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EXISTENCE OF WEAK SOLUTIONS FOR THE QUASI-STATE FLOW OF BLOOD AND MATHEMATICAL COAGULATION MODELLING

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Abstract. We consider a mathematical model that describes the quasi-state flow of blood involving the Bingham model. We derive a weak formulation of the system consisting of a stationary motion equation, a convection-diffusion-reaction equation and an energy conservation equation. We prove the existence of weak solutions and some properties of the solutions. We also study the mathematical modelling of blood coagulation, for which we prove a maximum principle.

1. Introduction

The study of blood flow is complicated in many aspects and thus simplifying assumptions are often made. Plasma behaves as a Newtonian fluid, but whole blood has non-Newtonian properties. In the large vessels where shear rates are high enough, it is also reasonable to assume that blood has a constant viscosity and a Newtonian behaviour. However, in smaller vessels, or in some diseased conditions, the presence of the cells induces low shear rate and whole blood exhibits remarkable non-Newtonian characteristics, like shear-thinning viscosity, thixotropy, viscoelasticity and possibly a yield stress. In particular, at rest or at low shear rates, blood seems to have a high apparent viscosity due

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to red blood cells (RBCs) aggregation into clusters, called “rouleaux”. While at high shear rates the cells become disaggregated and deform into an infinite variety of shapes without changing volume (deformability of RBCs), resulting in a reduction in the blood's viscosity. The deformed RBCs align with the flow field and tend to slide upon plasma layers formed in between. Attempts to recognize the shear-thinning nature of blood were initiated by Chien et al. [9, 10] in the 1960s. Empirical models like the power-law, Cross, Herschel-Bulkely, Bingham, Carreau or W-S generalized Newtonian fluid models, see. [3, 4, 28], have been obtained by fitting experimental data in one dimensional flow. Recently, Vlastos, Lerche and Koch [27] proposed a modified Carreau equation to capture the shear dependence of blood viscosity. The model of Bingham has been frequently used to describe the bahaviour of blood due to the yield limit and blockage phenomenon which can describe the blood coagulation. Such model is quite sensitive to a number of factors including hematocrit, temperature, plasma viscosity, exercise level and gender or disease state. Some numerical results concerning blood flow through a stenosis artery has been obtained in [1]. Convection-diffusion-reaction phenomena in the study of coagulation and formation blood clots are the topic of numerous papers, e.g. [2, 3, 23]. Reviews detailing the structure of the blood coagulation system are available elsewhere [4, 25].

The aim of this paper is to study the flow of blood involving the non-Newtonian model of Bingham. The paper is organized as follows. In Section 2, we present the bio-mechanical problem of blood flow. The problem is modelled by a mathematical system consisting of a motion equation for the incompressible viscous fluid of Bingham, a convection-diffusion-reaction equation and an energy conservation equation. Moreover, we introduce some notations and preliminaries. In Section 3, we derive the variational formulation of the quasi-state problem. We prove in Section 4 the existence of weak solutions and some properties of the solutions. Section 5 is devoted to the mathematical study of blood coagulation, for which we prove a maximum principle for the concentration.

2. Problem statement

Let $T > 0$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary Γ and let $Q = \Omega \times (0,T)$. We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \ \ \forall\, \mathbf{u},\mathbf{v}\in\mathbb{R}^n \ \text{ and } \ \sigma\cdot\tau = \sigma_{ij}\tau_{ij}, \ \ \forall\, \sigma,\tau\in\mathbb{S}_n,$$

$$|\mathbf{u}| = (\mathbf{u}\cdot\mathbf{u})^{\frac{1}{2}}, \ \ \forall\, \mathbf{u}\in\mathbb{R}^n \ \text{ and } \ |\sigma| = (\sigma\cdot\sigma)^{\frac{1}{2}}, \ \ \forall\, \sigma\in\mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We denote by $\tilde{\sigma}$ the deviator of $\sigma = (\sigma_{ij})$ given by

$$\tilde{\sigma} = (\tilde{\sigma}_{ij}), \quad \tilde{\sigma}_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{n}\delta_{ij},$$

We consider the rate of deformation operator defined for every $\mathbf{u} \in H^1(\Omega)^n$ by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

ν denotes the unit outward normal vector on the boundary Γ.

The bio-mechanical setting is the following. We adopt the viscous and incompressible Bingham fluid to describe the behaviour of blood, the domain Ω represents a part of the vascular system (an artery, a vessel or a simple vein). The fluid is acted upon by given volume forces of density $\mathbf{f}$. In addition, we admit a possible external heat source given by function h. The velocity is supposed equal to zero on $\Gamma \times (0,T)$. We suppose that on $\Gamma \times (0,T)$ the concentration and temperature are given by an homogeneous Neumann and a Fourier boundary conditions, respectively.

The quasi-state bio-mechanical problem may be formulated as follows.

Problem P1. Find the velocity field $\mathbf{u} = (u_i) : Q \longrightarrow \mathbb{R}^n$, the stress field $\sigma = (\sigma_{ij}) : Q \longrightarrow \mathbb{S}_n$, the concentration $C : Q \longrightarrow \mathbb{R}$ and the temperature $\theta : Q \longrightarrow \mathbb{R}$ such that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{div}(\sigma) + \mathbf{f} \text{ in } Q, \tag{2.1}$$

$$\begin{cases} \tilde{\sigma} = 2\mu(C,\theta)\,\varepsilon(\mathbf{u}) + g(C,\theta)\dfrac{\varepsilon(\mathbf{u})}{|\varepsilon(\mathbf{u})|} & \text{if } |\varepsilon(\mathbf{u})| \neq 0 \\ |\tilde{\sigma}| \leq g(C,\theta) & \text{if } |\varepsilon(\mathbf{u})| = 0 \end{cases} \text{ in } Q, \tag{2.2}$$

$$\operatorname{div}(\mathbf{u}) = 0 \text{ in } Q, \tag{2.3}$$

$$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C - \operatorname{div}(\eta(C,\theta)\nabla C) = R \text{ in } Q, \tag{2.4}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta - \operatorname{div}(\kappa(C,\theta)\nabla \theta) = \sigma \cdot \varepsilon(\mathbf{u}) + h \text{ in } Q, \tag{2.5}$$

$$\mathbf{u} = 0 \text{ on } \Gamma \times (0,T), \tag{2.6}$$

$$\frac{\partial C}{\partial \nu} = 0 \text{ on } \Gamma \times (0,T), \tag{2.7}$$

$$\kappa(C,\theta)\frac{\partial \theta}{\partial \nu} + \beta(C,\theta)\,\theta = 0 \text{ on } \Gamma \times (0,T), \tag{2.8}$$

$$C(0) = C_0, \; \theta(0) = \theta_0 \text{ in } \Omega, \tag{2.9}$$

where $\mathrm{div}(\sigma) = (\sigma_{ij,j})$ and $\mathrm{div}(\mathbf{u}) = u_{i,i}$. The flow during the time $(0,T)$ is given by equation (2.1) where the density is assumed equal to one. Equation (2.2) represents the constitutive law of a Bingham fluid whose the viscosity μ and yield limit g depend on the concentration and temperature. (2.3) represents the incompressibility condition. Equation (2.4) represents the convection-diffusion-reaction equation modelling the evolution in space of various enzymes, proteins and platelets involved in the extrinsic pathway of coagulation process, where C stands for the concentration of the different reactants, η denotes the diffusion coefficient of blood and R is non-linear reaction term which represents the production or depletion due to the enzymatic cascade of reactions. Equation (2.5) represents the energy conservation where the specific heat is assumed equal to one, k is the thermal conductivity and the term h denotes the external heat source. (2.6) gives the velocity on $\Gamma \times (0,T)$. (2.7) is an homogeneous Neumann boundary condition on $\Gamma \times (0,T)$ for the concentration. (2.8) represents a Fourier boundary condition on $\Gamma \times (0,T)$ for the temperature, where β represents the Robin coefficient. Finally, (2.9) gives the initial data. Our model, see [2], includes not only rheological factors but also biochemical indicators that are essential to describe coagulation and fibrinolysis and consequently the formation, growth and dissolution of clots.

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem. Denoting by q' and r' the conjugates of q and r, where $r \in [1, +\infty[$, $1 \leq q < \dfrac{n}{n-1}$ and let $s \geq n$. We define the function spaces

$$\mathcal{V} = \left\{ \mathbf{v} \in H_0^1(\Omega)^n : \mathrm{div}(\mathbf{v}) = 0 \text{ in } \Omega \right\},$$

$$\mathcal{X} = \left\{ \zeta \in L^2\left(0,T; H^1(\Omega)\right) \cap \mathcal{C}^0\left([0,T], L^2(\Omega)\right),\ \frac{\partial \zeta}{\partial t} \in L^2\left(0,T; H^1(\Omega)'\right) \right\},$$

$$\mathcal{Y}_{q,r} = \left\{ \zeta \in L^r\left(0,T; W^{1,q}(\Omega)\right),\ \frac{\partial \zeta}{\partial t} \in L^1\left(0,T; W^{-1,1}(\Omega)\right) \right\},$$

$$\mathcal{Z}_{q,r} = \left\{ \zeta \in W^{1,r}\left(0,T; W^{1,q}(\Omega)\right),\ \zeta(x,T) = 0 \text{ sur } \Omega \right\},$$

where $H^1(\Omega)'$ represents the topological dual of $H^1(\Omega)$, for more details about the definition of this space see, [17].

