uniqueness theorems for contractive type mappings in this setting. Consistent with [23] and [38], following denitions and results will be needed in the sequel.

Definition 1.1. ([23]) Let X be a nonempty set and $s \geq 1$ be given a real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric space if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space with the parameter s.
It should be noted that, the class of $b-$metric spaces is effectively larger than that of metric spaces, since a $b-$metric is a metric, when $s = 1$.

The following example shows that in general a $b-$metric need not necessarily be a metric. (see [38]).

Example 1.2. ([2]) Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a $b-$metric with $s = 2^{p-1}$. However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^s$ is a $b-$metric on $\mathbb{R}$ with $s = 2$, but is not a metric on $\mathbb{R}$.

Also, the following example of a b-metric space is given in [28].

Example 1.3. ([28]) Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^2 dx < \infty$. Define $D : X \times X \longrightarrow [0, \infty)$ by $D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx$. As $(\int_0^1 |f(x) - g(x)|^2 dx)^{\frac{1}{2}}$ is a metric on X, then, from the previous example, D is a $b-$metric on X, with $s = 2$.

Khamsi [29] also showed that each cone metric space over a normal cone has a b-metric structure.

Borcut and Berinde [19] introduced the concept of a tripled coincidence point of mappings $g : X \longrightarrow X$ and $T : X \times X \times X \longrightarrow X$.

Definition 1.4. ([19]) Let X be a nonempty set. Let $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings. An element $(x, y, z) \in X \times X \times X \longrightarrow X$ is called a tripled coincidence point of T and g if

$$gx = T(x, y, z), \quad gy = T(y, x, y), \quad gz = T(z, y, x).$$

Note that if g is the identity mapping, then Definition 1.4 reduces to Definition in [16].

Definition 1.5. ([19]) Let $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$. An element (x, y, z) is called a tripled common fixed point of T and g if

$$x = gx = T(x, y, z), \quad y = gy = T(y, x, y), \quad z = gz = T(z, y, x).$$

Definition 1.6. ([19]) Let X be a nonempty set. Let $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be mappings. Then T and g are said to be commutative, if

$$g(T(x, y, z)) = T(gx, gy, gz),$$

whenever $x, y, z \in X$.

Akin to the concept of g-mixed monotone property [31] for a bivariate mapping, $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$, Borcut and Berinde [19] introduced the concept g-mixed monotone property for a trivariate mapping $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$ in the following way.

Definition 1.7. ([19]) Let $(X, \leq)$ be a partially ordered set and $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say that T has the g-mixed monotone property if $T(x, y, z)$ is monotone nondecreasing in x and z, and if it is monotone nonincreasing in y, that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, \ \ g(x_1) \leq g(x_2) \implies T(x_1, y, z) \leq T(x_2, y, z),$$
$$y_1, y_2 \in X, \ \ g(y_1) \leq g(y_2) \implies T(x, y_1, z) \geq T(x, y_2, z)$$

and

$$z_1, z_2 \in X, \ \ g(z_1) \leq g(z_2) \implies T(x, y, z_1) \leq T(x, y, z_2).$$

Definition 1.8. ([22]) A mapping T is said to be $g-$nondecreasing if

$$gx \leq gy \implies Tx \leq Ty.$$

2. Main results

Throughout the paper, let Ψ be the family of all functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

(a) ψ is continuous,
(b) ψ is nondecreasing,

(c) $\psi(0) = 0 < \psi(t)$ for every $t > 0$.

We denote by Φ the set of all functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(i) φ is right continuous,
(ii) φ is nondecreasing,
(iii) $\varphi(t) < t$ for every $t > 0$.

For given mappings $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$, define

$$\begin{aligned} &M(x,y,z,u,v,w)\\ &= \max\Big\{d(gx,gu), d(gy,gv), d(gz,gw), \frac{1}{2s}d(T(x,y,z),gu),\\ &\qquad \frac{1}{2s}d(T(z,y,x),gw), \frac{1}{2s}d(T(u,v,w),gx), \frac{1}{2s}d(T(w,v,u),gz)\Big\} \end{aligned}$$

and

$$N(x,y,z,u,v,w) = \min\{d(gx,T(x,y,z)), d(gx,T(u,v,w)), d(gu,T(x,y,z))\}.$$

The first result in this paper is the following a tripled coincidence point theorem.

Theorem 2.1. *Let $(X,d,\leq)$ be a partially ordered b-metric space. Let $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings. Suppose that the following conditions are hold.*

(a_1) *$T(X \times X \times X) \subseteq g(X)$,*
(a_2) *g is continuous and g commutes with T,*
(a_3) *$g(X)$ is a complete subspace of X,*
(a_4) *T has the mixed g-monotone property.*

Assume that there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $L \geq 0$ such that

$$\begin{aligned} &\psi(s^3 d(T(x,y,z),T(u,v,w)))\\ &\leq \varphi(\psi(M(x,y,z,u,v,w))) + L\psi(N(x,y,z,u,v,w)), \end{aligned} \tag{2.1}$$

for all $x,y,z,u,v,w \in X$ with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Also suppose either

(a) *T is continuous*

or

(b) *X has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$ for all n.*

If there exists $x_0, y_0, z_0 \in X$ *such that* $gx_0 \leq T(x_0, y_0, z_0)$, $gy_0 \geq T(y_0, x_0, y_0)$ *and* $gz_0 \leq T(z_0, y_0, x_0)$, *then* T *and* g *have a tripled coincidence point.*

Proof. By the given assumptions, there exists $x_0, y_0, z_0 \in X$ such that $gx_0 \leq T(x_0, y_0, z_0)$, $gy_0 \geq T(y_0, x_0, y_0)$ and $gz_0 \leq T(z_0, y_0, x_0)$. Since $T(X \times X \times X) \subseteq g(X)$, we can define $x_1, y_1, z_1 \in X$ such that

$$gx_1 = T(x_0, y_0, z_0), \quad gy_1 = T(y_0, x_0, y_0), \quad gz_1 = T(z_0, y_0, x_0).$$

Then $gx_0 \leq gx_1$, $gy_0 \geq gy_1$ and $gz_0 \leq gz_1$. Again, define $gx_2 = T(x_1, y_1, z_1)$, $gy_2 = T(y_1, x_1, y_1)$ and $gz_2 = T(z_1, y_1, x_1)$. Since T has the mixed g-monotone property, we have $gx_0 \leq gx_1 \leq gx_2$, $gy_0 \geq gy_1 \geq gy_2$ and $gz_0 \leq gz_1 \leq gz_2$. Continuing this process we can construct the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that for all $n = 0, 1, 2, \cdots$,

$$gx_{n+1} = T(x_n, y_n, z_n), \quad gy_{n+1} = T(y_n, x_n, y_n), \quad gz_{n+1} = T(z_n, y_n, x_n), \tag{2.2}$$

for which

$$\begin{aligned} &gx_0 \leq gx_1 \leq gx_2 \leq \cdots \leq gx_n \leq gx_{n+1} \leq \cdots, \\ &gy_0 \geq gy_1 \geq gy_2 \geq \cdots \geq gy_n \geq gy_{n+1} \geq \cdots, \\ &gz_0 \leq gz_1 \leq gz_2 \leq \cdots \leq gz_n \leq gz_{n+1} \leq \cdots. \end{aligned} \tag{2.3}$$

If there exists $k_0 \in \mathbb{N}$ such that $gx_{k_0+1} = gx_{k_0}$, $gy_{k_0+1} = gy_{k_0}$ and $gz_{k_0+1} = gz_{k_0}$, then

$$gx_{k_0} = T(x_{k_0}, y_{k_0}, z_{k_0}), \quad gy_{k_0} = T(y_{k_0}, x_{k_0}, y_{k_0}), \quad gz_{k_0} = T(z_{k_0}, y_{k_0}, x_{k_0}).$$

