Nonlinear Functional Analysis and Applications
Vol. 20, No. 3 (2015), pp. 381-392

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

A NEW EXPLICIT ITERATIVE ALGORITHM FOR SOLVING SPLIT VARIATIONAL INCLUSION PROBLEM

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Abstract. In this paper, we introduce a new explicit iterative algorithm to approximate a common solution of split variational inclusion problem and fixed point problem for a nonexpansive mapping in Hilbert spaces. Further, we proved that the sequence generated by the iterative method strongly converges to the solution of the split variational inclusion problem.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let S is a nonexpansive operator. The set of fixed points of S is denoted by $Fix(S)$.

Throughout this paper, we suppose that $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two multi-valued maximal monotone operators, $A : H_1 \to H_2$ is a bounded linear operator. In 2011, Moudafi [10] introduced the following *split monotone variational inclusion problem* (SMVIP) which is to find $x^* \in H_1$ such that

$$0 \in f(x^*) + B_1(x^*), \tag{1.1}$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in g(y^*) + B_2(y^*), \tag{1.2}$$

[0]Received November 27, 2014. Revised February 24, 2015.
[0]2010 Mathematics Subject Classification: 47H09, 47H10, 49J40.
[0]Keywords: Nonexpansive mapping, maximal monotone mapping, split variational inclusion, k-strongly monotone.

where $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given single-valued operators. This formalism is also at the core of modeling of many inverse problems arising from phase retrieval, data compression, sensor networks and other real-world problem, see [2, 3]. As Moudafi notes in [10], SMVIP (1.1)-(1.2) includes as the following special cases, split minimization problem, split saddle-point problem and split equilibrium problem which have already been studied and used in practice as a model in some field, see e.g. [4, 5].

If $f \equiv 0$ and $g \equiv 0$, then SMVIP(1.1)-(1.2) reduces to the following *split variational inclusion problem (in short, SVIP)* which is to find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.3}$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in B_2(y^*). \tag{1.4}$$

We denote the solution set of SVIP(1.3), SVIP(1.4) by $SOLVIP(B_1)$ and $SOLVIP(B_2)$, respectively. The solution set of SVIP (1.3)-(1.4) is denoted by $\Gamma = \{x^* \in H_1 : x^* \in SOLVIP(B_1) \text{ and } Ax^* \in SOLVIP(B_2)\}$.

In 2013, Kazmi and Rizvi [9] proposed the following algorithm:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n. \end{cases} \tag{1.5}$$

If $\lambda > 0$, the sequences $\{u_n\}$ and $\{x_n\}$ generated by (1.5) both converge strongly to $\tilde{x} \in Fix(S) \bigcap \Gamma$, where $\tilde{x} = P_{Fix(S)\bigcap\Gamma} f(\tilde{x})$.

On the other hand, Tian [11] considered the following general viscosity type iterative method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n. \tag{1.6}$$

He proved that $\{x_n\}$ generated by (1.6) converges strongly to a fixed point $\tilde{x} \in Fix(T)$, which solves the variational inequality

$$\langle(\gamma f - \mu F)\tilde{x}, x - \tilde{x}\rangle \le 0, \quad \forall\, x \in Fix(T).$$

In [12], Tian generalized the iterative method (1.6) replacing the contraction operator f with an Lipschitzian continuous operator V to solve the following variational inequality

$$\langle(\gamma V - \mu F)\tilde{x}, x - \tilde{x}\rangle \le 0, \quad \forall\, x \in Fix(T).$$

In this paper, we will combine the iterative method (1.6) with the method (1.5) and consider the following general iterative method

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n \gamma V(x_n) + (I - \mu\alpha_n F)Su_n. \end{cases} \tag{1.7}$$

We will prove that if the parameters satisfy appropriate conditions, the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution $\tilde{x}$ of the variational inequality

$$\langle(\gamma V - \mu F)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall\, x \in Fix(S) \cap \Gamma. \tag{1.8}$$

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x \to x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and converges strongly to x, respectively. Let H be a real Hilbert space. The following definitions and lemmas are useful for our paper.

