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STRONG CONVERGENCE FOR BREGMAN RELATIVELY NONEXPANSIVE MAPPINGS IN REFLEXIVE BANACH SPACES AND APPLICATIONS

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Abstract. We study the convergence of two iterative algorithms for finding fixed points of a Bregman relatively nonexpansive mapping in reflexive Banach spaces. We establish two strong convergence theorems and then apply them to the problems of the solutions of convex feasibility, zeros of maximal monotone operator in reflexive Banach spaces.

1. Introduction

Throughout this paper, we always assume that X is a real reflexive Banach space with norm $\|\cdot\|$ and X^* is the topological dual of X endowed with the induced norm $\|\cdot\|_*$. We denote the value of the functional $\xi \in X^*$ at $x \in X$ by $\langle \xi, x\rangle$. The set of nonnegative integers will be denoted by $\mathbb{N}$.

Let C be a nonempty closed convex subset of a Banach space X and let T be a self-mapping of C. A point $p \in C$ is called an asymptotic fixed point of T ([11], [17]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{\mathrm{F}}(T)$ the set of asymptotic fixed points of T. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in C$. It turns out that nonexpansive fixed point theory can be applied to the solution of diverse problems such as finding zeroes of monotone

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operators and solutions to certain evolution equations, and solving convex feasibility, variational inequality and equilibrium problems. In some cases, it is enough to assume that the operator T is relatively nonexpansive, that is, $\|Tx-p\| \leq \|x-p\|$ for all $x \in C$ and $p \in F(T)$, where $F(T)=\{x \in C : Tx = x\} = \hat{\mathrm{F}}(T) \neq \emptyset$. There are many papers that deal with methods for finding fixed points of nonexpansive and relatively nonexpansive operators in Hilbert space and Banach spaces(see [24], [26], [27] and references therein).

Using the metric projection, Nakajo and Takahashi [15] introduced the following hybrid projection iterative algorithm in the frame work of Hilbert spaces: $x_0 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|z-y_n\| \leq \|z-x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \ \ \forall\, n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\} \subset [0,\alpha], \alpha \in [0,1)$ and $P_{C_n \cap Q_n}$ is the metric projection from a Hilbert space H onto $C_n \cap Q_n$. They proved that $\{x_n\}$ generated by (1.1) converges strongly to a fixed point of T. The authors [14] extended Nakajo and Takahashi's theorem to Banach spaces by using relatively nonexpansive mappings.

But we try to extend this theory to Banach spaces we encounter some difficulties because many of the useful examples of nonexpansive operators in Hilbert space are no longer nonexpansive in Banach spaces (for example, the resolvent $R_A = (I+A)^{-1}$ of a maximal monotone operator $A : H \to 2^H$ and the metric projection P_K onto a nonempty, closed and convex subset K of H). There are several ways to overcome these difficulties. One of them is to use the Bregman distance (see Section 2.2) instead of the norm and Bregman (relatively) nonexpansive operators instead of (relatively) nonexpansive operators (see Section 2.3 for more details). The Bregman projection (Section 2.2) and the generalized resolvent (Section 4.2) are examples of Bregman (relatively) nonexpansive operators.

Recently, Reich and Sabach [18] considered common fixed point problems of finitely many Bregman strongly nonexpansive mappings (see Section 2.3) in reflexive Banach spaces by hybrid and shrinking projection iterative algorithm. Suantai-Cho-Cholamjiak [23] studied strong convergence for Bregman strongly nonexpansive mappings by Halpern's iteration in reflexive Banach spaces.

In this paper we are concerned with Bregman relatively nonexpansive mapping. Our main goal is to study the convergence of two iterative algorithms for finding fixed point of Bregman relatively nonexpansive mapping in reflexive

Banach spaces. We establish two strong convergence theorems and then get as corollaries two methods for solving convex feasibility problem and finding zeroes of maximal monotone operator.