This means that $(x_{k_0}, y_{k_0}, z_{k_0})$ is a tripled coincidence point of T, g and the proof is finished. Thus, $(gx_{n+1}, gy_{n+1}, gz_{n+1}) \neq (gx_n, gy_n, gz_n)$ for all $n \in \mathbb{N}$. Since $gx_{n-1} \leq gx_n, gy_{n-1} \geq gy_n$ and $gz_{n-1} \leq gz_n$ from (2.1) and (2.2) with $(x, y, z) = (x_{n-1}, y_{n-1}, z_{n-1})$ and $(u, v, w) = (x_n, y_n, z_n)$, we have

$$\begin{aligned} \psi(d(gx_n, gx_{n+1})) &\leq \psi(s^3 d(gx_n, gx_{n+1})) \\ &= \psi(s^3 d(T(x_{n-1}, y_{n-1}, z_{n-1}), T(x_n, y_n, z_n))) \\ &\leq \varphi(\psi(M(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n))) \\ &\quad + L\psi(N(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n)), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &M(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n) \\ &= \max\Big\{ d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), \\ &\qquad \frac{1}{2s} d(T(x_{n-1}, y_{n-1}, z_{n-1}), gx_n), \frac{1}{2s} d(T(z_{n-1}, y_{n-1}, x_{n-1}), gz_n), \\ &\qquad \frac{1}{2s} d(T(x_n, y_n, z_n), gx_{n-1}), \frac{1}{2s} d(T(z_n, y_n, x_n), gz_{n-1}) \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), \frac{1}{2s}d(gx_n, gx_n),\\
&\qquad \frac{1}{2s}d(gz_n, gz_n), \frac{1}{2s}d(gx_{n+1}, gx_{n-1}), \frac{1}{2s}d(gz_{n+1}, gz_{n-1})\Big\}\\
&\le \max\Big\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n),\\
&\qquad \frac{1}{2s}d(gx_{n+1}, gx_{n-1}), \frac{1}{2s}d(gy_{n+1}, gy_{n-1}), \frac{1}{2s}d(gz_{n+1}, gz_{n-1})\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&N(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n)\\
&= \min\{d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_{n+1}), d(gx_n, gx_n)\} = 0.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{d(gx_{n-1}, gx_{n+1})}{2s} &\le \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2}\\
&\le \max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\},\\
\frac{d(gy_{n-1}, gy_{n+1})}{2s} &\le \frac{d(gy_{n-1}, gy_n) + d(gy_n, gy_{n+1})}{2}\\
&\le \max\{d(gy_{n-1}, gy_n), d(gy_n, gy_{n+1})\},\\
\frac{d(gz_{n-1}, gz_{n+1})}{2s} &\le \frac{d(gz_{n-1}, gz_n) + d(gz_n, gz_{n+1})}{2}\\
&\le \max\{d(gz_{n-1}, gz_n), d(gz_n, gz_{n+1})\},
\end{aligned}
$$

we have

$$
\begin{aligned}
&M(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n)\\
&\le \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n),\\
&\qquad d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\},\\
&N(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n) = 0.
\end{aligned} \tag{2.5}
$$

By (2.4) and (2.5), we have

$$
\begin{aligned}
&\psi(d(gx_n, gx_{n+1}))\\
&\le \varphi(\psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n),\\
&\qquad d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\})).
\end{aligned} \tag{2.6}
$$

Similarly, we can show that

$$
\begin{aligned}
&\psi(d(gy_n, gy_{n+1}))\\
&\le \varphi(\psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n),\\
&\qquad d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\})),
\end{aligned} \tag{2.7}
$$

and

$$\begin{aligned}&\psi(d(gz_n, gz_{n+1}))\\&\leq \varphi(\psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n),\\&\qquad d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\})).\end{aligned} \tag{2.8}$$

Combining (2.6), (2.7), (2.8) and the fact that

$$\max\{\psi(a), \psi(b), \psi(c)\} = \psi(\max\{a, b, c\})$$

for $a, b, c \in [0, +\infty)$, we have

$$\begin{aligned}&\psi(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\})\\&= \max\{\psi(d(gx_n, gx_{n+1})), \psi(d(gy_n, gy_{n+1})), \psi(d(gz_n, gz_{n+1}))\}\\&\leq \varphi(\psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n),\\&\qquad d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\})).\end{aligned}$$

Now denote

$$\delta_n := \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\}$$

and we prove

$$\delta_n \leq \delta_{n-1}. \tag{2.9}$$

For this purpose consider the following four cases.

Case 1. If

$$\begin{aligned}&\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gx_n, gx_{n+1}),\\&\qquad d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} = \delta_{n-1},\end{aligned}$$

then

$$\psi(\delta_n) \leq \varphi(\psi(\delta_{n-1})) < \psi(\delta_{n-1}), \tag{2.10}$$

so (2.9) obviously holds.

Case 2. If

$$\begin{aligned}&\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gx_n, gx_{n+1}),\\&\qquad d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} = d(gx_n, gx_{n+1}) > 0,\end{aligned}$$

then by (2.6),

$$\psi(d(gx_n, gx_{n+1})) \leq \varphi(\psi(d(gx_n, gx_{n+1}))) < \psi(d(gx_n, gx_{n+1})),$$

which is a contradiction.

Case 3. If

$$\begin{aligned}&\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gx_n, gx_{n+1}),\\&\qquad d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} = d(gy_n, gy_{n+1}) > 0,\end{aligned}$$

then by (2.7),

$$\psi(d(gy_n, gy_{n+1})) \leq \varphi(\psi(d(gy_n, gy_{n+1}))) < \psi(d(gy_n, gy_{n+1})),$$

which is a contradiction.

Case 4. If

$$\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gx_n, gx_{n+1}),\\ d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} = d(gz_n, gz_{n+1}) > 0,$$

then by (2.8),

$$\psi(d(gz_n, gz_{n+1})) \leq \varphi(\psi(d(gz_n, gz_{n+1}))) < \psi(d(gz_n, gz_{n+1})),$$

which is a contradiction.

Thus, in all cases, (2.9) holds for each $n \in \mathbb{N}$. It follows that the sequence $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $\delta \geq 0$ such that

$$\lim_{n \longrightarrow \infty} \delta_n = \delta. \tag{2.11}$$

We show that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$. Taking the limit as $n \to \infty$ in (2.10) and using the properties of the functions ψ and φ, we get

$$\psi(\delta) \leq \varphi(\psi(\delta)) < \psi(\delta),$$

which is a contradiction. Therefore, $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} = 0,$$

which implies that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0, \ \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0, \ \lim_{n\to\infty} d(gz_n, gz_{n+1}) = 0. \tag{2.12}$$

We shall show that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Suppose, on the contrary, that $\{gx_n\}$, $\{gy_n\}$ or $\{gz_n\}$ is not a Cauchy sequence, i.e., $\lim_{n,m\to\infty} d(gx_n, gx_m) \neq 0$, or $\lim_{n,m\to\infty} d(gy_n, gy_m) \neq 0$, or $\lim_{n,m\to\infty} d(gz_n, gz_m) \neq 0$. This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers $m(k)$ and $n(k)$ with $n(k) > m(k) \geq k$ such that

$$\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \geq \varepsilon. \tag{2.13}$$

Further, corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $m(k) < n(k)$ and satisfying (2.13). Then

$$\max\{d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)}), d(gz_{n(k)-1}, gz_{m(k)})\} < \varepsilon. \tag{2.14}$$

Using the triangle inequality in $b-$metric space and (2.13) and (2.14) we obtain that

$$\begin{aligned}\varepsilon \leq d(gx_{n(k)}, gx_{m(k)}) &\leq s\ d(gx_{n(k)}, gx_{n(k)-1}) + s\ d(gx_{n(k)-1}, gx_{m(k)})\\ &< s\ d(gx_{n(k)}, gx_{n(k)-1}) + s\varepsilon.\end{aligned}$$