$\mathcal{V}$ is a Hilbert space equipped with the inner product and the induced norm, respectively,

$$(\mathbf{u}, \mathbf{v})_{\mathcal{V}} = (u_i, v_i)_{H^1(\Omega)}, \quad \|\mathbf{v}\|_{\mathcal{V}} = (\mathbf{u}, \mathbf{u})_{\mathcal{V}}, \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

$\mathcal{X}$, $\mathcal{Y}_{q,r}$ and $\mathcal{Z}_{q,r}$ are Banach spaces equipped, respectively, with the norms

$$\|\zeta\|_{\mathcal{X}} = \|\zeta\|_{L^2(0,T;H^1(\Omega))} + \|\zeta\|_{\mathcal{C}^0([0,T],L^2(\Omega))} + \left\| \frac{\partial \zeta}{\partial t} \right\|_{L^2\left(0,T;H^1(\Omega)'\right)}, \quad \forall\, \zeta \in \mathcal{X},$$

$$\|\zeta\|_{\mathcal{Y}_{q,r}} = \|\zeta\|_{L^r(0,T;W^{1,q}(\Omega))} + \left\|\frac{\partial\zeta}{\partial t}\right\|_{L^1(0,T;W^{-1,1}(\Omega))}, \quad \forall\, \zeta \in \mathcal{Y}_{q,r},$$

$$\|\zeta\|_{\mathcal{Z}_{q,r}} = \|\zeta\|_{L^r(0,T;W^{1,q}(\Omega))} + \left\|\frac{\partial\zeta}{\partial t}\right\|_{L^r(0,T;W^{1,q}(\Omega))}, \quad \forall\, \zeta \in \mathcal{Z}_{q,r}.$$

We introduce the following functionals

$$\begin{cases} B : \mathcal{V}\times\mathcal{V}\times\mathcal{V} \to \mathbb{R}, \quad B(\mathbf{u},\mathbf{v},\mathbf{w}) = \displaystyle\int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\,dx, \\ E : W^{1,q}(\Omega)\times W^{1,q'}(\Omega)\times L^s(\Omega)^n \longrightarrow \mathbb{R}, \\ E(\xi,\zeta,\mathbf{v}) = \displaystyle\int_\Omega \xi\nabla\zeta\cdot\mathbf{v}dx. \end{cases} \tag{2.10}$$

In the study of the bio-mechanical problem (P1), we consider the following hypotheses

$$\begin{cases} \mu,\ g,\ \eta,\ \kappa,\ \beta \in \mathcal{C}^0\left(\mathbb{R}^2\right), \\ \exists\, \mu_*,\ \mu^* > 0 : \mu_* \ \le \mu(\xi,\zeta) \le \mu^*, \quad \forall\ (\xi,\zeta)\in\mathbb{R}^2, \\ \exists\, g^* > 0 : 0 \le g(\xi,\zeta) \le g^*, \quad \forall\ (\xi,\zeta)\in\mathbb{R}, \\ \exists\, \eta_*,\ \eta^* > 0 : \eta_* \ \le \eta(\xi,\zeta) \le \eta^*, \quad \forall\ (\xi,\zeta)\in\mathbb{R}^2, \\ \exists\, \kappa_*,\ \kappa^* > 0 : \kappa_* \ \le \kappa(\xi,\zeta) \le \kappa^*, \quad \forall\ (\xi,\zeta)\in\mathbb{R}^2, \\ \exists\, \beta_*,\ \beta^* > 0 : \beta_* \ \le \beta(\xi,\zeta) \le \beta^*, \quad \forall\ (\xi,\zeta)\in\mathbb{R}^2, \end{cases} \tag{2.11}$$

$$\mathbf{f} \in L^\infty\left(0,T;\mathcal{V}'\right),\ R \in L^2\left(0,T;H^1(\Omega)'\right) \text{ and } h \in L^1(Q), \tag{2.12}$$

$$C_0 \in L^2(\Omega),\ \theta_0 \in L^1(\Omega). \tag{2.13}$$

Lemma 2.1.

(1) *B is trilinear, continuous on $\mathcal{V}\times\mathcal{V}\times\mathcal{V}$. Moreover,* $B(\mathbf{u},\mathbf{v},\mathbf{w}) = -B(\mathbf{u},\mathbf{w},\mathbf{v}),\ \forall(\mathbf{u},\mathbf{v},\mathbf{w})\in\mathcal{V}\times\mathcal{V}\times\mathcal{V}$.

(2) *E is trilinear, continuous on $W^{1,q}(\Omega)\times W^{1,q'}(\Omega)\times L^s(\Omega)^n$ and on $H^1(\Omega)\times H^1(\Omega)\times\mathcal{V}$. Moreover, $E(\xi,\zeta,\mathbf{v}) = -E(\zeta,\xi,\mathbf{v}),\ \forall(\xi,\zeta,\mathbf{v})\in W^{1,q}(\Omega)\times W^{1,q'}(\Omega)\times L^s(\Omega)^n$ and $\forall(\xi,\zeta,\mathbf{v})\in H^1(\Omega)\times H^1(\Omega)\times\mathcal{V}$.*

Proof. For the proof of this lemma see for instance [18, 19, 20]. □

Remark 2.2. In the constitutive law (2.2) of Bingham fluid the blood pressure is given by the scalar

$$P = -\frac{1}{n}\mathrm{tr}(\sigma). \tag{2.14}$$

3. Variational formulation

The use of Green's formula permits us to derive the following variational formulation of bio-mechanical problem (P1).

Problem P2. Find the velocity field $\mathbf{u} = (u_i) : \Omega \times (0,T) \longrightarrow \mathbb{R}^n$, the concentration $C : \Omega \times (0,T) \longrightarrow \mathbb{R}$ and the temperature $\theta : \Omega \times (0,T) \longrightarrow \mathbb{R}$ satisfying the variational system

$$\begin{aligned} & B(\mathbf{u},\mathbf{u},\mathbf{v}) + 2\int_\Omega \mu(C,\theta)\,\varepsilon(\mathbf{u})\cdot\varepsilon(\mathbf{v}-\mathbf{u})\;dx \\ & \quad + \int_\Omega g(C,\theta)\,|\varepsilon(\mathbf{v})|\;dx - \int_\Omega g(C,\theta)\,|\varepsilon(\mathbf{u})|\;dx \\ & \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,dx, \quad \forall\, \mathbf{v}\in\mathcal{V}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} & \int_\Omega \frac{\partial C}{\partial t}\xi dx - E(C,\xi,\mathbf{u}) + \int_\Omega \eta(C,\theta)\,\nabla C\cdot\nabla\xi dx \\ & = \int_\Omega R\xi dx, \quad \forall\, \xi\in H^1(\Omega), \end{aligned} \tag{3.2}$$

$$\begin{aligned} & \int_\Omega \frac{\partial\theta}{\partial t}\zeta dx - E(\theta,\zeta,\mathbf{u}) + \int_\Omega \kappa(C,\theta)\,\nabla\theta\cdot\nabla\zeta dx + \int_\Gamma \beta(C,\theta)\,\theta\zeta ds \\ & = \int_\Omega F(\mathbf{u},C,\theta)\,\zeta dx + \int_\Omega h\zeta dx, \quad \forall\, \zeta\in W^{1,q'}(\Omega), \end{aligned} \tag{3.3}$$

where

$$F(\mathbf{u},C,\theta) = 2\mu(C,\theta)\,|\varepsilon(\mathbf{u})|^2 + g(C,\theta)\,|\varepsilon(\mathbf{u})| \tag{3.4}$$

and ds denotes the surface element.

Remark 3.1. In (3.3), the first and second terms on the right hand side has sense, since the injection $W^{1,q'}(\Omega) \longrightarrow \mathcal{C}^0\left(\bar{\Omega}\right)$ is continuous for $q' > n$, that is, $q < \dfrac{n}{n-1}$.

Definition 3.2. We will say that a function $\theta \in \mathcal{Y}_{q,r}$ is a weak solution of the variational equation (3.3) if

$$\begin{aligned} & -\int_Q \theta\frac{\partial\zeta}{\partial t}dxdt - \int_0^T E(\theta,\zeta,\mathbf{u})\,dt + \int_Q \kappa(C,\theta)\,\nabla\theta\cdot\nabla\zeta dxdt \\ & \quad + \int_{\Gamma\times(0,T)} \beta(C,\theta)\,\theta\zeta dsdt \\ & = \int_Q F(\mathbf{u},C,\theta)\,\zeta dxdt + \int_\Omega \theta_0\zeta(0)\,dx + \int_Q h\zeta dxdt, \quad \forall\, \zeta\in\mathcal{Z}_{q',r'}. \end{aligned} \tag{3.5}$$

We can then reformulate the variational problem (P2) as follows.