2. Preliminaries

In this paper, $f : X \to (-\infty, +\infty]$ is always a proper, lower semicontinuous and convex function. We denote by $domf$ the domain of f, that is, the set $\{x \in X : f(x) < +\infty\}$.

2.1. **Some fact about Legendre functions.** Let $x \in intdomf$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \quad \forall\, y \in X\},$$

where the Fenchel conjugate of f is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}.$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*), \quad \forall\, x \in X, \;\; x^* \in X^*.$$

Furthermore, equality holds if $x^* \in \partial f(x)$ (see also [22], Theorem 23.5).

A function f on X is said to be coercive [12] if $\lim_{\|x\|\to+\infty} f(x) = +\infty$ and f is said to be strongly coercive [25] if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in intdomf$ and $y \in X$, the right-hand derivative of f at x in the direction y is defined by

$$f^\circ(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be Gâteaux differentiable at x if

$$\lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}$$

exists for any y. In this case, $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in intdomf$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset C of X if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $intdomf$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on $intdomf$ (see also [2], [6]). We will need the following result.

Proposition 2.1. ([19], Proposition 2.1) *If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Definition 2.2. ([4], Definition 5.2) The function f is said to be:

(i) essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $dom\partial f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2.3. Let E be a reflexive Banach space. Then we have

(i) $(\partial f)^{-1} = \partial f^*$ (see [6]);
(ii) ef is Legendre if and only if f^* is Legendre (see [4], Corollary 5.5);
(iii) If f is Legendre, then ∇f is a bijection satisfying (see [4], Theorem 5.10 and [18])

$$\nabla f = (\nabla f^*)^{-1}, \quad ran\nabla f = dom\nabla f^* = intdomf^*,$$

and

$$ran\nabla f^* = dom\nabla f = intdomf.$$

Several interesting examples of Legendre functions are presented in [4] and [5]. Among them are the functions $\frac{1}{p}\|\cdot\|^p$ with $p \in (1,\infty)$, where the Banach space X is smooth and strictly convex. In this case the gradient ∇f of f is coincident with the generalized duality mapping of X, i.e., $\nabla f = J_p(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : X \to (-\infty, +\infty]$ is Legendre.

2.2. **Some fact about Bregman distance and totally convex functions.** Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : domf \times intdomf \to [0, +\infty)$ defined as follows:

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle,$$

is called the Bregman distance with respect to f ([10]). The Bregman distance has the following property, called the three point identity: for any $x \in domf$ and $y, z \in intdomf$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{2.1}$$

The modulus of total convexity of f at $x \in intdomf$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : \ y \in domf, \ \|y - x\| = t\}.$$

The function f is called totally convex at x if $\nu_f(x, t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in intdomf$ and is said to be totally convex on bounded sets if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of X and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : intdomf \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap domf\}.$$

We remark that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [9], Theorem 2.10).

Recall that the function f is said to be sequentially consistent [9] if, for any two sequences $\{x_n\} \subset intdomf$ and $\{y_n\} \subset domf$ in X such that the first is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \ \ \Rightarrow \ \ \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

The next proposition turns out to be very useful in the proof of our mail results.

Proposition 2.4. ([8], Lemma 2.1.2) *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Recall that the Bregman projection [7] of $x \in intdomf$ onto the nonempty closed and convex set $C \subset domf$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Similarly to the metric projection in Hilbert space, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Proposition 2.5. ([9], Lemma 2.1.2) *Suppose that f is Gâteaux differentiable and totally convex on $intdomf$. Let $x \in intdomf$ and let $C \subset intdomf$ be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:*

(i) *the vector $\hat{x}$ is the Bregman projection of x onto C with respect to f;*
(ii) *the vector $\hat{x}$ is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0, \quad \forall\, y \in C;$$

(iii) *the vector* $\hat{x}$ *is the unique solution of the inequality*

$$D_f(y,z)+D_f(z,x)\le D_f(y,x),\quad \forall\, y\in C.$$

The following two propositions exhibit two additional properties of totally convex functions.