Taking the upper limit as $k \longrightarrow \infty$ and using (2.12) we obtain

$$\varepsilon \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)}) \leq s\varepsilon. \tag{2.15}$$

Similarly, we have

$$\varepsilon \leq \limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)}) \leq s\varepsilon \tag{2.16}$$

and

$$\varepsilon \leq \limsup_{k\longrightarrow\infty} d(gz_{n(k)}, gz_{m(k)}) \leq s\varepsilon. \tag{2.17}$$

Also

$$\begin{aligned}\varepsilon \leq d(gx_{n(k)}, gx_{m(k)}) &\leq s\ d(gx_{n(k)}, gx_{m(k)+1}) + s\ d(gx_{m(k)+1}, gx_{m(k)})\\ &\leq s^2\ d(gx_{n(k)}, gx_{m(k)}) + s^2\ d(gx_{m(k)}, gx_{m(k)+1}) + s\ d(gx_{m(k)+1}, gx_{m(k)})\\ &\leq s^2\ d(gx_{n(k)}, gx_{m(k)}) + (s^2+s)\ d(gx_{m(k)}, gx_{m(k)+1}).\end{aligned}$$

So from (2.12) and (2.15), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)+1}) \leq s^2\varepsilon. \tag{2.18}$$

Similarly, we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)+1}) \leq s^2\varepsilon \tag{2.19}$$

and

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gz_{n(k)}, gz_{m(k)+1}) \leq s^2\varepsilon. \tag{2.20}$$

Also

$$\begin{aligned}\varepsilon \leq d(gx_{m(k)}, gx_{n(k)}) &\leq s\ d(gx_{m(k)}, gx_{n(k)+1}) + s\ d(gx_{n(k)+1}, gx_{n(k)})\\ &\leq s^2\ d(gx_{m(k)}, gx_{n(k)}) + s^2\ d(gx_{n(k)}, gx_{n(k)+1}) + s\ d(gx_{n(k)+1}, gx_{n(k)})\\ &\leq s^2\ d(gx_{m(k)}, gx_{n(k)}) + (s^2+s)\ d(gx_{n(k)}, gx_{n(k)+1}).\end{aligned}$$

So from (2.12) and (2.15), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)+1}) \leq s^2\varepsilon. \tag{2.21}$$

In a similar way, we obtain

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gy_{m(k)}, gy_{n(k)+1}) \leq s^2\varepsilon \tag{2.22}$$

and

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gz_{m(k)}, gz_{n(k)+1}) \leq s^2\varepsilon. \tag{2.23}$$

Also

$$d(gx_{n(k)+1}, gx_{m(k)}) \leq s\ d(gx_{n(k)+1}, gx_{m(k)+1}) + s\ d(gx_{m(k)+1}, gx_{m(k)}),$$

so from (2.12) and (2.21), we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)+1}, gx_{m(k)+1}). \tag{2.24}$$

Similarly, we obtain

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\longrightarrow\infty} d(gy_{n(k)+1}, gy_{m(k)+1}) \tag{2.25}$$

and

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\longrightarrow\infty} d(gz_{n(k)+1}, gz_{m(k)+1}). \tag{2.26}$$

$$\begin{aligned}
&M(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}) \\
&= \max\Big\{ d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)}), \\
&\qquad \frac{1}{2s} d(T(x_{n(k)}, y_{n(k)}, z_{n(k)}), gx_{m(k)}), \\
&\qquad \frac{1}{2s} d(T(z_{n(k)}, y_{n(k)}, x_{n(k)}), gz_{m(k)}), \\
&\qquad \frac{1}{2s} d(T(x_{m(k)}, y_{m(k)}, z_{m(k)}), gx_{n(k)}), \\
&\qquad \frac{1}{2s} d(T(z_{m(k)}, y_{m(k)}, x_{m(k)}), gz_{n(k)}) \Big\} \\
&= \max\Big\{ d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)}), \\
&\qquad \frac{1}{2s} d(gx_{n(k)+1}, gx_{m(k)}), \frac{1}{2s} d(gz_{n(k)+1}, gz_{m(k)}), \\
&\qquad \frac{1}{2s} d(gx_{m(k)}, gx_{n(k)}), \frac{1}{2s} d(gz_{m(k)+1}, gz_{n(k)}) \Big\}.
\end{aligned}$$

Linking (2.15),(2.16),(2.17),(2.20),(2.21) together with (2.23) we get

$$\limsup_{k\longrightarrow\infty} M(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, y_{m(k)}) \leq s\varepsilon. \tag{2.27}$$

Similarly, we have

$$\limsup_{k\longrightarrow\infty} M(y_{n(k)}, x_{n(k)}, y_{n(k)}, y_{m(k)}, x_{m(k)}, y_{m(k)}) \leq s\varepsilon \tag{2.28}$$

and

$$\limsup_{k\longrightarrow\infty} M(z_{n(k)}, y_{n(k)}, x_{n(k)}, z_{m(k)}, y_{m(k)}, x_{m(k)}) \leq s\varepsilon. \tag{2.29}$$

Also

$$\begin{aligned}&N(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})\\&= \min\{d(gx_{n(k)}, gx_{n(k)+1}), d(gx_{n(k)}, gx_{m(k)+1}), d(gx_{m(k)}, gx_{n(k)+1})\}.\end{aligned}$$

Letting $k \to \infty$ and using (2.12), we get

$$\limsup_{k \longrightarrow \infty} N(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}) = 0. \tag{2.30}$$

Similarly, we have

$$\limsup_{k \longrightarrow \infty} N(y_{n(k)}, x_{n(k)}, y_{n(k)}, y_{m(k)}, x_{m(k)}, y_{m(k)}) = 0 \tag{2.31}$$

and

$$\limsup_{k \longrightarrow \infty} N(z_{n(k)}, y_{n(k)}, x_{n(k)}, z_{m(k)}, y_{m(k)}, x_{m(k)}) = 0. \tag{2.32}$$

Since $n(k) > m(k)$, we have

$$gx_{m(k)} \leq gx_{n(k)},\ \ gy_{m(k)} \geq gy_{n(k)},\ \ gz_{m(k)} \leq gz_{n(k)}.$$

Now, using inequality (2.1) we obtain

$$\begin{aligned}&\psi(s^3 d(gx_{n(k)+1}, gx_{m(k)+1}))\\&= \psi(s^3\ d(T(x_{n(k)}, y_{n(k)}, z_{n(k)}), T(x_{m(k)}, y_{m(k)}, z_{m(k)})))\\&\leq \varphi(\psi(M(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})))\\&\quad + L\psi(N(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}).\end{aligned}$$

Passing to the upper limit as $k \longrightarrow \infty$, and using (2.24), (2.27) and (2.30), we get

$$\begin{aligned}\psi(s\varepsilon) &\leq \psi(s^3 \limsup_{k \longrightarrow \infty} d(gx_{n(k)+1}, gx_{m(k)+1}))\\&= \limsup_{k \longrightarrow \infty} \psi(s^3 d(gx_{n(k)+1}, gx_{m(k)+1}))\\&= \limsup_{k \longrightarrow \infty} \psi(s^3\ d(T(x_{n(k)}, y_{n(k)}, z_{n(k)}), T(x_{m(k)}, y_{m(k)}, z_{m(k)})))\\&\leq \limsup_{k \longrightarrow \infty} \varphi(\psi(M(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})))\\&\quad + \limsup_{k \longrightarrow \infty} L\psi(N(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})\\&= \varphi(\psi(\limsup_{k \longrightarrow \infty} M(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})))\\&\quad + L\psi(\limsup_{k \longrightarrow \infty} N(x_{n(k)}, y_{n(k)}, z_{n(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})\\&\leq \varphi(\psi(\varepsilon s)) < \psi(s\varepsilon),\end{aligned}$$