Problem P3. Find the velocity field $\mathbf{u}(t) \in \mathcal{V}$, the concentration $C(t) \in H^1(\Omega)$ and the temperature $\theta \in \mathcal{Y}_{q,r}$ satisfying the variational system

$$\begin{aligned} & B(\mathbf{u},\mathbf{u},\mathbf{v}) + 2\int_\Omega \mu(C,\theta)\,\varepsilon(\mathbf{u})\cdot\varepsilon(\mathbf{v}-\mathbf{u})\,dx \\ & \qquad + \int_\Omega g(C,\theta)\,|\varepsilon(\mathbf{v})|\,dx - \int_\Omega g(C,\theta)\,|\varepsilon(\mathbf{u})|\,dx \\ & \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,dx, \quad \forall\, \mathbf{v}\in\mathcal{V}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} & \int_\Omega \frac{\partial C}{\partial t}\xi dx - E(C,\xi,\mathbf{u}) + \int_\Omega \eta(C,\theta)\nabla C\cdot\nabla\xi dx \\ & = \int_\Omega R\xi dx, \quad \forall\, \xi\in H^1(\Omega), \end{aligned} \tag{3.6}$$

$$\begin{aligned} & -\int_Q \theta\frac{\partial\zeta}{\partial t}dxdt - \int_0^T E(\theta,\zeta,\mathbf{u})\,dt + \int_Q \kappa(C,\theta)\nabla\theta\cdot\nabla\zeta dxdt \\ & \quad + \int_{\Gamma\times(0,T)} \beta(C,\theta)\,\theta\zeta dsdt \\ & = \int_Q F(\mathbf{u},C,\theta)\,\zeta dxdt + \int_\Omega \theta_0\zeta(0)\,dx + \int_Q h\zeta dxdt, \quad \forall\, \zeta\in\mathcal{Z}_{q',r'}. \end{aligned} \tag{3.7}$$

4. Main results

In this section we establish an existence theorem to the problem (P3) and some properties of the solutions.

Theorem 4.1. *Under the assumptions* (2.11), (2.12) *and* (2.13)*, the problem* (P3) *admits at least one solution* $(\mathbf{u},C,\theta)$ *satisfying the regularity*

$$\mathbf{u}\in L^\infty(0,T;\mathcal{V}), \tag{4.1}$$

$$C\in\mathcal{X}, \tag{4.2}$$

$$\theta\in\mathcal{Y}_{q,r}, \tag{4.3}$$

where r *is such that*

$$1\leq r<2 \quad \text{and} \quad \frac{2}{r}+\frac{n}{q}>n+1. \tag{4.4}$$

The proof of Theorem 4.1 is based on the application of the Kakutani-Glicksberg fixed point theorem for multivalued mappings, see [15, 20], using three auxiliary existence results. The first one results from the theory of elliptic inequalities with convex functionals, see [8, 13]. The second one results from the theory of parabolic equations, see [15], and the third one results from the L^1-Data theory for linear parabolic equations, see [6]. Finally, compactness arguments are used to conclude the proof.

The first auxiliary existence result is given by.

Proposition 4.2. *For every* $(\mathbf{w},\alpha,\gamma) \in L^\infty\left(0,T;L^s\left(\Omega\right)^n\right) \times \mathcal{X} \times \mathcal{Y}_{q,r}$, *there exists a unique solution* $\mathbf{u} = \mathbf{u}\left(\mathbf{w},\alpha,\gamma\right) \in L^\infty\left(0,T;\mathcal{V}\right)$ *to the problem*

$$\begin{aligned} &B\left(\mathbf{w},\mathbf{u},\mathbf{v}\right) + 2\int_\Omega \mu\left(\alpha,\gamma\right)\varepsilon(\mathbf{u})\cdot\varepsilon\left(\mathbf{v}-\mathbf{u}\right)\,dx \\ &\quad + \int_\Omega g\left(\alpha,\gamma\right)\left|\varepsilon\left(\mathbf{v}\right)\right|\,dx - \int_\Omega g\left(\alpha,\gamma\right)\left|\varepsilon\left(\mathbf{u}\right)\right|dx \\ &\geq \int_\Omega \mathbf{f}\cdot\left(\mathbf{v}-\mathbf{u}\right)dx, \quad \forall\, \mathbf{v}\in\mathcal{V}, \end{aligned} \tag{4.5}$$

and it satisfies the estimate

$$\left\|\mathbf{u}\right\|_{L^\infty(0,T;\mathcal{V})} \leq d_1, \tag{4.6}$$

where d_1 is a positive constant.

Proof. Introducing for every $(\mathbf{w},\alpha,\gamma) \in L^\infty\left(0,T;L^s\left(\Omega\right)^n\right) \times \mathcal{X} \times \mathcal{Y}_{q,r}$ the following form

$$_{(\mathbf{w},\alpha,\gamma)} : \mathcal{V}\times\mathcal{V} \longrightarrow \mathbb{R},$$

$$_{(\mathbf{w},\alpha,\gamma)}\left(\mathbf{u},\mathbf{v}\right) = B\left(\mathbf{w},\mathbf{u},\mathbf{v}\right) + 2\int_\Omega \mu\left(\alpha,\gamma\right)\varepsilon(\mathbf{u})\cdot\varepsilon\left(\mathbf{v}-\mathbf{u}\right)\,dx. \tag{4.7}$$

It follows from Lemma 2.1 that $_{(\mathbf{w},\alpha,\gamma)}$ is bilinear, continuous and coercive on $\mathcal{V}\times\mathcal{V}$. Furthermore, the functional $\mathbf{v} \longmapsto \int_\Omega g\left(\alpha,\gamma\right)\left|\varepsilon\left(\mathbf{v}\right)\right|\,dx$ is continuous and convex on $\mathcal{V}$, it is then lower semi-continuous on $\mathcal{V}$. Consequently, the existence and uniqueness of the solution result from the classical theorems, see [8, 13] on elliptic variational inequalities.

To prove the estimate (4.6) we proceed as follows, by choosing $\mathbf{v}=0$ as test function in (4.5) and using assumption (2.11), we find

$$\mu_*\int_\Omega \left|\varepsilon\left(\mathbf{u}\left(t\right)\right)\right|^2 dx \leq \left\|\mathbf{f}\right\|_{\mathcal{V}'}\left\|\mathbf{u}\left(t\right)\right\|_{\mathcal{V}} \quad \text{a.e. } t\in\left(0,T\right).$$

Hence, Korn's inequality and assumption (2.12) permit us to conclude the proof. □

The second auxiliary existence result is given by.

Proposition 4.3. *Let* $\mathbf{u} = \mathbf{u}(\mathbf{w}, \alpha, \gamma) \in L^{\infty}(0, T; \mathcal{V})$ *be the solution of problem* (4.5) *given by Proposition* 4.2. *Then, there exists a unique solution* $C = C(\mathbf{w}, \alpha, \gamma) \in \mathcal{X}$ *to the problem*

$$\begin{aligned} &\int_{\Omega} \frac{\partial C}{\partial t} \xi dx - E(C, \xi, \mathbf{u}) + \int_{\Omega} \eta(\alpha, \gamma) \nabla C \cdot \nabla \xi dx \\ &= \int_{\Omega} R \xi dx, \quad \forall\, \xi \in H^1(\Omega) \end{aligned} \tag{4.8}$$

and it satisfies the estimate

$$\|C\|_{\mathcal{X}} \leq d_2, \tag{4.9}$$

where d_2 *is a positive constant.*

Proof. The continuity of E on $H^1(\Omega) \times H^1(\Omega) \times \mathcal{V}$ leads, using Hölder's inequality with respect to the time variable and the estimate (4.6), to

$$\left| \int_0^T E(C, \xi, \mathbf{u})\, dt \right| \leq \|C\|_{L^2(0,T;H^1(\Omega))} \|\xi\|_{L^2(0,T;H^1(\Omega))} \|\mathbf{u}\|_{L^{\infty}(0,T;\mathcal{V})}. \tag{4.10}$$

Thus, E is continuous on $L^2\left(0, T; H^1(\Omega)\right)^2 \times L^{\infty}(0, T; \mathcal{V})$. Now, let us consider the bilinear form

$$G_{(\mathbf{u},\alpha,\gamma)} : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R},$$

$$G_{(\mathbf{u},\alpha,\gamma)}(C, \xi) = -E(C, \xi, \mathbf{u}) + \int_{\Omega} \eta(\alpha, \gamma) \nabla C \cdot \nabla \xi dx. \tag{4.11}$$