Proposition 2.6. ([20], Lemma 3.1) *Let* $f:X\to\mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0\in X$ *and the sequence* $\{D_f(x_n,x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is bounded too.*

Proposition 2.7. ([20], Lemma 3.2) *Let* $f:X\to\mathbb{R}$ *be a Gâteaux differentiable and totally convex function,* $x_0\in X$ *and let* C *be a nonempty, closed and convex subset of* X*. Suppose that the sequence* $\{x_n\}$ *is bounded and any weak subsequential limit of* $\{x_n\}$ *belongs to* C*. If* $D_f(x_n,x_0)\le D_f(P_C^f(x_0),x_0)$ *for any* $n\in\mathbb{N}$*, then* $\{x_n\}$ *converges strongly to* $P_C^f(x_0)$*.*

Let $f:X\to\mathbb{R}$ be a convex, Legendre and Gateaux differentiable function. Following [10] and [1], we make use of the function $V_f:X\times X^*\to[0,+\infty)$ associated with f, which is defined by

$$V_f(x,x^*)=f(x)-\langle x^*,x\rangle+f^*(x^*),\quad \forall\, x\in X,\ \ x^*\in X^*.$$

Then V_f is nonnegative and $V_f(x,x^*)=D_f(x,\nabla f^*(x^*))$ for all $x\in X$ and $x^*\in X^*$. Moreover, by the subdifferential inequality,

$$V_f(x,x^*)+\langle y^*,\nabla f^*(x^*)-x\rangle\le V_f(x,x^*+y^*)$$

for all $x\in X$ and $x^*,y^*\in X^*$ (see also [13], Lemmas 3.2 and 3.3). In addition, if $f:X\to(-\infty,+\infty]$ is a proper lower semi-continuous function, then $f^*:E^*\to(-\infty,+\infty]$ is a proper weak* lower semi-continuous and convex function (see [16]). Hence V_f is convex in the second variable. Thus, for all $z\in X$,

$$D_f\bigg(z,\nabla f^*\bigg(\sum_{i=1}^N t_i\nabla f(x_i)\bigg)\bigg)\le\sum_{i=1}^N t_iD_f(z,x_i),$$

where $\{x_i\}_{i=1}^N\subset X$ and $\{t_i\}_{i=1}^N\subset(0,1)$ with $\sum_{i=1}^N t_i=1$.

2.3. **Some fact about Bregman relatively nonexpansive operators.** Let C be a convex subset of $intdomf$ and let T be a self-mapping of C. We say that the operator T is Bregman relatively nonexpansive if $\hat{\mathrm{F}}(T)=F(T)\neq\emptyset$ and

$$D_f(p,Tx)\le D_f(p,x)$$

for all $x\in C$ and $p\in F(T)$. T is Bregman strongly nonexpansive with respect to nonempty $\hat{\mathrm{F}}(T)$ if

$$D_f(p,Tx)\le D_f(p,x)$$

for all $p \in \hat{\mathrm{F}}(T)$ and $x \in C$, and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{\mathrm{F}}(T)$, and

$$\lim_{n\to\infty}(D_f(p,x_n) - D_f(p,Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0.$$

T is Bregman firmly nonexpansive if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle$$

for all $x, y \in C$. It is also known that if T is Bregman firmly nonexpansive, then $F(T)=\hat{\mathrm{F}}(T)$ and $F(T)$ is closed and convex when f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X (see [21]). In this case it also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) =\hat{\mathrm{F}}(T)$. If $F(T) =\hat{\mathrm{F}}(T)$, we can see that Bregman strongly nonexpansive is Bregman relatively nonexpansive operators. So every Bregman firmly nonexpansive mapping is Bregman relatively nonexpansive operators when f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X.

We note, by Reich and Sabach ([21], Lemma 15.5), that $F(T)$ is closed and convex for Bregman relatively nonexpansive mapping when f is Gâteaux differentiable.