which is a contradiction. Similarly, we have

$$\begin{aligned}
\psi(s\varepsilon) &\leq \psi(s^3 \limsup_{k\longrightarrow\infty} d(gy_{n(k)+1}, gy_{m(k)+1}))\\
&= \limsup_{k\longrightarrow\infty} \psi(s^3 d(gy_{n(k)+1}, gy_{m(k)+1}))\\
&= \limsup_{k\longrightarrow\infty} \psi(s^3\ d(T(y_{n(k)}, x_{n(k)}, y_{n(k)}), T(y_{m(k)}, x_{m(k)}, y_{m(k)})))\\
&\leq \limsup_{k\longrightarrow\infty} \varphi(\psi(M(y_{n(k)}, x_{n(k)}, y_{n(k)}, y_{m(k)}, x_{m(k)}, y_{m(k)})))\\
&\quad + \limsup_{k\longrightarrow\infty} L\psi(N(y_{n(k)}, x_{n(k)}, y_{n(k)}, y_{m(k)}, x_{m(k)}, y_{m(k)})\\
&= \varphi(\psi(\limsup_{k\longrightarrow\infty} M(y_{n(k)}, x_{n(k)}, y_{n(k)}, y_{m(k)}, x_{m(k)}, y_{m(k)})))\\
&\quad + L\psi(\limsup_{k\longrightarrow\infty} N(y_{n(k)}, x_{n(k)}, y_{n(k)}, y_{m(k)}, x_{m(k)}, y_{m(k)})\\
&\leq \varphi(\psi(\varepsilon s)) < \psi(s\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
\psi(s\varepsilon) &\leq \psi(s^3 \limsup_{k\longrightarrow\infty} d(gz_{n(k)+1}, gz_{m(k)+1}))\\
&= \limsup_{k\longrightarrow\infty} \psi(s^3 d(gz_{n(k)+1}, gz_{m(k)+1}))\\
&= \limsup_{k\longrightarrow\infty} \psi(s^3\ d(T(z_{n(k)}, y_{n(k)}, x_{n(k)}), T(z_{m(k)}, y_{m(k)}, x_{m(k)})))\\
&\leq \limsup_{k\longrightarrow\infty} \varphi(\psi(M(z_{n(k)}, y_{n(k)}, x_{n(k)}, z_{m(k)}, y_{m(k)}, x_{m(k)})))\\
&\quad + \limsup_{k\longrightarrow\infty} L\psi(N(z_{n(k)}, y_{n(k)}, x_{n(k)}, z_{m(k)}, y_{m(k)}, x_{m(k)})\\
&= \varphi(\psi(\limsup_{k\longrightarrow\infty} M(z_{n(k)}, y_{n(k)}, x_{n(k)}, z_{m(k)}, y_{m(k)}, x_{m(k)})))\\
&\quad + L\psi(\limsup_{k\longrightarrow\infty} N(z_{n(k)}, y_{n(k)}, x_{n(k)}, z_{m(k)}, y_{m(k)}, x_{m(k)})\\
&\leq \varphi(\psi(\varepsilon s)) < \psi(s\varepsilon),
\end{aligned}$$

which are contradiction. Hence $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in gX. Since gX is complete, there exist $a = gx, b = gy, c = gz \in gX$ such that

$$\lim_{n\to\infty} gx_{n+1} = a, \quad \lim_{n\to\infty} gy_{n+1} = b, \quad \lim_{n\to\infty} gz_{n+1} = c.$$

Now, we show that (a, b, c) is a coincidence point of T and g. Suppose that the assumption (a) holds. From the commutativity of T and g, we have

$$\begin{aligned}
g(gx_{n+1}) &= g(T(x_n, y_n, z_n)) = T(gx_n, gy_n, gz_n),\\
g(gy_{n+1}) &= g(T(y_n, x_n, y_n)) = T(gy_n, gx_n, gy_n),\\
g(gz_{n+1}) &= g(T(z_n, y_n, x_n)) = T(gz_n, gy_n, gx_n).
\end{aligned} \tag{2.33}$$

Letting $n \longrightarrow \infty$ in (2.33) and from the continuity of T and g, we get

$$\begin{aligned} ga &= \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} T(gx_n, gy_n, gz_n) \\ &= T(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gz_n) = T(a,b,c), \\ gb &= \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} T(gy_n, gx_n, gy_n) \\ &= T(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n) = T(b,a,b), \\ gc &= \lim_{n\to\infty} g(gz_{n+1}) = \lim_{n\to\infty} T(gz_n, gy_n, gx_n) \\ &= T(\lim_{n\to\infty} gz_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n) = T(c,b,a). \end{aligned}$$

So (a,b,c) is a tripled coincidence point of T and g. Suppose now that (b) holds. From (2.3) and hypothesis (b), we have

$$gx_n \leq gx, \quad gy_n \geq gy, \quad gz_n \leq gz \text{ for all } n.$$

Our claim is

$$\max\{\psi(d(T(x,y,z),gx)), \psi(d(T(z,y,x),gz)), \psi(d(gy, T(y,x,y))\} = 0.$$

To prove our claim, suppose that

$$\max\{\psi(d(T(x,y,z),gx)), \psi(d(T(z,y,x),gz)), \psi(d(gy, T(y,x,y))\} \neq 0.$$

So, we have

$$\begin{aligned} &M(x_n, y_n, z_n, x, y, z)) \\ &= \max\Big\{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz), \frac{1}{2s} d(T(x_n,y_n,z_n), gx), \\ &\qquad \frac{1}{2s} d(T(z_n,y_n,x_n), gz), \frac{1}{2s} d(T(x,y,z), gx_n), \frac{1}{2s} d(T(z,y,x), gz_n)\Big\} \\ &= \max\Big\{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz), \frac{1}{2s} d(gx_{n+1}, gx), \\ &\qquad \frac{1}{2s} d(gz_{n+1}, gz), \frac{1}{2s} d(T(x,y,z), gx_n), \frac{1}{2s} d(T(z,y,x), gz_n)\Big\} \\ &\leq \max\Big\{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz), \frac{1}{2s} d(gx_{n+1}, gx), \frac{1}{2s} d(gz_{n+1}, gz), \\ &\qquad d(T(x,y,z), gx), d(gx, gx_n), d(T(z,y,x), gz), d(gz, gz_n)\Big\}. \end{aligned}$$

So,

$$\begin{aligned} &\limsup_{n\longrightarrow\infty} M(x_n, y_n, z_n, x, y, z)) \\ &\leq \max\{d(T(x,y,z), gx), d(T(z,y,x), gz)\} \\ &\leq \max\{d(T(x,y,z), gx), d(T(z,y,x), gz), d(gy, T(y,x,y)\}. \end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}&\limsup_{n\longrightarrow\infty} M(y_n,x_n,y_n,y,x,y))\\&\leq \max\{d(T(x,y,z),gx),d(T(z,y,x),gz),d(gy,T(y,x,y)\}\end{aligned}$$

and

$$\begin{aligned}&\limsup_{n\longrightarrow\infty} M(z_n,y_n,x_n,z,y,x))\\&\leq \max\{d(T(x,y,z),gx),d(T(z,y,x),gz),d(gy,T(y,x,y)\}.\end{aligned}$$

Also

$$\begin{aligned}&N(x_n,y_n,z_n,x,y,z)\\&=\min\{d(gx_n,T(x_n,y_n,z_n)),d(gx_n,T(x,y,z)),d(gx,T(x_n,y_n,z_n))\}\\&=\min\{d(gx_n,gx_{n+1}),d(gx_n,T(x,y,z)),d(gx,gx_{n+1})\}.\end{aligned}$$

So,

$$\limsup_{n\longrightarrow\infty} N(x_n,y_n,z_n,x,y,z))=0.$$

Similarly, we have

$$\limsup_{n\longrightarrow\infty} N(y_n,x_n,y_n,y,x,y))=0,\quad \limsup_{n\longrightarrow\infty} N(z_n,y_n,x_n,z,y,x))=0.$$