We know, due to the Neumann boundary condition, that the form $G_{(\mathbf{u},\alpha,\gamma)}$ is not $H^1(\Omega)-$elliptic. To solve this problem we introduce the functions

$$\tilde{C}(t) = e^{-t} C(t),$$

$$\tilde{\xi}(t) = e^{-t} \xi(t).$$

Consequently, (3.12) is equivalent to the following equation

$$\begin{aligned} &\int_{\Omega} \frac{\partial \tilde{C}}{\partial t} \tilde{\xi} dx + \int_{\Omega} \tilde{C} \tilde{\xi} dx - E\left(\tilde{C}, \tilde{\xi}, \mathbf{u}\right) + \int_{\Omega} \eta(\alpha, \gamma) \nabla \tilde{C} \cdot \nabla \tilde{\xi} dx \\ &= \int_{\Omega} e^{-t} R \tilde{\xi} dx, \quad \forall\, \tilde{\xi} \in H^1(\Omega), \end{aligned} \tag{4.12}$$

Hypothesis (2.11) and Lemma 2.1 imply that

$$G_{(\mathbf{u},\alpha,\gamma)}\left(\tilde{\xi}, \tilde{\xi}\right) + \left(\tilde{\xi}, \tilde{\xi}\right)_{L^2(\Omega)} \geq c_1 \left\|\tilde{\xi}\right\|^2_{H^1(\Omega)}, \quad \forall\, \tilde{\xi} \in H^1(\Omega),$$

where

$$c_1 = \min(\eta_*, 1).$$

Which permits us to deduce, using classical arguments of functional analysis concerning linear parabolic equations, see [16] and assumptions (2.12) and (2.13), that equation (4.12) admits a unique solution $\tilde{C} \in L^2\left(0,T;H^1(\Omega)\right) \cap L^\infty\left((0,T);L^2(\Omega)\right)$. Setting now $\tilde{\xi} = \tilde{C}$ as test function in (4.12), integrating over the interval time $(0,t)$ and using Hölder's and Young's inequalities, the following energy inequality holds

$$\left\|\tilde{C}(t)\right\|^2_{L^2(\Omega)} + c_1 \int\limits_0^t \left\|\tilde{C}(a)\right\|^2_{H^1(\Omega)} da \leq \frac{1}{c_1}\|R\|^2_{L^2(Q)} + \|C_0\|^2_{L^2(\Omega)}.$$

On the other hand, by virtue of Lemma 2.1 there exists a linear and continuous operator $A : H^1(\Omega) \longrightarrow H^1(\Omega)'$, defined for every $\tilde{\zeta}$, $\tilde{\xi} \in H^1(\Omega)$ by

$$A\left(\tilde{\zeta},\tilde{\xi}\right) = \int_\Omega \tilde{\zeta}\tilde{\xi}dx - E\left(\tilde{\zeta},\tilde{\xi},\mathbf{u}\right) + \int_\Omega \eta(\alpha,\gamma)\nabla\tilde{\zeta}\cdot\nabla\tilde{\xi}dx.$$

Therefore, (4.12) can be rewritten, using the operator A

$$\frac{\partial\tilde{C}}{\partial t} + A\tilde{C} = e^{-t}R \ \text{ in } H^1(\Omega)'.$$

Then

$$\frac{\partial\tilde{C}}{\partial t} \in L^2\left(0,T;H^1(\Omega)'\right).$$

Hence, from trace theorems, see [17], after a possible modification on a set of measure zero, $\tilde{C}$ is continuous from $[0,T]$ into $L^2(\Omega)$. The estimate (4.9) becomes a simple consequence of the energy inequality and the previous estimate. □

The third auxiliary existence result is given by.

Proposition 4.4. *Let $\mathbf{u} = \mathbf{u}(\mathbf{w},\alpha,\gamma) \in L^\infty(0,T;\mathcal{V})$ be the solution of problem (4.5) given by Proposition 4.2. Then, there exists $\theta = \theta(\mathbf{w},\alpha,\gamma) \in \mathcal{Y}_{q,r}$, r given by (4.4), a solution to the weak problem*

$$\begin{aligned}
&-\int_Q \theta\frac{\partial\zeta}{\partial t}dxdt - \int\limits_0^T E(\theta,\zeta,\mathbf{u})\,dt + \int_Q \kappa(\alpha,\gamma)\nabla\theta\cdot\nabla\zeta dxdt \\
&\quad + \int_{\Gamma\times(0,T)} \beta(\alpha,\gamma)\,\theta\zeta dsdt \\
&= \int_Q F(\mathbf{u},\alpha,\gamma)\,\zeta dxdt + \int_\Omega \theta_0\zeta(0)\,dx + \int_Q h\zeta dxdt, \quad \forall\, \zeta \in \mathcal{Z}_{q',r'},
\end{aligned} \tag{4.13}$$

and it satisfies the estimate

$$\|\theta\|_{\mathcal{Y}_{q,r}} \leq d_3, \tag{4.14}$$

where d_3 is a positive constant.

Proof. Since $s \geq n$, $1 \leq q < \dfrac{n}{n-1}$ and $W^{1,q'}(\Omega) \subset \mathcal{C}^0\left(\bar{\Omega}\right)$, we obtain using Hölder's inequality and the antisymmetry of E with respect to the two first variables

$$\left|\int_0^T E(\theta,\zeta,\mathbf{u})\,dt\right| \leq \int_0^T \|\theta\|_{W^{1,q}(\Omega)} \|\zeta\|_{W^{1,q'}(\Omega)} \|\mathbf{u}\|_{L^s(\Omega)^n}\,dt. \tag{4.15}$$

Thus, we can infer, using Hölder's inequality with respect to the time variable

$$\left|\int_0^T E(\theta,\zeta,\mathbf{u})\,dt\right| \leq \|\theta\|_{L^r(0,T;W^{1,q}(\Omega))} \|\zeta\|_{L^{r'}\left(0,T;W^{1,q'}(\Omega)\right)} \|\mathbf{u}\|_{L^\infty\left(0,T;L^s(\Omega)^n\right)}.$$

This entrains by exploiting the Sobolev imbedding $\mathcal{V} \subset L^s(\Omega)^n$, $n \leq s \leq \dfrac{2n}{n-2}$

$$\begin{aligned}\left|\int_0^T E(\theta,\zeta,\mathbf{u})\,dt\right| &\leq \|\theta\|_{L^r(0,T;W^{1,q}(\Omega))} \|\zeta\|_{L^{r'}\left(0,T;W^{1,q'}(\Omega)\right)} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{V})}\\ &\leq \|\theta\|_{\mathcal{Y}_{q,r}} \|\zeta\|_{\mathcal{Z}_{q',r'}} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{V})}.\end{aligned}$$

Consequently, E is continuous on $\mathcal{Y}_{q,r} \times \mathcal{Z}_{q',r'} \times L^\infty(0,T;\mathcal{V})$. However, technically, it is difficult to obtain a solution of problem (4.13). To this end we introduce for each $m \in \mathbb{N}$ the following approximate standard weak equations

$$\left\{\begin{array}{l}\displaystyle\int_\Omega \frac{\partial\theta_m}{\partial t}\zeta dx - E(\theta_m,\zeta,\mathbf{u}) + \int_\Omega \kappa(\alpha,\gamma)\nabla\theta_m\cdot\nabla\zeta dx\\ \displaystyle\quad + \int_\Gamma \beta(\alpha,\gamma)\theta_m\zeta ds = \int_\Omega (F_m + h_m)\zeta dx, \quad \forall\, \zeta \in H^1(\Omega),\\ \theta_m(0) = \theta_{0m} \text{ in } \Omega,\end{array}\right. \tag{4.16}$$

where

$$F_m = \frac{mF(\mathbf{u},\lambda,\mu)}{m + F(\mathbf{u},\lambda,\mu)} \in L^\infty(Q),\ h_m = \frac{mh}{m+h} \in L^\infty(Q), \tag{4.17}$$

(if is also possible to take $h_m \in L^2(Q)$ such that h_m converges to h in $L^1(Q)$). Therefore, we choose $\theta_{0m} \in L^2(\Omega)$ such that θ_{0m} converges to θ_0 in $L^1(\Omega)$.