3. Main results

In this section we study the following algorithm when $F(T) \neq \emptyset$:

$$\begin{cases} x_1 \in X = C_1, \\ y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)), \\ C_n = \{z \in X : D_f(z, y_n) \leq D_f(z, x_n)\}, \\ Q_n = \{z \in X : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = P^f_{C_n\cap Q_n}(x_1), \quad \forall\, n \geq 1. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let X be a real reflexive Banach space and $f : X \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let T be a Bregman relatively nonexpansive mapping on X such that $F = F(T) =\hat{F}(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence given by* (3.1) *with* $\{\alpha_n\} \subset (0, 1)$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$. *Then the sequence $\{x_n\}$ strongly converges to $P^f_F(x_1)$ as $n \to \infty$.*

Proof. We begin with the following claim.

Claim 1. $C_n \cap Q_n$ is closed and convex for each $n \geq 1$.

It is obvious that Q_n is closed and convex. Observe that the set

$$C_n = \{z \in X : D_f(z, y_n) \leq D_f(z, x_n)\}$$

can be written to

$$C_n = \{z \in X : \langle \nabla f(x_n), z - x_n\rangle - \langle \nabla f(y_n), z - y_n\rangle \leq f(y_n) - f(x_n)\}.$$

It is obvious that C_n is closed for each $n \geq 1$. For z_1, $z_2 \in C_n$ and $t \in (0,1)$, denote $w = tz_1 + (1-t)z_2$, we obtain

$$\begin{aligned}
&\langle \nabla f(x_n), w - x_n\rangle - \langle \nabla f(y_n), w - y_n\rangle \\
&= t\langle \nabla f(x_n), z_1 - x_n\rangle + (1-t)\langle \nabla f(x_n), z_2 - x_n\rangle \\
&\quad - t\langle \nabla f(y_n), z_1 - y_n\rangle - (1-t)\langle \nabla f(y_n), z_2 - y_n\rangle \\
&\leq t(f(y_n) - f(x_n)) + (1-t)(f(y_n) - f(x_n)) \\
&= f(y_n) - f(x_n)
\end{aligned}$$

which implies that $w \in C_n$, so we get C_n is convex. Thus $C_n \cap Q_n$ is closed and convex for each $n \geq 1$.

Claim 2. $F \subset C_n \cap Q_n$ for all $n \geq 1$.

Let $p \in F$. Since T is Bregman relatively nonexpansive, we have

$$\begin{aligned}
D_f(p, y_n) &= D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n))) \\
&= V_f(p, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)) \\
&\leq \alpha_n V_f(p, \nabla f(x_n)) + (1-\alpha_n)V_f(p, \nabla f(Tx_n)) \\
&= \alpha_n D_f(p, \nabla f^*(\nabla f(x_n))) + (1-\alpha_n)D_f(p, \nabla f^*(\nabla f(Tx_n))) \\
&= \alpha_n D_f(p, x_n) + (1-\alpha_n)D_f(p, Tx_n) \\
&\leq D_f(p, x_n).
\end{aligned} \tag{3.2}$$

Hence, we have $p \in C_n$ for all $n \geq 1$. Next we show by induction that $F \subset C_n \cap Q_n$ for all $n \geq 1$. From $Q_1 = X$, we have $F \subset C_1 \cap Q_1$. Suppose $F \subset C_{n-1} \cap Q_{n-1}$ for some $n \geq 2$. We have that $x_n = P^f_{C_{n-1}\cap Q_{n-1}}(x_1)$ is well defined because $C_{n-1} \cap Q_{n-1}$ is nonempty, closed and convex subset of X. So from Proposition 2.5 we obtain

$$\langle \nabla f(x_1) - \nabla f(x_n), y - x_n\rangle \leq 0, \quad \forall\, y \in C_{n-1} \cap Q_{n-1}.$$

Hence we have $F \subset Q_n$. Therefore $F \subset C_n \cap Q_n$ and hence $x_{n+1} = P^f_{C_n \cap Q_n}(x_1)$ is also well defined. Consequently, we see that $F \subset C_n \cap Q_n$ for all $n \geq 1$. Thus the sequence we constructed is indeed well defined, as claimed.