Definition 2.1. A mapping $T : H \to H$ is said to be
• nonexpansive if for all $x, y \in H$

$$\|Tx - Ty\| \leq \|x - y\|,$$

• firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently for all $x, y \in H$

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2,$$

• monotone if for all $x, y \in H$

$$\langle Tx - Ty, x - y\rangle \geq 0,$$

• η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in H$,
• a multi-valued mapping $M : H \to 2^H$ is called monotone if

$$\langle u - v, x - y\rangle \geq 0 \;\; whenever \;\; u \in M(x), \;\; v \in M(y),$$

• a multi-valued mapping $M : H \to 2^H$ is maximal if, in addition, its graph,

$$gph\ M := \{(x, y) \in H \times H : y \in M(x)\},$$

is not properly contained in the graph of any other monotone operator.

It is well known that every nonexpansive operator $T : H \to H$ satisfies the inequality

$$\langle(x - Tx) - (y - Ty), Ty - Tx\rangle \leq \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2, \tag{2.1}$$

for all $(x, y) \in H \times H$, and therefore, we get, for all $(x, y) \in H \times Fix(T)$,

$$\langle x - Tx, y - Tx\rangle \leq \frac{1}{2}\|Tx - x\|^2, \tag{2.2}$$

see e.g., ([7], Theorem 3.1) and ([6], Theorem 2.1).

Definition 2.2. A mapping T is said to be an averaged mapping on a real Hilbert space H if there exists some number $\alpha \in (0,1)$ such that

$$T = (1-\alpha)I + \alpha S, \tag{2.3}$$

where $I : H \to H$ is the identity mapping and $S : H \to H$ is nonexpansive. More precisely, when (2.3) holds, we say that T is α-averaged.

Lemma 2.3. ([12]) *Let H be a real Hilbert space, let $V : H \to H$ be a α-Lipschitzian operator with $\alpha > 0$, and let $F : H \to H$ be a k-Lipschitzian continuous operator and η-strongly monotone operator with $k > 0$, $\eta > 0$. Then, for $0 < \gamma < \frac{\mu\eta}{\alpha}$, $\mu F - \gamma V$ is strongly monotone with coefficient $\mu\eta - \gamma\alpha$.*

Lemma 2.4. ([8]) *Assume that T is nonexpansive self mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I-T)x_n\}$ converges strongly to some y, it follows that $(I-T)x = y$. Here I is the identity mapping on H.*

Lemma 2.5. ([13]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. ([2]) *In a Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad x, y \in H.$$

Lemma 2.7. ([9]) *Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the resolvent mapping $J_\lambda^M : H \to H$ is defined by*

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \quad \forall\, x \in H,$$

for some $\lambda > 0$, the resolvent operator J_λ^M is single-valued and firmly nonexpansive. It is easily deduced that J_λ^M is nonexpansive and $\frac{1}{2}$-averaged.

Lemma 2.8. ([14]) *Assume S is a λ-strictly pseudo-contractive mapping on a Hilbert space H. Define a mapping T by $Tx = \alpha x + (1-\alpha)Sx$ for all $x \in H$ and $\alpha \in [\lambda, 1)$. Then T is a nonexpansive mapping such that $Fix(T) = Fix(S)$.*

3. Main results

Now we state and prove our main results in this paper.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Suppose that V is α-Lipschitzian continuous on H_1 with coefficient $\alpha > 0$ and $F : H_1 \to H_1$ is a k-Lipschitzian continuous and η-strongly monotone operator with $k > 0$ and $\eta > 0$. Let S is a nonexpansive mapping such that $Fix(S) \cap \Gamma \neq \emptyset$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \tau/\alpha$ with $\tau = \mu(\eta - \mu k^2/2)$. Suppose that $\lambda > 0$ and $\beta \in (0, 1/L)$ where L is the spectral radius of the operator A^*A and A^* is the adjoint of A. If $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then for a given $x_0 \in H_1$ arbitrarily, the sequences $\{u_n\}$ and $\{x_n\}$ be generated by (1.7) both converge strongly to the unique solution $\tilde{x}$ of the variational inequality (1.8), where $\tilde{x} = P_{Fix(S)\cap\Gamma}(I - \mu F + \gamma V)\tilde{x}$.