Let us consider the bilinear form

$$T_{(\mathbf{u},\alpha,\gamma)} : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R},$$

$$T_{(\mathbf{u},\alpha,\gamma)}(\theta,\zeta) = -E(\theta,\zeta,\mathbf{u}) + \int_\Omega \kappa(\alpha,\gamma)\nabla\theta\cdot\nabla\zeta dx + \int_\Gamma \beta(\alpha,\gamma)\theta\zeta ds. \tag{4.18}$$

Combining Lemma 2.1 and hypothesis (2.11) with the Poincaré-Freidrics type inequality, it follows that $T_{(\mathbf{u},\alpha,\gamma)}$ is continuous and coercive on $H^1(\Omega) \times H^1(\Omega)$. Consequently, we deduce from classical arguments of functional analysis concerning linear parabolic equations, see [16], via assumptions (2.12) and (2.13), that equation (4.16) admits a unique solution θ_m satisfying the regularity

$$\begin{cases} \theta_m \in L^2\left(0,T;H^1(\Omega)\right) \cap \mathcal{C}^0\left([0,T], L^2(\Omega)\right), \\ \dfrac{\partial \theta_m}{\partial t} \in L^2\left(0,T;H^1(\Omega)'\right). \end{cases} \tag{4.19}$$

Furthermore, the use of Hölder's inequality for $1 \leq q < \dfrac{n}{n-1}$ and $\lambda > 1$, leads for almost every $t \in [0,T]$ to

$$\begin{aligned} &\int_\Omega |\nabla\theta_m(x,t)|^q\,dx \\ &\leq \left(\int_\Omega \frac{|\nabla\theta_m(x,t)|^2}{(1+|\theta_m(x,t)|)^\lambda}dx\right)^{\frac{q}{2}} \left(\int_\Omega (1+|\theta_m(x,t)|)^{\frac{\lambda q}{2-q}}\,dx\right)^{\frac{2-q}{2}}. \end{aligned}$$

Which eventually gives

$$\|\nabla\theta_m(t)\|^q_{L^q(\Omega)^n} \leq \left(\int_\Omega \frac{|\nabla\theta_m|(x,t)^2}{(1+|\theta_m(x,t)|)^\lambda}dx\right)^{\frac{q}{2}} \left(\int_\Omega (1+|\theta_m(x,t)|)^{\frac{\lambda q}{2-q}}\,dx\right)^{\frac{2-q}{2}}.$$

Raising to the power $\dfrac{r}{q}$ where $1 \leq r < 2$ and using Hölder's inequality with respect to the time variable, the following inequality holds

$$\begin{aligned} &\int_0^T \|\nabla\theta_m(t)\|^r_{L^q(\Omega)^n}\,dt \\ &\leq \left(\int_Q \frac{|\nabla\theta_m(x,t)|^2}{(1+|\theta_m(x,t)|)^\lambda}dxdt\right)^{\frac{r}{2}} \\ &\quad\times \left\{\int_0^T \left(\left(\int_\Omega (1+|\theta_m(x,t)|)^{\frac{\lambda q}{2-q}}\,dx\right)^{\frac{(2-q)r}{(2-r)q}}\right) dt\right\}^{\frac{2-r}{2}}. \end{aligned} \tag{4.20}$$

On the other hand, the use of estimate (4.19) permits us to find, for almost every $t \in [0,T]$

$$\int_\Omega \frac{|\nabla\theta_m(x,t)|^2}{(1+|\theta_m(x,t)|)^\lambda}dx \leq \int_\Omega |\nabla\theta_m(x,t)|^2\,dx \leq c. \tag{4.21}$$

From, (2.20), (2.21) and some algebraic calculations, we can infer

$$\int_0^T \|\nabla\theta_m(t)\|^r_{L^q(\Omega)^n}\,dt \le c\left\{1+\left(\int_0^T \|\theta_m(t)\|^{\frac{\lambda r}{2-r}}_{L^{\frac{\lambda q}{2-q}}(\Omega)}\,dt\right)^{\frac{2-r}{2}}\right\}. \tag{4.22}$$

Gagliardo-Nirenberg's imbedding, see [21] gives for almost every $t\in[0,T]$

$$\|\theta_m(t)\|_{L^{\frac{\lambda q}{2-q}}(\Omega)} \le c\,\|\nabla\theta_m(t)\|^{\sigma}_{L^q(\Omega)^n}\,\|\theta_m(t)\|^{1-\sigma}_{L^1(\Omega)},$$

for every σ satisfying

$$\begin{cases} \dfrac{2-q}{\lambda q} = \sigma\dfrac{n-q}{nq}+1-\sigma, \\ 0\le\sigma\le 1. \end{cases} \tag{4.23}$$

It follows using Hölder's inequality and the fact that $\theta_m\in\mathcal{C}^0\left([0,T],L^2(\Omega)\right)$

$$\|\theta_m(t)\|_{L^{\frac{q\lambda}{2-q}}(\Omega)} \;\le\; c\,\|\nabla\theta_m(t)\|^{\sigma}_{L^q(\Omega)^n}.$$

Raising the inequality above to the power $\dfrac{r}{\sigma}$ and integrating over the interval time $(0,T)$, we obtain

$$\int_0^T \|\theta_m(t)\|^{\frac{r}{\sigma}}_{L^{\frac{q\lambda}{2-q}}(\Omega)}\,dt \;\le\; c\int_0^T \|\nabla\theta_m(t)\|^r_{L^q(\Omega)^n}\,dt. \tag{4.24}$$

Now, we assume that

$$\frac{\lambda r}{2-r}=\frac{r}{\sigma}. \tag{4.25}$$

Hence, (4.22) and (4.24) imply

$$\int_0^T \|\nabla\theta_m(t)\|^r_{L^q(\Omega)^n}\,dt \le c\left\{1+\left(\int_0^T \|\nabla\theta_m(t)\|^r_{L^q(\Omega)^n}\,dt\right)^{\frac{2-r}{2}}\right\}. \tag{4.26}$$

Since $\dfrac{2-r}{2}<1$, we can obtain an a priori estimate on the term $\nabla\theta_m$ in the space $L^r(0,T;L^q(\Omega))$. Putting together (4.23) and (4.25), we obtain after some manipulations

$$\begin{cases} 1\le r<2, \\ \lambda = \dfrac{2q+nq+nr-rq-nrq}{nq}. \end{cases}$$

We deduce, using the fact that $\lambda > 1$

$$1 \leq r < 2, \qquad \frac{2}{r} + \frac{n}{q} > n + 1.$$

This gives sense to hypothesis (4.4). Which, together with the fact that $1 \leq q < \dfrac{n}{n-1}$ and standard imbedding theorems between Lebesgue spaces, the following estimate holds

$$\theta_m \in L^r\left(0, T; W^{1,q}(\Omega)\right). \tag{4.27}$$

We can then extract, using the estimate (4.27) a subsequence of θ_m, still denoted by θ_m satisfying

$$\theta_m \longrightarrow \theta \text{ in } L^r\left(0, T; W^{1,q}(\Omega)\right) \text{ weakly.} \tag{4.28}$$

Furthermore, the estimate (4.27) can also gives

$$\nabla\theta_m \in L^r\left(0, T; L^q(\Omega)^n\right) \subset L^1\left(0, T; L^1(\Omega)^n\right).$$

Thus, we find using hypothesis (2.11)

$$\operatorname{div}\left(\kappa(\alpha, \gamma)\nabla\theta_m\right) \in L^1\left(0, T; W^{-1,1}(\Omega)\right). \tag{4.29}$$

On the other hand, the use of the Sobolev imbedding $\mathcal{V} \subset L^n(\Omega)^n$, Hölder's inequality and estimates (4.6) and (4.27), permits us to obtain, keeping in mind the fact that $1 \leq q < \dfrac{n}{n-1}$

$$\mathbf{u}\theta_m \in L^r\left(0, T; L^1(\Omega)^n\right) \subset L^1\left(0, T; L^1(\Omega)^n\right).$$

This leads, recalling the incompressibility condition (2.3), to

$$\mathbf{u} \cdot \nabla\theta_m = \operatorname{div}\left(\mathbf{u}\theta_m\right) \in L^1\left(0, T; W^{-1,1}(\Omega)\right). \tag{4.30}$$

Moreover, by virtue of the approximate strong system

$$\begin{cases} \dfrac{\partial\theta_m}{\partial t} + \mathbf{u} \cdot \nabla\theta_m - \operatorname{div}\left(\kappa(\alpha, \gamma)\nabla\theta_m\right) = F_m + h_m \text{ in } Q, \\ \kappa(\alpha, \gamma)\dfrac{\partial\theta_m}{\partial\nu} + \beta(\alpha, \gamma)\theta_m = 0 \text{ on } \Gamma \times (0, T), \\ \theta_m(0) = \theta_{0m} \text{ in } \Omega, \end{cases} \tag{4.31}$$

and by the definition of the functions F_m and h_m due to (4.29) and (4.30),we can infer

$$\frac{\partial\theta_m}{\partial t} \in L^1\left(0, T; W^{-1,1}(\Omega)\right). \tag{4.32}$$

Using (4.32) and compactness arguments, see [24] it is easy to see that

$$\theta_m \longrightarrow \theta \text{ in } L^1(Q) \text{ strongly and a.e.in } Q. \tag{4.33}$$

From Sobolev's theorem, the trace of θ_m belongs to $L^r\left(0,T;W^{1-\frac{1}{q},q}(\Omega)\right)$, and via the Sobolev imbedding $W^{1-\frac{1}{q},q}(\Gamma)\longrightarrow L^q(\Gamma)$, we obtain, after a new extraction, still denoted by θ_m

$$\theta_m \longrightarrow \theta \text{ in } L^r\left(0,T;L^q(\Gamma)\right) \text{ weakly.} \tag{4.34}$$

We conclude from (4.16), (4.17), (4.28), (4.33) and (4.34) that problem (4.13) admits a solution $\theta=\theta(\mathbf{u},\alpha,\gamma)\in\mathcal{Y}_{q,r}$. Moreover, the estimate (4.14) can be obtained as a consequence of (4.27) and (4.32). □