Claim 3. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

It follows from the definition of Q_n and Proposition 2.5 (i) $\Leftrightarrow$ (ii) that $P^f_{Q_n}(x_1) = x_n$. Furthermore, by Proposition 2.5 (i) $\Leftrightarrow$ (iii), for each $p \in F$, we have

$$D_f(x_n, x_1) = D_f(P^f_{Q_n}(x_1), x_1) \leq D_f(p, x_1) - D_f(p, P^f_{Q_n}(x_1)) \leq D_f(p, x_1).$$

Hence the sequence $\{D_f(x_n, x_1)\}_{n\in\mathbb{N}}$ is bounded. By Proposition 2.6 the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too, as claimed.

Claim 4. Every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to F.

Since $x_{n+1} \in Q_n$ and $x_n = P^f_{Q_n}(x_1)$, from the definition of $P^f_{Q_n}$ we have

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1), \quad \forall\, n \geq 1.$$

Thus, $\{D_f(x_n, x_1)\}_{n\in\mathbb{N}}$ is nondecreasing and since it is also bounded (see Claim 3), $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. From $x_n = P^f_{Q_n}(x_1)$ and Proposition 2.5 (i) $\Leftrightarrow$ (iii), we also have

$$D_f(x_{n+1}, P^f_{Q_n}(x_1)) + D_f(P^f_{Q_n}(x_1), x_1) \leq D_f(x_{n+1}, x_1)$$

and hence

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1)$$

for all $n \in \mathbb{N}$. This means that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.3}$$

Proposition 2.4 now implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Since the definition of x_{n+1}, we have $x_{n+1} \in C_n$ and

$$D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n).$$

Hence,

$$\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0. \tag{3.5}$$

It follows from the three point identity (2.2) that

$$\begin{aligned} D_f(x_{n+1}, y_n) = D_f(x_{n+1}, x_n) + D_f(x_n, y_n) \\ + \langle \nabla f(x_n) - \nabla f(y_n), x_{n+1} - x_n \rangle \end{aligned}$$

and hence

$$\begin{aligned} D_f(x_n, y_n) = D_f(x_{n+1}, y_n) - D_f(x_{n+1}, x_n) \\ - \langle \nabla f(x_n) - \nabla f(y_n), x_{n+1} - x_n \rangle. \end{aligned} \tag{3.6}$$

Since f is bounded on bounded subsets of X, ∇f is also bounded on bounded subsets of X (see [8], Proposition 1.1.11). So $\{\nabla f(x_n)\}_{n\in\mathbb{N}}$ and $\{\nabla f(Tx_n)\}_{n\in\mathbb{N}}$ are bounded. Since f is strongly coercive, f^* is bounded on bounded sets (see [25], Lemma 3.6.1 and [4], Theorem 3.3). Hence ∇f^* is also bounded on bounded subsets of X. Therefore $\{y_n\}_{n\in\mathbb{N}}$ is bounded. It follows from (3.3), (3.4), (3.5) and (3.6) that

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0.$$

Proposition 2.4 now implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

It follows from Proposition 2.1 that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0.$$

On the other hand,

$$\begin{aligned}\|\nabla f(x_n) - \nabla f(y_n)\| &= \|\nabla f(x_n) - \alpha_n \nabla f(x_n) - (1-\alpha_n)\nabla f(Tx_n)\| \\ &= (1-\alpha_n)\|\nabla f(x_n) - \nabla f(Tx_n)\|.\end{aligned}$$

From $\limsup_{n\to\infty} \alpha_n < 1$, we have

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(Tx_n)\| = 0. \tag{3.7}$$

Since f is strongly coercive and uniformly convex on bounded subsets of X, f^* is uniformly Fréchet differentiable on bounded subsets of X^* (see [25], Lemma 3.6.2). Moreover, f^* is bounded on bounded sets (see [25], Lemma 3.6.1 and [4], Theorem 3.3). Applying Proposition 2.1 and (3.7), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|\nabla f^*(\nabla f(x_n)) - \nabla f^*(\nabla f(Tx_n))\| = 0.$$

Now let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Then $v \in \hat{\mathrm{F}}(T) = F$.