Proof. In fact, using Lemma 2.3, we know that $\mu F - \gamma V$ is strongly monotone. So, the variational inequality (1.8) has only one solution. We set $\tilde{x}$ to indicate the unique solution of (1.8). The variational inequality (1.8) can be written as

$$\langle (I - \mu F + \gamma V)\tilde{x} - \tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall\, x \in Fix(S) \cap \Gamma.$$

So, it is equivalent to the fixed point equation

$$P_{Fix(S)\cap\Gamma}(I - \mu F + \gamma V)\tilde{x} = \tilde{x}.$$

Take a $p \in Fix(S) \cap \Gamma$, then we have $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $Sp = p$. Using the fact that $J_\lambda^{B_1}$ is nonexpansive, we have

$$\begin{aligned}
&\|u_n - p\|^2 \\
&= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - p\|^2 \\
&= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - J_\lambda^{B_1} p\|^2 \\
&\leq \|x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\
&= \|x_n - p\|^2 + \beta^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 + 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle.
\end{aligned} \tag{3.1}$$

It follows that

$$\begin{aligned}
\|u_n - p\|^2 \leq \|x_n - p\|^2 &+ \beta^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \\
&+ 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle.
\end{aligned} \tag{3.2}$$

Now, we have

$$\beta^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \leq L\beta^2\|(J_\lambda^{B_2} - I)Ax_n\|^2. \tag{3.3}$$

Denoting $\Lambda = 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle$ and using (2.2), we have

$$\begin{aligned}\Lambda &= 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\beta\langle A(x_n - p), (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\beta\langle A(x_n - p) + (J_\lambda^{B_2} - I)Ax_n - (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\beta\{\langle J_\lambda^{B_2}Ax_n - Ap, (J_\lambda^{B_2} - I)Ax_n\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2\} \\ &\leq 2\beta\{\frac{1}{2}\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|(J_\lambda^{B_2} - I)Ax_n\|^2\} \\ &\leq -\beta\|(J_\lambda^{B_2} - I)Ax_n\|^2.\end{aligned} \tag{3.4}$$

Using (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \beta(L\beta - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2.\end{aligned} \tag{3.5}$$

Since $\beta \in (0, \frac{1}{L})$, we have

$$\|u_n - p\| \leq \|x_n - p\|. \tag{3.6}$$

Next, we estimate

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n\gamma Vx_n + (I - \mu\alpha_n F)Su_n - p\| \\ &= \|\alpha_n\gamma Vx_n - \alpha_n\gamma Vp + \alpha_n\gamma Vp + (I - \mu\alpha_n F)Su_n \\ &\quad - (I - \mu\alpha_n F)Sp - \mu\alpha_n Fp\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + (1 - \alpha_n\tau)\|u_n - p\| + \alpha_n\|\gamma Vp - \mu Fp\| \\ &\leq \big(1 - \alpha_n(\tau - \gamma\alpha)\big)\|x_n - p\| + \alpha_n\|\gamma Vp - \mu Fp\| \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma\alpha}\right\} \\ &\ \ \vdots \\ &\leq \max\left\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma\alpha}\right\}.\end{aligned} \tag{3.7}$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$, $\{Vx_n\}$ and $\{Su_n\}$ are bounded.

Next, we show that the sequence $\{x_n\}$ is asymptotically regular, i.e., $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. It follows from (1.7) that

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \|\alpha_n\gamma Vx_n + (I-\mu\alpha_n F)Su_n - \alpha_{n-1}\gamma Vx_{n-1} \\
&\quad - (I-\mu\alpha_{n-1}F)Su_{n-1}\| \\
&= \|\alpha_n\gamma Vx_n - \alpha_n\gamma Vx_{n-1} + \alpha_n\gamma Vx_{n-1} - \alpha_{n-1}\gamma Vx_{n-1} \\
&\quad + (I-\mu\alpha_n F)Su_n - (I-\mu\alpha_n F)Su_{n-1} + (I-\mu\alpha_n F)Su_{n-1} \\
&\quad - (I-\mu\alpha_{n-1}F)Su_{n-1}\| \\
&\le \alpha_n\gamma\alpha\|x_n - x_{n-1}\| + (1-\alpha_n\tau)\|u_n - u_{n-1}\| \\
&\quad + \gamma|\alpha_n - \alpha_{n-1}|\|Vx_{n-1}\| + \mu|\alpha_n - \alpha_{n-1}|\|FSu_{n-1}\| \\
&\le \alpha_n\gamma\alpha\|x_n - x_{n-1}\| + (1-\alpha_n\tau)\|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}|K,
\end{aligned} \tag{3.8}$$

where $K := \sup\{\gamma\|Vx_{n-1}\| + \mu\|FSu_{n-1}\| : n \in N\}$. Since, for $\beta \in (0, \frac{1}{L})$, the mapping $J_\lambda^{B_1}(I + \beta A^*(J_\lambda^{B_2} - I)A)$ is nonexpansive, we have