Proof of Theorem 4.1. In order to apply the Kakutani-Glicksberg fixed point theorem, let us consider the closed convex ball

$$\begin{aligned} K=\Big\{&(\mathbf{w},\alpha,\gamma)\in L^\infty(0,T;\mathcal{V})\times\mathcal{X}\times\mathcal{Y}_{q,r}:\|\mathbf{w}\|_{L^\infty(0,T;\mathcal{V})}\le d_1,\\ &\|\alpha\|_{\mathcal{X}}\le d_2 \text{ and } \|\gamma\|_{\mathcal{Y}_{q,r}}\le d_3\Big\},\end{aligned} \tag{4.35}$$

where d_1, d_2 and d_3 are the constants given by estimates (4.6), (4.9) and (4.14). The ball K is compact when the topological vector space is provided by the weak star topology of $L^\infty(0,T;\mathcal{V})$ and the weak topology of $\mathcal{X}\times\mathcal{Y}_{q,r}$. Let us built the mapping $\mathcal{L}:K\longrightarrow 2^K$, as follows

$$(\mathbf{w},\alpha,\gamma)\longmapsto\mathcal{L}(\mathbf{w},\alpha,\gamma)=\{(\mathbf{u},C,\theta)\}\subset K. \tag{4.36}$$

For every $(\mathbf{w},\alpha,\gamma)\in K$, equations (4.8), (4.13) are linear with respect to the functions C and θ, respectively. Moreover, $\mathbf{u}$ is the unique solution of problem (4.5) in the space $L^\infty(0,T;\mathcal{V})$. Consequently the set $\mathcal{L}(\mathbf{w},\alpha,\gamma)$ is convex. To conclude the proof it remains to verify the closeness in $K\times K$ of the graph set

$$G(\mathcal{L})=\{((\mathbf{w},\alpha,\gamma),(\mathbf{u},C,\theta))\in K\times K:(\mathbf{u},C,\theta)\in\mathcal{L}(\mathbf{w},\alpha,\gamma)\}. \tag{4.37}$$

To do so, we consider a sequence $(\mathbf{w}_m,\alpha_m,\gamma_m)\in K$, such that

$$\left\{\begin{array}{c}\mathbf{w}_m\longrightarrow\mathbf{w} \text{ in } L^\infty(0,T;\mathcal{V}) \text{ weakly*},\\ (\alpha_m,\gamma_m)\longrightarrow(\alpha,\gamma) \text{ in } \mathcal{X}\times\mathcal{Y}_{q,r} \text{ weakly,}\end{array}\right. \tag{4.38}$$

and let $(\mathbf{u}_m,C_m,\theta_m)\in\mathcal{L}(\mathbf{w}_m,\alpha_m,\gamma_m)$. Let us remember that $(\mathbf{u}_m,C_m,\theta_m)$ is solution to the following system

$$\begin{aligned}&B(\mathbf{w}_m,\mathbf{u}_m,\mathbf{v})+2\int_\Omega\mu(\alpha_m,\gamma_m)\,\varepsilon(\mathbf{u}_m)\cdot\varepsilon(\mathbf{v}-\mathbf{u}_m)\,dx\\ &\quad+\int_\Omega g(\alpha_m,\gamma_m)\,|\varepsilon(\mathbf{v})|\,dx-\int_\Omega g(\alpha_m,\gamma_m)\,|\varepsilon(\mathbf{u}_m)|\,dx\\ &\ge\int_\Omega\mathbf{f}\cdot(\mathbf{v}-\mathbf{u}_m)\,dx,\quad\forall\,\mathbf{v}\in\mathcal{V},\end{aligned} \tag{4.39}$$

$$\left\{\begin{array}{l}\displaystyle\int_{\Omega}\frac{\partial C_m}{\partial t}\xi dx - E\left(C_m,\xi,\mathbf{u}_m\right) + \int_{\Omega}\eta\left(\alpha_m,\gamma_m\right)\nabla C_m\cdot\nabla\xi dx\\ \displaystyle= \int_{\Omega}R\xi dx,\quad \forall\,\xi\in H^1\left(\Omega\right),\\ C_m\left(0\right) = C_{0m},\end{array}\right. \tag{4.40}$$

$$\begin{aligned}&-\int_Q\theta_m\frac{\partial\zeta}{\partial t}dxdt - \int_0^T E\left(\theta_m,\zeta,\mathbf{u}_m\right)dt + \int_Q\kappa\left(\alpha_m,\gamma_m\right)\nabla\theta_m\cdot\nabla\zeta dxdt\\ &\quad+\int_{\Gamma\times(0,T)}\beta\left(\alpha_m,\gamma_m\right)\theta_m\zeta dsdt\\ &= \int_Q F\left(\mathbf{u}_m,\alpha_m,\gamma_m\right)\zeta dxdt + \int_{\Omega}\theta_{0m}\zeta\left(0\right)dx + \int_Q h\zeta dxdt,\quad \forall\,\zeta\in\mathcal{Z}_{q',r'}.\end{aligned} \tag{4.41}$$

Then, from Propositions 4.2, 4.3 and 4.4

$$\left\|\mathbf{u}_m\right\|_{L^\infty(0,T;\mathcal{V})} \le d_1,\quad \left\|C_m\right\|_{\mathcal{X}} \le d_2 \ \text{ and } \ \left\|\theta_m\right\|_{\mathcal{Y}_{q,r}} \le d_3.$$

Thus, we can extract a subsequences, still denoted by $\mathbf{u}_m$, C_m and θ_m such that

$$\mathbf{u}_m \longrightarrow \mathbf{u} \text{ in } L^\infty\left(0,T;\mathcal{V}\right) \text{ weakly*}, \tag{4.42}$$

$$C_m \longrightarrow C \text{ in } \mathcal{X} \text{ weakly}, \tag{4.43}$$

$$\theta_m \longrightarrow \theta \text{ in } L^r\left(0,T;W^{1,q}\left(\Omega\right)\right) \text{ weakly}, \tag{4.44}$$

$$\frac{\partial\theta_m}{\partial t} \in L^1\left(0,T;W^{-1,1}\left(\Omega\right)\right). \tag{4.45}$$

It follows by compactness theorems, that we can also extract subsequences, still denoted by C_m, θ_m, α_m and γ_m such that

$$C_m \longrightarrow C \text{ in } L^2\left(Q\right) \text{ strongly and a.e. in } Q, \tag{4.46}$$

$$\theta_m \longrightarrow \theta \text{ in } L^1\left(Q\right) \text{ strongly and a.e. in } Q, \tag{4.47}$$

$$\theta_m \longrightarrow \theta \text{ in } L^1\left(\Gamma\times(0,T)\right) \text{ strongly and a.e. on } \Gamma\times\left(0,T\right), \tag{4.48}$$

$$\alpha_m \longrightarrow \alpha \text{ in } L^2\left(Q\right) \text{ strongly and a.e. in } Q, \tag{4.49}$$

$$\gamma_m \longrightarrow \gamma \text{ in } L^1\left(Q\right) \text{ strongly and a.e. in } Q. \tag{4.50}$$

The inequality (4.39) gives

$$\begin{aligned}&B\left(\mathbf{w}_m,\mathbf{u}_m,\mathbf{v}\right) + 2\int_{\Omega}\mu\left(\alpha_m,\gamma_m\right)\varepsilon(\mathbf{u}_m)\cdot\varepsilon\left(\mathbf{v}\right)dx\\ &\quad+\int_{\Omega}g\left(\alpha_m,\gamma_m\right)\left|\varepsilon\left(\mathbf{v}\right)\right|\,dx - \int_{\Omega}\mathbf{f}\cdot\left(\mathbf{v}-\mathbf{u}_m\right)dx\\ &\ge 2\int_{\Omega}\mu\left(\alpha_m,\gamma_m\right)\left|\varepsilon(\mathbf{u}_m)\right|^2dx + \int_{\Omega}g\left(\alpha_m,\gamma_m\right)\left|\varepsilon(\mathbf{u}_m)\right|dx,\quad \forall\,\mathbf{v}\in\mathcal{V}.\end{aligned} \tag{4.51}$$

The use of Lemma 2.1 and Lebesgue's dominated convergence theorem permits us to find the following limit, see [20]

$$B(\mathbf{w}_m, \mathbf{u}_m, \mathbf{v}) \longrightarrow B(\mathbf{w}, \mathbf{u}, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathcal{V} \text{ in } L^\infty(0,T). \tag{4.52}$$