By property of ψ, φ, (2.1), the inequality above and using the triangle inequality in b-metric space, we have

$$\begin{aligned}&\psi(\max\{d(T(x,y,z),gx),d(T(z,y,x),gz),d(gy,T(y,x,y)\})\\&=\max\{\psi(d(T(x,y,z),gx)),\psi(d(T(z,y,x),gz)),\psi(d(gy,T(y,x,y))\}\\&\leq \max\{\limsup_{n\longrightarrow\infty}\psi(d(T(x_n,y_n,z_n),T(x,y,z))),\\&\qquad\limsup_{n\longrightarrow\infty}\psi(d(T(y_n,x_n,y_n),T(y,x,y))),\\&\qquad\limsup_{n\longrightarrow\infty}\psi(d(T(z_n,y_n,x_n),T(z,y,x)))\}\\&\leq \max\{\limsup_{n\longrightarrow\infty}\psi(s^3d(T(x_n,y_n,z_n),T(x,y,z))),\\&\qquad\limsup_{n\longrightarrow\infty}\psi(s^3d(T(y_n,x_n,y_n),T(y,x,y))),\\&\qquad\limsup_{n\longrightarrow\infty}\psi(s^3d(T(z_n,y_n,x_n),T(z,y,x)))\}\\&\leq \max\{\limsup_{n\longrightarrow\infty}[\varphi(\psi(M(x_n,y_n,z_n,x,y,z)))+L\psi(N(x_n,y_n,z_n,x,y,z))],\\&\qquad\limsup_{n\longrightarrow\infty}[\varphi(\psi(M(y_n,x_n,y_n,y,x,y)))+L\psi(N(y_n,x_n,y_n,x,y,z))],\\&\qquad\limsup_{n\longrightarrow\infty}[\varphi(\psi(M(z_n,y_n,x_n,z,y,x)))+L\psi(N(z_n,y_n,x_n,z,y,x))]\}.\end{aligned}$$

Then,

$$\begin{aligned}&\max\{\psi(d(T(x,y,z),gx)),\psi(d(T(z,y,x),gz)),\psi(d(gy,T(y,x,y))\}\\&\leq\varphi(\max\{\psi(d(T(x,y,z),gx)),\psi(d(T(z,y,x),gz)),\psi(d(gy,T(y,x,y))\})\\&<\max\{\psi(d(T(x,y,z),gx)),\psi(d(T(z,y,x),gz)),\psi(d(gy,T(y,x,y)))\},\end{aligned}$$

which is contradiction. Therefore

$$\max\{\psi(d(T(x,y,z),gx)),\psi(d(T(z,y,x),gz)),\psi(d(gy,T(y,x,y))\}=0$$

and hence $d(T(x,y,z),gx)=0$, $d(T(z,y,x),gz)=0$ and $d(gy,T(y,x,y))=0$. Thus $T(x,y,z)=gx$, $T(y,x,y)=gy$ and $T(z,y,x)=gz$. That is (x,y,z) is a tripled coincidence point of T and g. □

Corollary 2.2. *Let $(X,d,\leq)$ be a partially ordered b-metric space. Let $T: X\times X\times X\longrightarrow X$ and $g: X\longrightarrow X$ be two mappings. Suppose that the followings are hold:*

(a_1) $T(X\times X\times X)\subseteq g(X)$,

(a_2) *g is continuous and g commutes with T,*

(a_3) *$g(X)$ is a complete subspace of X,*

(a_4) *T has the mixed g-monotone property.*

Assume that there exist $\varphi\in\Phi$ and $L\geq 0$ such that

$$\begin{aligned}&s^3d(T(x,y,z),T(u,v,w))\\&\leq\varphi(M(x,y,z,u,v,w))+L\,N(x,y,z,u,v,w),\end{aligned}\tag{2.34}$$

for all $x,y,z,u,v,w\in X$ with $gx\leq gu$, $gy\geq gv$ and $gz\leq gw$. Also suppose either

(a) *T is continuous*

or

(b) *X has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n\leq x$ for all n,*

(ii) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n\geq y$ for all n.*

If there exists $x_0,y_0,z_0\in X$ such that $gx_0\leq T(x_0,y_0,z_0)$, $gy_0\geq T(y_0,x_0,y_0)$ and $gz_0\leq T(z_0,y_0,x_0)$, then T and g have a tripled coincidence point.

Proof. It suffices to take $\psi(t)=t$ in Theorem 2.1. □

Corollary 2.3. *Let $(X,d,\leq)$ be a partially ordered b-metric space. Let $T: X\times X\times X\longrightarrow X$ and $g: X\longrightarrow X$ be two mappings. Suppose that the followings are hold:*

(a_1) $T(X\times X\times X)\subseteq g(X)$,

(a_2) *g is continuous and g commutes with T,*
(a_3) *$g(X)$ is a complete subspace of X,*
(a_4) *T has the mixed g-monotone property.*

Assume that there exist $\lambda \in [0,1)$ and $L \geq 0$ such that

$$s^3 d(T(x,y,z),T(u,v,w)) \leq \lambda M(x,y,z,u,v,w) + LN(x,y,z,u,v,w),$$

for all $x,y,z,u,v,w \in X$ with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Also suppose either

(a) *T is continuous*

or

(b) *X has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$ for all n.*

If there exists $x_0, y_0, z_0 \in X$ such that $gx_0 \leq T(x_0,y_0,z_0)$, $gy_0 \geq T(y_0,x_0,y_0)$ and $gz_0 \leq T(z_0,y_0,x_0)$, then T and g have a tripled coincidence point.

Proof. It suffices to take $\varphi(t) = \lambda\, t$ for all $t \geq 0$ in Corollary 2.2. □

Corollary 2.4. *Let $(X,d,\leq)$ is a partially ordered b-metric space. Let $T : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings. Suppose that the followings are hold:*

(a_1) *$T(X \times X \times X) \subseteq g(X)$,*
(a_2) *g is continuous and g commutes with T,*
(a_3) *$g(X)$ is a complete subspace of X,*
(a_4) *T has the mixed g-monotone property.*

Assume that there exist $\varphi \in \Phi$ and $L \geq 0$ such that

$$s^3 d(T(x,y,z),T(u,v,w))$$
$$\leq \varphi(max\{d(gx,gu),d(gy,gv),d(gz,gw)\}) + LN(x,y,z,u,v,w),$$

for all $x,y,z,u,v,w \in X$ with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Also suppose either

(a) *T is continuous*

or

(b) *X has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$ for all n.*

If there exists $x_0, y_0, z_0 \in X$ *such that* $gx_0 \leq T(x_0, y_0, z_0)$, $gy_0 \geq T(y_0, x_0, y_0)$ *and* $gz_0 \leq T(z_0, y_0, x_0)$, *then* T *and* g *have a tripled coincidence point.*

Proof. It suffices to remark that

$$max\{d(gx, gu), d(gy, gv), d(gz, gw)\} \leq M(x, y, z, u, v, w).$$

Then, we apply Theorem 2.1 because that φ is non-decreasing. □

Corollary 2.5. *Let* $(X, d, \leq)$ *be a partially ordered b-metric space. Let* $T : X \times X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings. Suppose that the followings are hold:*

(a_1) $T(X \times X \times X) \subseteq g(X)$,
(a_2) *g is continuous and g commutes with T,*
(a_3) *$g(X)$ is a complete subspace of X,*
(a_4) *T has the mixed g-monotone property.*

Assume that there exist $\varphi \in \Phi$ *and* $L \geq 0$ *such that*

$$s^3 d(T(x, y, z), T(u, v, w)) \leq \varphi\left(\frac{d(gx, gu) + d(gy, gv) + d(gz, gw)}{3}\right) + LN(x, y, z, u, v, w),$$

for all $x, y, z, u, v, w \in X$ *with* $gx \leq gu$, $gy \geq gv$ *and* $gz \leq gw$. *Also suppose either*

(a) *T is continuous*

or

(b) *X has the following property:*
(i) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$ for all n,*
(ii) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$ for all n.*