$$\begin{aligned}
&\|u_n - u_{n-1}\| \\
&= \|J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1}(x_{n-1} + \beta A^*(J_\lambda^{B_2} - I)Ax_{n-1})\| \\
&\le \|J_\lambda^{B_1}(I + \beta A^*(J_\lambda^{B_2} - I)A)x_n - J_\lambda^{B_1}(I + \beta A^*(J_\lambda^{B_2} - I)A)x_{n-1}\| \\
&\le \|x_n - x_{n-1}\|.
\end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\|x_{n+1} - x_n\| \le (1 - \alpha_n(\tau - \gamma\alpha))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K.$$

By Lemma 2.5 with

$$a_n := \|x_n - x_{n-1}\|, \quad \gamma_n := \alpha_n(\tau - \gamma\alpha)$$

and

$$\delta_n := |\alpha_n - \alpha_{n-1}|K,$$

we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$. By Lemma 2.6, we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n\gamma Vx_n + (I - \mu\alpha_n F)Su_n - p\|^2 \\
&= \|\alpha_n(\gamma Vx_n - \mu Fp) + (I - \mu\alpha_n F)Su_n - (I - \mu\alpha_n F)Sp\|^2 \\
&\le (1 - \alpha_n\tau)^2\|u_n - p\|^2 + 2\alpha_n\langle\gamma Vx_n - \mu Fp, x_{n+1} - p\rangle.
\end{aligned} \tag{3.11}$$

By (3.5), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\tau)^2\big(\|x_n-p\|^2+\beta(L\beta-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\big)\\ &\quad+2\alpha_n\langle\gamma Vx_n-\mu Fp, x_{n+1}-p\rangle\\ &\le \|x_n-p\|^2+\beta(L\beta-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\ &\quad+2\alpha_n\|\gamma Vx_n-\mu Fp\|\|x_{n+1}-p\|.\end{aligned}\tag{3.12}$$

Then it follows that

$$\begin{aligned}&\beta(1-L\beta)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\|\gamma Vx_n-\mu Fp\|\|x_{n+1}-p\|\\ &\le \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\alpha_n\|\gamma Vx_n-\mu Fp\|\|x_{n+1}-p\|.\end{aligned}$$

Since $1-L\beta>0$, $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\|(J_\lambda^{B_2}-I)Ax_n\|=0.\tag{3.13}$$

Furthermore, using (3.5) and $\beta\in(0,\frac{1}{L})$, we observe that

$$\begin{aligned}\|u_n-p\|^2 &= \|J_\lambda^{B_1}\big(x_n+\beta A^*(J_\lambda^{B_2}-I)Ax_n\big)-p\|^2\\ &= \|J_\lambda^{B_1}\big(x_n+\beta A^*(J_\lambda^{B_2}-I)Ax_n\big)-J_\lambda^{B_1}p\|^2\\ &\le \langle u_n-p, x_n+\beta A^*(J_\lambda^{B_2}-I)Ax_n-p\rangle\\ &= \frac{1}{2}\{\|u_n-p\|^2+\|x_n+\beta A^*(J_\lambda^{B_2}-I)Ax_n-p\|^2\\ &\quad-\|(u_n-p)-\big(x_n+\beta A^*(J_\lambda^{B_2}-I)Ax_n-p\big)\|^2\}\\ &\le \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2+\beta(L\beta-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\ &\quad-\|u_n-x_n-\beta A^*(J_\lambda^{B_2}-I)Ax_n\|^2\}\\ &\le \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-\big(\|u_n-x_n\|^2\\ &\quad+\beta^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2-2\beta\langle u_n-x_n, A^*(J_\lambda^{B_2}-I)Ax_n\rangle\big)\}\\ &\le \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2\\ &\quad+2\beta\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|u_n-p\|^2 &\le \|x_n-p\|^2-\|u_n-x_n\|^2\\ &\quad+2\beta\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|.\end{aligned}\tag{3.14}$$