Claim 5. The sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $P^f_F(x_1)$ as $n \to \infty$.

Let $u = P^f_F(x_1)$. Since $x_{n+1} = P^f_{C_n\cap Q_n}(x_1)$ and F is contained in $C_n \cap Q_n$, we have $D_f(x_{n+1}, x_1) \le D_f(u, x_1)$. Therefore Proposition 2.7 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $u = P^f_F(x_1)$, as claimed. This completes the proof. □

We now present another result which is similar to Theorem 3.1, but with a different construction of the sequence $\{C_n\}_{n\in\mathbb{N}}$. we study the following so-called shrinking projection algorithm when $F = F(T) \neq \emptyset$:

$$\begin{cases} x_1 \in X = C_1, \\ y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)), \\ C_{n+1} = \{z \in C_n : D_f(z, y_n) \le D_f(z, x_n)\}, \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \quad \forall\, n \ge 1. \end{cases} \tag{3.8}$$

Theorem 3.2. *Let X be a real reflexive Banach space and $f : X \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let T be a Bregman relatively nonexpansive mapping on X such that $F = F(T) = \hat{F}(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence given by* (3.8) *with $\{\alpha_n\} \subset (0,1)$ such that* $\limsup_{n\to\infty} \alpha_n < 1$. *Then the sequence $\{x_n\}$ strongly converges to $P_F^f(x_1)$ as $n \to \infty$.*

Proof. Similar to the proof of Theorem 3.1, we can show the following claims:

Claim 1. C_n is closed and convex for each $n \geq 1$.

Claim 2. $F \subset C_n$ for all $n \geq 1$ and hence $P_F^f x_1$ is well defined for $x_1 \in C$.

This can be proved by induction on n. For $n = 1$, we have $F \subset X = C_1$. Assume that $F \subset C_n$ for some $n > 1$. From the induction assumption, (3.2) and the definition of C_{n+1}, we conclude that $F \subset C_{n+1}$ and hence $F \subset C_n$ for all $n \geq 1$.

Claim 3. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

It follows from Proposition 2.5 (i)$\Leftrightarrow$(iii) that, for each $p \in F$, we have

$$D_f(x_n, x_1) = D_f(P_{C_n}^f(x_1), x_1) \leq D_f(p, x_1) - D_f(p, P_{C_n}^f(x_1)) \leq D_f(p, x_1).$$

Hence the sequence $\{D_f(x_n, x_1)\}_{n\in\mathbb{N}}$ is bounded. By Proposition 2.6 the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too, as claimed.

Claim 4. Every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to F.

Since $x_{n+1} \in C_{n+1} \subset C_n$, it follows from Proposition 2.5 (iii) that

$$D_f(x_{n+1}, P_{C_n}^f(x_1)) + D_f(P_{C_n}^f(x_1), x_1) \leq D_f(x_{n+1}, x_1)$$

and hence

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1). \tag{3.9}$$

Therefore the sequence $\{D_f(x_n, x_1)\}_{n\in\mathbb{N}}$ is increasing and since it is also bounded (see Claim 2), $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Thus it follows from (3.9) that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0.$$

Now, using an argument similar to the one we employed in the proof of Theorem 3.1 (see Claim 4 there), we get the conclusion of Claim 4.

Claim 5. The sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $P_F^f(x_1)$ as $n \to \infty$.