On the other hand, since $\alpha_m \longrightarrow \alpha$ and $\gamma_m \longrightarrow \gamma$ a.e. in Q, the functions μ and g are continuous and due to the weak lower semicontinuity of the continuous and convex functional $v \in \mathcal{V} \longmapsto \int_\Omega g(\alpha_m, \gamma_m) |\varepsilon(\mathbf{v})|\, dx$, combined with the convergence result (4.42), we deduce from Fatou's lemma that

$$\liminf \int_\Omega \mu(\alpha_m, \gamma_m) |\varepsilon(\mathbf{u}_m)|^2\, dx \geq \int_\Omega \mu(\alpha, \gamma) |\varepsilon(\mathbf{u})|^2\, dx \text{ in } L^\infty(0,T), \tag{4.53}$$

$$\liminf \int_\Omega g(\alpha_m, \gamma_m) |\varepsilon(\mathbf{u}_m)|\, dx \ \geq \int_\Omega g(\alpha, \gamma) |\varepsilon(\mathbf{u})|\, dx \text{ in } L^\infty(0,T) \tag{4.54}$$

and, by using Lebesgue's dominated convergence theorem, we find

$$\begin{aligned} &\int_\Omega \mu(\alpha_m, \gamma_m)\, \varepsilon(\mathbf{u}_m) \cdot \varepsilon(\mathbf{v})\, dx \\ &\longrightarrow \int_\Omega \mu(\alpha, \gamma)\, \varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v})\, dx, \quad \forall\, \mathbf{v} \in \mathcal{V} \text{ in } L^\infty(0,T), \end{aligned} \tag{4.55}$$

$$\int_\Omega g(\alpha_m, \gamma_m) |\varepsilon(\mathbf{v})|\ dx \longrightarrow \int_\Omega g(\alpha, \gamma) |\varepsilon(\mathbf{v})|\ dx, \quad \forall\, \mathbf{v} \in \mathcal{V}. \tag{4.56}$$

We conclude, from (4.39), (4.42), (4.52), (4.53), (4.54), (4.55) and (4.56), that $\mathbf{u}$ solves the problem (4.5). Moreover, if we choose $\mathbf{v} = \dfrac{\mathbf{u}_m + \mathbf{u}}{2}$ as test function in (4.5) and (4.39), and we subtract the obtained inequalities, we find after simplification

$$\begin{aligned} &\int_\Omega \mu(\alpha_m, \gamma_m) |\varepsilon(\mathbf{u}_m - \mathbf{u})|^2\, dx + \int_\Omega (g(\alpha_m, \gamma_m) - g(\alpha, \gamma)) |\varepsilon(\mathbf{u}_n)|\, dx \\ &\leq B(\mathbf{w} - \mathbf{w}_m, \mathbf{u}, \mathbf{u}_m) + \int_\Omega (\mu(\alpha, \gamma) - \mu(\alpha_m, \gamma_m))\, \varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{u}_m - \mathbf{u})\, dx \\ &\quad + \int_\Omega (g(\alpha_m, \gamma_m) - g(\alpha, \gamma)) |\varepsilon(\mathbf{u})|\, dx. \end{aligned}$$

Keeping in mind the fact that $\alpha_m \longrightarrow \alpha$ and $\gamma_m \longrightarrow \gamma$ a.e. in Q and that the functions μ and g are continuous and due to hypothesis (2.11), we can apply Fatou's lemma to the second term on the left hand side and Lebesgue's dominated convergence theorem to the right hand side, thanks to Korn's inequality, we obtain the following strong convergence

$$\mathbf{u}_m \longrightarrow \mathbf{u} \text{ in } L^\infty(0,T;\mathcal{V}) \ \text{ strongly.} \tag{4.57}$$

Furthermore, the Sobolev imbedding $\mathcal{V} \subset L^4(\Omega)^n$ and $H^1(\Omega) \subset L^4(\Omega)$ leads easily to

$$E(C_m, \xi, \mathbf{u}_m) \longrightarrow E(C, \xi, \mathbf{u}), \quad \forall\, \xi \in H^1(\Omega). \tag{4.58}$$

Hence, by virtue of (4.40) and (4.58), it easily follows that C solves the problem (4.8). For the passage to the limit in equation (4.41) we proceed as follows. Sobolev's imbedding asserts that

$$L^{\infty}\left(0,T;\mathcal{V}\right) \subset L^{\infty}\left(0,T;L^{n}\left(\Omega\right)^{n}\right),$$
$$L^{r}\left(0,T;W^{1,q}\left(\Omega\right)\right) \subset L^{r}\left(0,T;L^{\frac{nq}{n-q}}\left(\Omega\right)\right).$$

Then, the fact that $\mathbf{u}_m \in L^{\infty}\left(0,T;\mathcal{V}\right)$ and $\theta_m \in L^{r}\left(0,T;W^{1,q}\left(\Omega\right)\right)$ permits us to get, by application of the Hölder inequality

$$\theta_m \mathbf{u}_m \in L^{r}\left(0,T;L^{q}\left(\Omega\right)^{n}\right).$$

Thus, since $\nabla\zeta \in L^{r'}\left(0,T;L^{q'}\left(\Omega\right)^{n}\right)$, the following limit holds

$$\int_0^T E\left(\theta_m,\zeta,\mathbf{u}_m\right)dt \longrightarrow \int_0^T E\left(\theta,\zeta,\mathbf{u}\right)dt, \quad \forall\,\zeta \in \mathcal{Z}_{q',r'}. \tag{4.59}$$

Therefore, since $\theta_m \in L^{r}\left(0,T;W^{1,q}\left(\Omega\right)\right)$ and $\dfrac{\partial\zeta}{\partial t} \in L^{r'}\left(0,T;W^{1,q'}\left(\Omega\right)\right)$, we find the following limit

$$\int_Q \theta_m \frac{\partial\zeta}{\partial t}dxdt \longrightarrow \int_Q \theta_m \frac{\partial\zeta}{\partial t}dxdt, \quad \forall\,\zeta \in \mathcal{Z}_{q',r'}. \tag{4.60}$$

Moreover, since $\alpha_m \longrightarrow \alpha$ and $\gamma_m \longrightarrow \gamma$ a.e. in Q, the functions μ and g are continuous and due to the fact that $\mathcal{Z}_{q',r'} \subset L^{r'}\left(0,T;\mathcal{C}^0\left(\bar{\Omega}\right)\right)$ (by the Sobolev imbedding $W^{1,q'}\left(\Omega\right) \subset \mathcal{C}^0\left(\bar{\Omega}\right)$), we get, by application of the Lebesgue dominated convergence theorem and thanks to (4.57)

$$\int_Q F\left(\mathbf{u}_m,\lambda_m,\mu_m\right)\zeta dxdt \longrightarrow \int_Q F\left(\mathbf{u},\lambda,\mu\right)\zeta dxdt, \quad \forall\,\zeta \in \mathcal{Z}_{q',r'}. \tag{4.61}$$

Now, it remains to prove the following limit

$$\int_\Omega \theta_{0m}\zeta\left(0\right)dx \longrightarrow \int_\Omega \theta_0\zeta\left(0\right)dx, \quad \forall\,\zeta \in \mathcal{Z}_{q',r'}. \tag{4.62}$$

To this aim, let us remark, by the definition of the space function $\mathcal{Y}_{q,r}$, that $\theta_m \in L^{r}\left(0,T;W^{1,q}\left(\Omega\right)\right)$ and $\dfrac{\partial\theta_m}{\partial t} \in L^{1}\left(0,T;W^{-1,1}\left(\Omega\right)\right)$. Consequently, from trace theorems, after a possible modification on a set of measure zero, θ_m is continuous from $[0,T]$ into $L^1\left(\Omega\right)$, which asserts that the condition $\theta_m\left(0\right) = \theta_{0m}$ has sense in the space $L^1\left(\Omega\right)$ and

$$\theta_{0m} \longrightarrow \theta_0 \text{ in } L^1\left(\Omega\right) \text{ weakly}. \tag{4.63}$$

On the other hand, the condition $\zeta \in \mathcal{Z}_{q',r'}$ entrains that

$$\zeta \in L^{r'}\left(0,T;W^{1,q'}\left(\Omega\right)\right) \quad \text{and} \quad \frac{\partial\zeta}{\partial t} \in L^{r'}\left(0,T;W^{1,q'}\left(\Omega\right)\right).$$

Hence, after a possible modification on a set of measure zero, ζ is continuous from $[0,T]$ into $W^{1,q'}(\Omega)$, then, via the Sobolev imbedding $W^{1,q'}(\Omega) \subset \mathcal{C}^0\left(\bar{\Omega}\right)$, we get

$$\zeta(0) \in \mathcal{C}^0\left(\bar{\Omega}\right). \tag{4.64}$$

Thus, (4.62) is an immediate consequence of (4.63) and (4.64). From (4.41), (4.59), (4.60), (4.61) and (4.62), we deduce that θ is solution to problem (4.14). We conclude finally that $\mathbf{u}_m \longrightarrow \mathbf{u}$ in $L^\infty(0,T;\mathcal{V})$ strongly and $(C_m, \theta_m) \longrightarrow (C, \theta)$ in $\mathcal{X} \times \mathcal{Y}_{q,r}$ weakly, where $(\mathbf{u}, C, \theta) \in \mathcal{L}(\mathbf{w}, \alpha, \gamma)$. By virtue of Kakutani-Glicksberg's fixed point theorem, the mapping $\mathcal{L}$ admits a fixed point $(\mathbf{u}, C, \theta) \in \mathcal{L}(\mathbf{u}, C, \theta)$, which solves the problem (P3). $\square$

Remark 4.5. This proof permits also to verify the continuous dependence of the solution $\left(\mathbf{u}(\mathbf{w}, \alpha, \gamma), \tilde{C}(\mathbf{w}, \alpha, \gamma), \theta(\mathbf{w}, \alpha, \gamma)\right) \in L^\infty(0,T;\mathcal{V}) \times \mathcal{X} \times \mathcal{Y}_{q,r}$ of problems (4.5), (4.8) and (4.14) with respect to the auxiliary function $(\mathbf{w}, \alpha, \gamma) \in L^\infty(0,T;\mathcal{V}) \times \mathcal{X} \times \mathcal{Y}_{q,r}$.