It follows from (3.11) and (3.14) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\tau)^2\|u_n-p\|^2+2\alpha_n\langle\gamma Vx_n-\mu Fp,x_{n+1}-p\rangle\\ &\le (1-\alpha_n\tau)^2\{\|x_n-p\|^2-\|u_n-x_n\|^2\\ &\quad +2\beta\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\}\\ &\quad +2\alpha_n\langle\gamma Vx_n-\mu Fp,x_{n+1}-p\rangle\\ &\le \|x_n-p\|^2-\|u_n-x_n\|^2+2\beta\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\\ &\quad +2\alpha_n\|\gamma Vx_n-\mu Fp\|\|x_{n+1}-p\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|u_n-x_n\|^2 &\le \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)\\ &\quad +2\beta\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\\ &\quad +2\alpha_n\|\gamma Vx_n-\mu Fp\|\|x_{n+1}-p\|.\end{aligned}$$

Since $\alpha_n\to 0$ as $n\to\infty$, from (3.10) and (3.13), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.15}$$

Next, we estimate

$$\begin{aligned}\|x_{n+1}-Su_n\| &= \|\alpha_n\gamma Vx_n+(I-\mu\alpha_nF)Su_n-Su_n\|\\ &= \alpha_n\|\gamma Vx_n-\mu FSu_n\|.\end{aligned}$$

Since $\alpha_n\to 0$ as $n\to\infty$, we get

$$\lim_{n\to\infty}\|x_{n+1}-Su_n\|=0. \tag{3.16}$$

Now, we can write

$$\|Su_n-u_n\|\le\|Su_n-x_{n+1}\|+\|x_{n+1}-x_n\|+\|x_n-u_n\|.$$

By (3.10), (3.15) and (3.16), it follows that

$$\lim_{n\to\infty}\|Su_n-u_n\|=0.$$

Since $\{u_n\}$ is bounded, so, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu F)\tilde{x},u_n-\tilde{x}\rangle=\lim_{j\to\infty}\langle(\gamma V-\mu F)\tilde{x},u_{n_j}-\tilde{x}\rangle$$

and $u_{n_j}\rightharpoonup u^*$. Now, S being nonexpansive, by Lemma 2.4, we obtain that $u^*\in Fix(S)$. On the other hand, by Lemma 2.7,

$$u_{n_j}=J_\lambda^{B_1}(x_{n_j}+\beta A^*(J_\lambda^{B_2}-I)Ax_{n_j})$$

can be rewritten as

$$\frac{(x_{n_j}-u_{n_j})+\beta A^*(J_\lambda^{B_2}-I)Ax_{n_j}}{\lambda}\in B_1u_{n_j}. \tag{3.17}$$

By passing to limit $j \to \infty$ in (3.17) and by taking into account (3.13), (3.15) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(u^*)$, i.e., $u^* \in SOLVIP(B_1)$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $\{Ax_{n_j}\}$ weakly converges to Au^*. Again, using the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive, from (3.13) and Lemma 2.4, we obtain that $0 \in B_2(Au^*)$, i.e., $Au^* \in SOLVIP(B_2)$. Thus $u^* \in Fix(S) \cap \Gamma$. Hence

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, Su_n - \tilde{x}\rangle &= \limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, u_n - \tilde{x}\rangle \\ &= \lim_{j\to\infty}\langle(\gamma V - \mu F)\tilde{x}, u_{n_j} - \tilde{x}\rangle \\ &= \langle(\gamma V - \mu F)\tilde{x}, u^* - \tilde{x}\rangle \\ &\le 0.\end{aligned} \tag{3.18}$$

Finally, we show that $x_n \to \tilde{x}$. Since $\tilde{x} \in Fix(S) \cap \Gamma$, we have

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ &= \|\alpha_n \gamma V x_n + (I - \mu\alpha_n F)Su_n - \tilde{x}\|^2 \\ &= \|\alpha_n(\gamma V x_n - \mu F\tilde{x}) + (I - \mu\alpha_n F)Su_n - (I - \mu\alpha_n F)S\tilde{x}\|^2 \\ &\le (1 - \alpha_n\tau)^2\|u_n - \tilde{x}\|^2 + \alpha_n^2\|\gamma V x_n - \mu F\tilde{x}\|^2 \\ &\quad + 2\alpha_n\langle \gamma V x_n - \mu F\tilde{x}, (I - \mu\alpha_n F)Su_n - (I - \mu\alpha_n F)S\tilde{x}\rangle \\ &\le (1 - \alpha_n\tau)^2\|u_n - \tilde{x}\|^2 + \alpha_n^2\|\gamma V x_n - \mu F\tilde{x}\|^2 \\ &\quad + 2\alpha_n\big(\langle Su_n - S\tilde{x}, \gamma V x_n - \mu F\tilde{x}\rangle \\ &\quad - \alpha_n\mu\langle FSu_n - F\tilde{x}, \gamma V x_n - \mu F\tilde{x}\rangle\big).\end{aligned}$$