Let ũ$= P_F^f(x_1)$. Since $x_n = P_{C_n}^f(x_1)$ and F is contained in C_n, we have $D_f(x_n, x_1) \leq D_f(\tilde{u}, x_1)$. Therefore Proposition 2.7 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to ũ$= P_F^f(x_1)$, as claimed. This completes the proof. □

4. Applications

In this section, we give some applications of Theorem 3.1 and 3.2 in the frame work of reflexive Banach spaces.

4.1. **Convex feasibility problems.** Let K be a nonempty, closed and convex subset of X. The convex feasibility problem is to find an element in K. It is clear that $F(P_K^f) = K$. If the Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X, then the Bregman projection P_K^f is Bregman firmly nonexpansive mapping, hence Bregman relatively nonexpansive mapping, and $F(P_K^f) = \hat{F}(P_K^f)$. Therefore, if we take $T = P_K^f$ in Theorem 3.1 and 3.2, then we get the following algorithms for solving convex feasibility problems.

Corollary 4.1. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X and $f : X \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $\{x_n\}$ be the sequence given by* (3.1) *with $T = P_K^f$. If $\{\alpha_n\} \subset (0, 1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ strongly converges to $P_K^f(x_1)$ as $n \to \infty$.*

Corollary 4.2. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X and $f : X \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $\{x_n\}$ be the sequence given by* (3.8) *with $T = P_K^f$. If $\{\alpha_n\} \subset (0, 1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ strongly converges to $P_K^f(x_1)$ as $n \to \infty$.*

4.2. **Zeros of maximal monotone operators.** Let $A : X \to 2^{X^*}$ be a set-valued mapping. The domain of A is denoted by $\text{dom}A = \{x \in X : Ax \neq \emptyset\}$ and also the graph of A is denote by $G(A) = \{(x, x^*) \in X \times 2^{X^*} : x^* \in Ax\}$. A is said to be monotone if $\langle x^* - y^*, x - y\rangle \geq 0$ for each $(x, x^*), (y, y^*) \in G(A)$. It is said to be maximal monotone if its graph is not contained in the graph of any other monotone operators on X. It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{z \in X : 0^* \in Az\}$ is closed and convex. The problem of finding an element $x \in A^{-1}(0^*)$ is very important in optimization theory and related fields. In this section we present two different algorithms for finding zeros of maximal monotone operator.

The resolvent of A, denoted by $Res_A^f : X \to 2^{X^*}$, is defined as follows [3]:

$$Res_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

It is known that $F(Res_A^f) = A^{-1}(0^*)$, and Res_A^f is single-valued and Bregman firmly nonexpansive (see [3]). In addition, if f is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X, then $F(Res_A^f) = \hat{F}(Res_A^f)$ (see [21]). If we take $T = Res_A^f$ in Theorem 3.1 and 3.2, then we obtain two algorithms for finding a zero of a maximal monotone operator. Note that since A is a maximal monotone operator, $X^* = ran(\nabla f) = ran(\nabla f + A)$ (see [20]) and therefore T is defined on all of X.

Corollary 4.3. *Let $A : X \to 2^{X^*}$ be a maximal monotone operator with $Z = A^{-1}(0^*) \neq \emptyset$ and $f : X \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $\{x_n\}$ be the sequence given by (3.1) with $T = Res_A^f$. If $\{\alpha_n\} \subset (0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ strongly converges to $P_Z^f(x_1)$ as $n \to \infty$.*

Corollary 4.4. *Let $A : X \to 2^{X^*}$ be a maximal monotone operator with $Z = A^{-1}(0^*) \neq \emptyset$ and $f : X \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $\{x_n\}$ be the sequence given by (3.8) with $T = Res_A^f$. If $\{\alpha_n\} \subset (0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ strongly converges to $P_Z^f(x_1)$ as $n \to \infty$.*

Remark 4.5. Theorem 3.1 and 3.2 can be applied to equilibrium problems, variational inequality problems and the problems of finding zeros of Bregman inverse strongly monotone operators in reflexive Banach spaces (see Sections 6,7,8 in [18]).

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