If there exists $x_0, y_0, z_0 \in X$ *such that* $gx_0 \leq T(x_0, y_0, z_0)$, $gy_0 \geq T(y_0, x_0, y_0)$ *and* $gz_0 \leq T(z_0, y_0, x_0)$, *then* T *and* g *have a tripled coincidence point.*

Proof. It suffices to remark that

$$\frac{d(gx, gu) + d(gy, gv) + d(gz, gw)}{3} \leq max\{d(gx, gu), d(gy, gv), d(gz, gw)\}.$$

Then, we apply Corollary 2.4 because that φ is non-decreasing. □

Corollary 2.6. *Let* $(X, d, \leq)$ *be a partially ordered b-metric space. Let* $T : X \times X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings. Suppose the following:*

(a_1) $T(X \times X \times X) \subseteq g(X)$,
(a_2) *g is continuous and g commutes with T,*

(a_3) *$g(X)$ is a complete subspace of X,*
(a_4) *T has the mixed g-monotone property.*

Assume that there exist $\lambda \in [0,1)$ and $L \geq 0$ such that

$$s^3 d(T(x,y,z),T(u,v,w)) \leq \frac{\lambda}{3}\Big[d(gx,gu)+d(gy,gv)+d(gz,gw)\Big] + LN(x,y,z,u,v,w),$$

for all $x,y,z,u,v,w \in X$ with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Also suppose either

(a) *T is continuous*

or

(b) *X has the following properties:*
  (i) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$ for all n.*

If there exists $x_0, y_0, z_0 \in X$ such that $gx_0 \leq T(x_0,y_0,z_0)$, $gy_0 \geq T(y_0,x_0,y_0)$ and $gz_0 \leq T(z_0,y_0,x_0)$, then T and g have a tripled coincidence point.

Proof. It suffices to take that $\varphi(t) = \lambda t$ in Corollary 2.5. □

Remark 2.7.

(1) Theorem 2.1 and 2.2 of [37] is the analogous of Corollary 2.2.
(2) Corollary 2.3 generalizes Theorem 7 and 8 of Berinde and Borcut [16].
(3) Theorem 7 of [16] is a special case of Corollary 2.6.
(4) Theorem 4 of [19] is a special case of Corollary 2.6.
(5) Corollary 2.6 is the analogous of Theorem 2.1 and Theorem 2.2 of Lakshmikantham and *Ćirić* [31] for coupled fixed point results by taking $s = 1$ and $L = 0$.
(6) Theorem 5 of [20] is a special case of Corollary 2.4.
(7) If we take $g = I$, $L = 0$ and $s = 1$ in Corollary 2.4 then we get the main result (Theorem 7) in [16] regarding the existence of tripled fixed points.
(8) Corollary 2.4 generalizes Theorem 2.1 and 2.2 of [9].

Remark 2.8. Other corollaries could be derived from Theorem 2.1 and Corollaries 2.2, 2.3, 2.4, 2.5 and 2.6 by taking $g = I$.

Now, we shall state and prove the corresponding result regarding the existence and uniqueness of tripled common fixed point. We endow the product

space $X \times X \times X$ with the following partial order:
For all (x, y, z) and (u, v, w) in X

$$(x, y, z) \leq (u, v, w) \iff x \leq u, \ \ y \geq v, \ \ z \leq w.$$

We say that (x, y, z) and (u, v, w) are comparable if

$$(x, y, z) \leq (u, v, w) \text{ or } (u, v, w) \leq (x, y, z).$$

Theorem 2.9. *In addition to the hypothesis of Theorem* 2.1*, suppose that for all* (x, y, z) *and* (x^*, y^*, z^*) *in* $X \times X \times X$*, there exists a* $(u, v, w) \in X \times X \times X$ *such that* $(T(u, v, w), T(v, u, v), T(w, v, u))$ *is comparable to* (gx, gy, gz) *and to* (gx^*, gy^*, gz^*)*. Then* T *and* g *have a unique tripled common fixed point.*

Proof. It follows from Theorem 2.1 that the set of tripled coincidence points is nonempty. Suppose (x, y, z) and (x^*, y^*, z^*) are coincidence points of T and g, that is, $gx = T(x, y, z)$, $gy = T(y, x, y)$, $gz = T(z, y, x)$, $gx^* = T(x^*, y^*, z^*)$, $gy^* = T(y^*, x^*, y^*)$ and $gz^* = T(z^*, y^*, x^*)$. We shall now show that $gx = gx^*, gy = gy^*$ and $gz = gz^*$. By assumption, there exists $(u, v, w) \in X \times X \times X$ that is comparable to (gx, gy, gz) and (gx^*, gy^*, gz^*).
Put $u_0 = u, v_0 = v, w_0 = w$ and choose $(u_1, v_1, w_1) \in X \times X \times X$ such that

$$gu_1 = T(u_0, v_0, w_0), \ \ gv_1 = T(v_0, u_0, v_0), \ \ gw_1 = T(w_0, v_0, u_0).$$

For $n \geq 1$, continuing this process we can construct sequences $\{gu_n\}$, $\{gv_n\}$ and $\{gw_n\}$ such that

$$gu_{n+1} = T(u_n, v_n, w_n), \ \ gv_{n+1} = T(v_n, u_n, v_n), \ \ gw_{n+1} = T(w_n, v_n, u_n)$$

for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $x_0^* = x^*$, $y_0^* = y^*$, $z_0^* = z^*$ and on the same way define sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, $\{gx_n^*\}$, $\{gy_n^*\}$ and $\{gz_n^*\}$. Then, it is easy to see that

$$\begin{aligned} &gx_n \longrightarrow T(x, y, z), \quad gy_n \longrightarrow T(y, x, y), \quad gz_n \longrightarrow T(z, y, x), \\ &gx_n^* \longrightarrow T(x^*, y^*, z^*), \quad gy_n^* \longrightarrow T(y^*, x^*, y^*), \quad gz_n^* \longrightarrow T(z^*, y^*, x^*), \end{aligned} \tag{2.35}$$

for all $n \geq 1$. Since

$$(T(x, y, z), T(y, x, y), T(z, y, x)) = (gx, gy, gz) = (gx_1, gy_1, gz_1)$$

is comparable to

$$(T(u, v, w), T(v, u, v), T(w, v, u)) = (gu_1, gv_1, gw_1),$$

then $(gx, gy, gz) \leq (gu_1, gv_1, gw_1)$. Recursively, we get that

$$gx \leq gu_n, \quad gy \geq gy_n, \quad gz \leq gw_n \quad \textit{for all } n. \tag{2.36}$$

Thus from (2.1), we have

$$\begin{aligned} \psi(d(gx, gu_{n+1})) &\leq \psi(s^3 d(gx, gu_{n+1})) = \psi(s^3 d(T(x, y, z), T(u_n, v_n, w_n))) \\ &\leq \varphi(\psi(M(x, y, z, u_n, v_n, w_n))) + L \ \psi(N(x, y, z, u_n, v_n, w_n)), \end{aligned}$$

where

$$M(x,y,z,u_n,v_n,w_n)$$
$$= \max\Big\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n), \frac{1}{2s}d(T(x,y,z),gu_n),$$
$$\frac{1}{2s}d(T(z,y,x),gw_n), \frac{1}{2s}d(T(u_n,v_n,w_n),gx), \frac{1}{2s}d(T(w_n,v_n,u_n),gz)\Big\},$$

and

$$N(x,y,z,u_n,v_n,w_n)$$
$$= \min\{d(gx,T(x,y,z)), d(gx,T(u_n,v_n,w_n)), d(gu_n,T(x,y,z))\}.$$

It is easy to show that

$$M(x,y,z,u_n,v_n,w_n) \le \max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\}$$

and

$$N(x,y,z,u_n,v_n,w_n) = 0.$$

Hence

$$\psi(d(gx,gu_{n+1})) \le \varphi(\psi(\max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\})). \quad (2.37)$$