By (3.6), we obtain

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ &\le \big((1 - \alpha_n\tau)^2 + 2\alpha_n\gamma\alpha\big)\|x_n - \tilde{x}\|^2 + \alpha_n^2\|\gamma V x_n - \mu F\tilde{x}\|^2 \\ &\quad + 2\alpha_n(\langle Su_n - S\tilde{x}, \gamma V\tilde{x} - \mu F\tilde{x}\rangle \\ &\quad - \alpha_n\mu\langle FSu_n - F\tilde{x}, \gamma V x_n - \mu F\tilde{x}\rangle) \\ &= \big(1 - 2\alpha_n(\tau - \gamma\alpha)\big)\|x_n - \tilde{x}\|^2 + \alpha_n\big(\alpha_n\|\gamma V x_n - \mu F\tilde{x}\|^2 \\ &\quad + 2\langle Su_n - \tilde{x}, \gamma V\tilde{x} - \mu F\tilde{x}\rangle - 2\alpha_n\mu\langle FSu_n - F\tilde{x}, \gamma V x_n - \mu F\tilde{x}\rangle \\ &\quad + \alpha_n\tau^2\|x_n - \tilde{x}\|^2\big) \\ &= (1 - \bar{\alpha_n})\|x_n - \tilde{x}\|^2 + \bar{\alpha_n}\bar{\beta_n},\end{aligned}$$

where $\bar{\alpha_n} = 2\alpha_n(\tau - \gamma\alpha)$ and

$$\bar{\beta_n} = \frac{1}{2(\tau - \gamma\alpha)}\Big(2\langle Su_n - \tilde{x}, \gamma V\tilde{x} - \mu F\tilde{x}\rangle - 2\alpha_n\mu\|FSu_n - F\tilde{x}\|\|\gamma Vx_n - \mu F\tilde{x}\| + \alpha_n\|\gamma Vx_n - \mu F\tilde{x}\|^2 + \alpha_n\tau^2\|x_n - \tilde{x}\|^2\Big).$$

Consequently, according to the condition (i) and (ii), (3.18) and Lemma 2.5, we conclude that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

4. An extension of our result

In this section, we extend our result to the more broad λ-strictly pseudo-contractive mapping. It is well-known that a mapping $S : H_1 \to H_1$ is said to be λ-strictly pseudo-contractive if there exists a constant $\lambda \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \lambda\|(I - S)x - (I - S)y\|^2, \quad \forall\, x, y \in H_1.$$

Define the operator

$$\hat{T} = \omega I + (1 - \omega)S, \tag{4.1}$$

where $0 \leq \lambda \leq \omega < 1$. By virtue of Lemma 2.8, we know that $\hat{T}$ is a nonexpansive operator and $Fix(T) = Fix(S)$. Thus we extend theorem 3.1 to a λ-strictly pseudo-contractive mapping.

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert spaces. Suppose that V is α-Lipschitzian continuous on H_1 with coefficient $\alpha > 0$ and $F : H_1 \to H_1$ a k-Lipschitzian continuous and η-strongly monotone operator with $k > 0$, $\eta > 0$. Let S be a λ-strictly pseudo-contractive mapping on H_1 such that $Fix(S) \cap \Gamma \neq \emptyset$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \tau/\alpha$ with $\tau = \mu(\eta - \mu k^2/2)$. Suppose that $\lambda > 0$ and $\beta \in (0, 1/L)$ where L is the spectral radius of the operator A^*A and A^* is the adjoint of A. If the condition* (i)-(iii) *of Theorem* 3.1 *are satisfied, then the sequence $\{x_n\}_{n\geq 0}$ and $\{u_n\}_{n\geq 0}$ defined by* (1.7) *with S replaced by $\hat{T}$ in* (4.1)*, converges strongly to the unique solution $\tilde{x}$ of the following variational inequality:*

$$\langle(\gamma V - \mu F)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall\, x \in Fix(S) \cap \Gamma,$$

where $\tilde{x} = P_{Fix(S)\cap\Gamma}(I - \mu F + \gamma V)\tilde{x}$.

Acknowledgments: This research is supported by the Fundamental Science Research Funds for the Central Universities (Program No. 3122014k010).

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