Similarly one can prove that

$$\begin{aligned}\psi(d(gy,gv_{n+1})) &\le \varphi(\psi(\max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\})),\\ \psi(d(gz,gw_{n+1})) &\le \varphi(\psi(\max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\})).\end{aligned} \quad (2.38)$$

Combining (2.37),(2.38) and the fact that

$$\max\{\psi(a),\psi(b),\psi(c)\} = \psi(\max\{a,b,c\})$$

for $a,b,c \in [0,+\infty)$, we have

$$\begin{aligned}&\psi(\max\{d(gx,gu_{n+1}), d(gy,gv_{n+1}), d(gz,gw_{n+1})\})\\ &= \max\{\psi(d(gx,gu_{n+1})), \psi(d(gy,gv_{n+1})), \psi(d(gz,gw_{n+1}))\}\\ &\le \varphi(\psi(\max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\}))\\ &< \psi(\max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\}).\end{aligned} \quad (2.39)$$

Using the non-decreasing property of ψ, we get

$$\begin{aligned}&\max\{d(gx,gu_{n+1}), d(gy,gv_{n+1}), d(gz,gw_{n+1})\}\\ &\le \max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\},\end{aligned}$$

implies that $\max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\}$ is a non-increasing sequence. Hence, there exists $r \ge 0$ such that

$$\lim_{n\longrightarrow\infty} \max\{d(gx,gu_n), d(gy,gv_n), d(gz,gw_n)\} = r.$$

Passing the upper limit in (2.39) as $n \longrightarrow \infty$, we obtain

$$\psi(r) \leq \varphi(\psi(r)) < \psi(r),$$

which implies that $r = 0$. We deduce that

$$\lim_{n\longrightarrow\infty} \max\{d(gx, gu_n), d(gy, gv_n), d(gz, gw_n)\} = 0,$$

which concludes

$$\lim_{n\longrightarrow\infty} d(gx, gu_n) = \lim_{n\longrightarrow\infty} d(gy, gv_n) = \lim_{n\longrightarrow\infty} d(gz, gw_n) = 0. \qquad (2.40)$$

Similarly, one can prove that

$$\lim_{n\longrightarrow\infty} d(gx^*, gu_n) = \lim_{n\longrightarrow\infty} d(gy^*, gv_n) = \lim_{n\longrightarrow\infty} d(gz^*, gw_n) = 0. \qquad (2.41)$$

From (2.39) and (2.40), we have $gx = gx^*, gy = gy^*$ and $gz = gz^*$. Since $gx = T(x, y, z), gy = T(y, x, y)$ and $gz = T(z, y, x)$, by commutativity of T and g, we have

$$\begin{aligned} g(gx) &= g(T(x, y, z)) = T(gx, gy, gz), \\ g(gy) &= g(T(y, x, y)) = T(gy, gx, gy), \\ g(gz) &= g(T(z, y, x)) = T(gz, gy, gx). \end{aligned} \qquad (2.42)$$

Denote $gx = a$, $gy = b$ and $gz = c$, then from (2.42),

$$g(a) = T(a, b, c), \quad g(b) = T(b, a, b), \quad g(c) = T(c, b, a). \qquad (2.43)$$

Thus, (a, b, c) is a tripled coincidence point, it follows that $ga = gx^*, gb = gy^*$ and $gc = gz^*$, that is,

$$g(a) = a, \quad g(b) = b, \quad g(c) = c. \qquad (2.44)$$

From (2.43) and (2.44) ,

$$a = g(a) = T(a, b, c), \quad b = g(b) = T(b, a, b), \quad c = g(c) = T(c, b, a). \qquad (2.45)$$

Therefore, (a, b, c) is a tripled common fixed point of T and g. To prove the uniqueness of the point (a, b, c), assume that (a^*, b^*, c^*) is another tripled common fixed point of T and g. Then we have

$$a^* = ga^* = T(a^*, b^*, c^*), \;\; b^* = gb^* = T(b^*, a^*, b^*), \;\; c^* = gc^* = T(c^*, b^*, a^*).$$

Since (a^*, b^*, c^*) is a tripled coincidence point of T and g, we have $ga^* = gx = a, gb^* = gy = b$ and $gc^* = gz = c$. Thus $a^* = ga^* = ga = a, b^* = gb^* = gb = b$ and $c^* = gc^* = gc = c$, which is the desired result. □

3. Application to integral equations

Here, in this section, we wish to study the existence of solutions to a nonlinear integral equations, as an application to the our tripled fixed point theorem. Consider the integral equations in the following system:

$$\begin{aligned} x(t) &= h(t) + \lambda \int_0^1 k(t,s)[f_1(s,x(s)) + f_2(s,y(s)) + f_3(s,z(s))]ds, \\ y(t) &= h(t) + \lambda \int_0^1 k(t,s)[f_1(s,y(s)) + f_2(s,x(s)) + f_3(s,y(s))]ds, \\ z(t) &= h(t) + \lambda \int_0^1 k(t,s)[f_1(s,z(s)) + f_2(s,y(s)) + f_3(s,x(s))]ds, \end{aligned} \tag{3.1}$$

$t \in I = [0,1]$, $\lambda \geq 0$. Let Γ denote the class of those functions $\gamma : [0,+\infty) \longrightarrow [0,+\infty)$ which $\gamma \in \Phi$ and $(\gamma(t))^p \leq \gamma(t^p)$, for all $p \geq 1$. For example, $\gamma_1(t) = kt$, where $0 \leq k < 1$ and $\gamma_2(t) = \frac{t}{t+1}$ are in Γ.
We will analyze Eq. (3.1) under the following assumptions:

- (a_1) $h : I \longrightarrow \mathbb{R}$ is a continuous function.
- (a_2) $f_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ $(i = 1,2,3)$ are continuous and there exists constant $0 \leq L < 1$ and $\gamma \in \Gamma$ such that for all $x, y \in \mathbb{R}$ and $x \geq y$
  - (i) $0 \leq |f_1(t,x) - f_1(t,y)| \leq L\gamma(x-y)$;
  - (ii) $0 \leq |f_2(t,y) - f_2(t,x)| \leq L\gamma(x-y)$;
  - (iii) $0 \leq |f_3(t,x) - f_3(t,y)| \leq L\gamma(x-y)$.
- (a_3) $k : I \times I \longrightarrow \mathbb{R}$ is continuous in $t \in I$ for every $s \in I$ and measurable in $s \in I$ for all $t \in I$ such that

$$3\int_0^1 k(t,s)ds \leq K,$$

  and $k(t,s) \geq 0$.
- (a_4) There exist $\alpha, \beta, \gamma \in C(I)$ such that

$$\begin{aligned} \alpha(t) &\leq h(t) + \lambda \int_0^1 k(t,s)[f_1(s,\alpha(s)) + f_2(s,\beta(s)) + f_3(s,\gamma(s))]ds, \\ \beta(t) &\geq h(t) + \lambda \int_0^1 k(t,s)[f_1(s,\beta(s)) + f_2(s,\alpha(s)) + f_3(s,\beta(s))]ds, \\ \gamma(t) &\leq h(t) + \lambda \int_0^1 k(t,s)[f_1(s,\gamma(s)) + f_2(s,\beta(s)) + f_3(s,\alpha(s))]ds. \end{aligned}$$

- (a_5) $L^p \lambda^p K^p \leq \frac{1}{2^{3p-3}}$.

Considered the space $X = C(I)$ of continuous functions defined on $I = [0,1]$ with the standard metric given by

$$\rho(x,y) = \sup_{t \in I} |x(t) - y(t)|, \quad for\ x, y \in C(I).$$

This space can also be equipped with a partial order given by

$$x, y \in C(I),\ x \le y \Longleftrightarrow x(t) \le y(t), \quad for\ any\ t \in I.$$

Now for $p \ge 1$, we define

$$\begin{aligned} d(x,y) = (\rho(x,y))^p &= \left(\sup_{t\in I} |x(t) - y(t)| \right)^p \\ &= \sup_{t\in I} |x(t) - y(t)|^p, \quad for\ x, y \in C(I). \end{aligned}$$

It is easy to see that (X, d) is a complete $b-$metric space with $s = 2^{p-1}$ [2]. For any $x, y \in X$ and each $t \in I$, $\max\{x(t), y(t)\}$ and $\min\{x(t), y(t)\}$ belong to X and are upper and lower bounds of x, y, respectively. Therefore, for every $x, y \in X$, one can take $\max\{x, y\}, \min\{x, y\} \in X$ which are comparable to x, y.

Now, we formulate the main result of this section.

Theorem 3.1. *Under assumptions* $(a_1) - (a_5)$*, Eq.* (3.1) *has a solution in* $C(I) \times C(I) \times C(I)$*.*

Proof. We consider the operator $T : X \times X \times X \longrightarrow X$ and $g : X \to X$ defined by

$$\begin{aligned} T(x,y,z)(t) &= h(t) + \lambda \int_0^1 k(t,s)[f_1(s,x(s)) + f_2(s,y(s)) + f_3(s,z(s))]ds, \\ g(x) &= x, \quad t \in I, \end{aligned}$$

for all $x, y, z \in X$. By virtue of our assumptions, T is well defined (this means that if $x, y, z \in X$ then $T(x, y, z) \in X$). For $x_1 \le x_2$ and $t \in I$ we have

$$\begin{aligned} &T(x_1,y,z)(t) - T(x_2,y,z)(t) \\ &= h(t) + \lambda \int_0^1 k(t,s)[f_1(s,x_1(s)) + f_2(s,y(s)) + f_3(s,z(s))]ds \\ &\quad - h(t) - \lambda \int_0^1 k(t,s)[f_1(s,x_2(s)) + f_2(s,y(s)) + f_3(s,z(s))]ds \\ &= \lambda \int_0^1 k(t,s)[f_1(s,x_1(s)) - f_1(s,x_2(s))]ds, \end{aligned}$$

so by (i), we have

$$f_1(s,x_1(s)) - f_1(s,x_2(s)) \le 0,$$

that is,

$$T(x_1,y,z) \le T(x_2,y,z).$$

Now, for $y_1 \le y_2$, $t \in I$ and by (ii), we have

$$\begin{aligned} &T(x, y_1, z)(t) - T(x, y_2, y, z)(t) \\ &= h(t) + \lambda \int_0^1 k(t,s)[f_1(s, x(s)) + f_2(s, y_1(s)) + f_3(s, z(s))]ds \\ &\quad - h(t) - \lambda \int_0^1 k(t,s)[f_1(s, x(s)) + f_2(s, y_2(s)) + f_3(s, z(s))]ds \\ &= \lambda \int_0^1 k(t,s)[f_2(s, y_1(s)) - f_2(s, y_2(s))]ds \ge 0, \end{aligned}$$

that is,

$$T(x, y_1, z) \ge T(x, y_2, z).$$

Similarly we show that, for $z_1 \le z_2$, $t \in I$ and by (iii),

$$T(x, y, z_1) \le T(x, y, z_2).$$

Therefore, T has the mixed g-monotone property. Also, for $x \le u$, $y \ge v$ and $z \le w$, we have

$$\begin{aligned} &|T(x, y, z)(t) - T(u, v, w)(t)| \\ &= |h(t) + \lambda \int_0^1 k(t,s)[f_1(s, x(s)) + f_2(s, y(s)) + f_3(s, z(s))]ds \\ &\quad - h(t) - \lambda \int_0^1 k(t,s)[f_1(s, u(s)) + f_2(s, v(s)) + f_3(s, w(s))]ds| \\ &= \lambda \int_0^1 k(t,s)|f_1(s, x(s)) - f_1(s, u(s))|ds \\ &\quad + \lambda \int_0^1 k(t,s)|f_2(s, y(s)) - f_2(s, v(s))|ds \\ &\quad + \lambda \int_0^1 k(t,s)|f_3(s, z(s)) - f_3(s, w(s))|ds \\ &\le \lambda \int_0^1 k(t,s) L\gamma(u(s) - x(s))ds + \lambda \int_0^1 k(t,s) L\gamma(y(s) - v(s))ds \\ &\quad + \lambda \int_0^1 k(t,s) L\gamma(w(s) - z(s))ds. \end{aligned}$$

Since the function γ is non-decreasing and $x \le u$, $y \ge v$ and $z \le w$, we have

$$\begin{aligned} \gamma(u(s) - x(s)) &\le \gamma(\sup_{t\in I} |x(s) - u(s)|) = \gamma(\rho(x, u)), \\ \gamma(y(s) - v(s)) &\le \gamma(\sup_{t\in I} |y(s) - v(s)|) = \gamma(\rho(y, v)), \\ \gamma(w(s) - z(s)) &\le \gamma(\sup_{t\in I} |z(s) - w(s)|) = \gamma(\rho(z, w)). \end{aligned}$$

Hence

$$\begin{aligned}
&|T(x,y,z)(t)-T(u,v,w)(t)|\\
&\le \lambda\int_0^1 k(t,s)L\gamma(\rho(x,u))ds+\lambda\int_0^1 k(t,s)L\gamma(\rho(y,v))ds\\
&\quad +\lambda\int_0^1 k(t,s)L\gamma(\rho(z,w))ds\\
&\le \lambda L[\gamma(\rho(x,u))+\gamma(\rho(y,v))+\gamma(\rho(z,w))]\int_0^1 k(t,s)ds\\
&\le 3\lambda L\max\{\gamma(\rho(x,u)),\gamma(\rho(y,v)),\gamma(\rho(z,w))\}\times\frac{K}{3}\\
&\le \lambda KL\max\{\gamma(\rho(x,u)),\gamma(\rho(y,v)),\gamma(\rho(z,w))\}.
\end{aligned}$$

Then, we can obtain

$$\begin{aligned}
&d(T(x,y,z),T(u,v,w))\\
&=\sup_{t\in I}|T(x,y,z)(t)-T(u,v,w)(t)|^p\\
&\le \{\lambda KL\max\{\gamma(\rho(x,u)),\gamma(\rho(y,v)),\gamma(\rho(z,w))\}\}^p\\
&\le \lambda^pK^pL^p\max\{\gamma(\rho(x,u))^p,\gamma(\rho(y,v))^p,\gamma(\rho(z,w))^p\}\\
&\le \lambda^pK^pL^p\max\{\gamma(d(x,u),\gamma(d(y,v)),\gamma(d(z,w)\}\\
&\le \lambda^pK^pL^p\varphi\Bigg(\max\Bigg\{d(gx,gu),d(gy,gv),d(gz,gw),\frac{1}{2s}d(T(x,y,z),gu),\\
&\qquad\frac{1}{2s}d(T(z,y,x),gw),\frac{1}{2s}d(T(u,v,w),gx),\frac{1}{2s}d(T(w,v,u),gz)\Bigg\}\Bigg)\\
&\le \frac{1}{2^{3p-3}}\varphi\Bigg(\max\Bigg\{d(gx,gu),d(gy,gv),d(gz,gw),\frac{1}{2s}d(T(x,y,z),gu),\\
&\qquad\frac{1}{2s}d(T(z,y,x),gw),\frac{1}{2s}d(T(u,v,w),gx),\frac{1}{2s}d(T(w,v,u),gz)\Bigg\}\Bigg).
\end{aligned}$$

This proves that the operator T satisfies the contractive condition (2.34) appearing in Corollary 2.2. Also, let α,β,γ be the functions appearing in assumption (a_4); then, by (a_4), we get

$$\alpha\le T(\alpha,\beta,\gamma),\quad \beta\ge T(\beta,\alpha,\beta),\quad \gamma\le T(\gamma,\beta,\alpha).$$

So, the Eq. (3.1) has a solution and the proof is completed. □

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