Theorem 4.6. (Positivity of the concentration) Let the hypotheses of Theorem 4.1 hold and suppose in addition that

$$R \geq 0 \text{ a.e. in } Q, \tag{4.65}$$

$$C_0 \geq 0 \text{ a.e. in } \Omega. \tag{4.66}$$

Then, the concentration C is such that

$$C(x,t) \geq 0 \text{ for a.e. in } \mathbb{Q}. \tag{4.67}$$

Proof. Let us replace in equation (3.7) C and ξ by the functions $C(t) = e^t C(t)$ and $\xi(t) = e^t \tilde{\xi}(t)$ (as in the proof of Proposition 4.3), it follows that

$$\int_\Omega \frac{\partial C}{\partial t}\tilde{\xi} dx + \int_\Omega C\tilde{\xi} dx - E\left(C, \tilde{\xi}, \mathbf{u}\right) + \int_\Omega \eta(\alpha, \gamma) \nabla C \cdot \nabla\tilde{\xi} dx$$
$$= \int_\Omega e^{-t} R\tilde{\xi} dx, \quad \forall\, \tilde{\xi} \in H^1(\Omega),$$

Testing the equation above by the function $-C^-$ and integrate over the interval time $(0,t)$. We find, using Lemma 2.1 and (4.66)

$$\left\|C^-(t)\right\|^2_{L^2(\Omega)} + c_1 \int_0^t \left\|C^-(a)\right\|^2_{H^1(\Omega)} da$$
$$\leq -\int_0^t \int_\Omega e^{-a} R(x,a)\, C^-(x,a)\, dx da \quad \text{a.e. } t \in (0,T).$$

Then, via (4.65), one can easily obtain

$$\left\| C^- \right\|_{L^2(0,T;H^1(\Omega)) \cap L^2(0,T;L^\infty(\Omega))} \leq 0.$$

Which permits us to conclude the proof. □

We show in the following Theorem a property of integral of the temperature.

Theorem 4.7. *Let the hypotheses of Theorem* 4.1 *hold and suppose in addition that the Robin coefficient* $\beta = 0$ *and*

$$h \geq 0 \quad a.e.\ in\ Q, \tag{4.68}$$

$$\theta_0 \geq 0 \quad a.e.\ in\ \ \Omega. \tag{4.69}$$

Then, there exists a positive constant d *depending only on the data of problem, such that*

$$\int_\Omega \theta_0(x)\, dx \leq \int_\Omega \theta(x,t)\, dx \leq d \quad a.e.\ t \in (0,T). \tag{4.70}$$

Proof. We proceed by testing the inequality (3.1) by the constant vector $\mathbf{0}$ and the equation (3.3) by the constant function 1. It follows

$$\int_\Omega \Big(2\mu(C,\theta) \left| \varepsilon(\mathbf{u}) \right|^2 dx + g(C,\theta) \left| \varepsilon(\mathbf{u}) \right| \Big)\, dx \leq \int_\Omega \mathbf{f} \cdot \mathbf{u} dx, \tag{4.71}$$

$$\int_\Omega \frac{\partial \theta}{\partial t} dx = \int_\Omega \Big(2\mu(C,\theta) \left| \varepsilon(\mathbf{u}) \right|^2 dx + g(C,\theta) \left| \varepsilon(\mathbf{u}) \right| \Big)\, dx + \int_\Omega h dx. \tag{4.72}$$

Hence, from (4.68) and hypothesis (2.11), we eventually get

$$\int_\Omega \frac{\partial \theta}{\partial t} dx \geq 0 \quad \text{a.e. } t \in (0,T). \tag{4.73}$$

Furthermore, combining (4.71) with (4.72) and integrating over the interval time $(0,t)$ one obtains, using standard arguments

$$\begin{aligned} 0 &\leq \int_\Omega (\theta(x,t) - \theta_0(x))\, dx \\ &\leq \|\mathbf{f}\|_{L^\infty(0,T;\mathcal{V}')} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{V})} + \|h\|_{L^1(Q)} \quad \text{a.e. } t \in (0,T). \end{aligned}$$

Consequently, (4.70) can be easily deduced, recalling (4.1) and (4.69). □

5. Coagulation modelling

We study in this section the mathematical modelling of blood coagulation. To this aim, we consider the steady-state flow without taking into account the thermal effects and we suppose that the blood diffusion coefficient η depends only on the space variable x and such that $\eta > 0$ and $\eta \in L^{\infty}(\Omega)$. In addition, we replace the Neumann boundary condition for the concentration by an homogeneous Dirichlet boundary condition. The bio-mechanical problem can be formulated as follows.

Problem P4. Find the velocity field $\mathbf{u} = (u_i) : \Omega \longrightarrow \mathbb{R}^n$, the stress field $\sigma = (\sigma_{ij}) : \Omega \longrightarrow \mathbb{S}_n$ and the concentration $C : \Omega \longrightarrow \mathbb{R}$ such that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{div}(\sigma) + \mathbf{f} \text{ in } \Omega, \tag{5.1}$$

$$\begin{cases} \tilde{\sigma} = \mu(C)\,\varepsilon(\mathbf{u}) + g(C)\dfrac{\varepsilon(\mathbf{u})}{|\varepsilon(\mathbf{u})|} & \text{if } |\varepsilon(\mathbf{u})| \neq 0 \\ |\tilde{\sigma}| \leq g(C,\theta) & \text{if } |\varepsilon(\mathbf{u})| = 0 \end{cases} \text{ in } \Omega, \tag{5.2}$$

$$\operatorname{div}(\mathbf{u}) = 0 \text{ in } \Omega, \tag{5.3}$$

$$\mathbf{u} \cdot \nabla C - \operatorname{div}(\eta \nabla C) = R \text{ in } \Omega, \tag{5.4}$$

$$\mathbf{u} = 0 \text{ on } \Gamma, \tag{5.5}$$

$$C = 0 \text{ on } \Gamma. \tag{5.6}$$

We can easily prove, using Green's formula, that the variational formulation of bio-mechanical problem (P4) can be written.

Problem P5. For prescribed data $\mathbf{f} \in \mathcal{V}'$, $R \in H^{-1}(\Omega)$. Find the velocity field $\mathbf{u} \in \mathcal{V}$ and the concentration $C \in H_0^1(\Omega)$ satisfying the system

$$\begin{aligned} B(\mathbf{u},\mathbf{u},\mathbf{v}) + 2\int_{\Omega} \mu(C)\,\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v}-\mathbf{u})\,dx + \int_{\Omega} g(C)\,|\varepsilon(\mathbf{v})|\,dx \\ - \int_{\Omega} g(C)\,|\varepsilon(\mathbf{u})|\,dx \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{v}-\mathbf{u})\,dx, \quad \forall\, \mathbf{v} \in \mathcal{V}, \end{aligned} \tag{5.7}$$

$$-E(C,\xi,\mathbf{u}) + \int_{\Omega} \eta \nabla C \cdot \nabla \xi dx = \int_{\Omega} R\xi dx, \quad \forall\, \xi \in H_0^1(\Omega), \tag{5.8}$$

Kakutani-Gliksberg's fixed point theorem permits also to deduce that the variational problem P5 admits a solution $(\mathbf{u}, C) \in \mathcal{V} \times H_0^1(\Omega)$, see [19, 20]. Our goal now is to give the mathematical interpretation of blood coagulation for the steady-state problem. To do this, let us recall the following standard definition.

Definition 5.1. The blood is coagulated in the domain Ω, if the fluid is blocked in Ω, it means that $\mathbf{u} \equiv 0$ is solution of the variational inequality (5.7).

Thus, one can easily verify that the blood is coagulated in the domain Ω if and only if the following system holds

$$\int_\Omega g\left(C_g\right)\left|\varepsilon\left(\mathbf{v}\right)\right|\,dx \geq \int_\Omega \mathbf{f}\cdot\mathbf{v}dx, \quad \forall\,\mathbf{v}\in\mathcal{V}, \tag{5.9}$$

$$\int_\Omega \eta\nabla C_g\cdot\nabla\xi dx = \int_\Omega R\xi dx, \quad \forall\,\xi\in H_0^1\left(\Omega\right). \tag{5.10}$$

It is easy to check that equation (5.10) admits one and one solution $C_g \in H_0^1\left(\Omega\right)$. Hence, the mathematical study of the blood coagulation consists in finding the link between $g\left(C_g\right)$ and $\mathbf{f}$ such that the inequality (5.9) holds.

The following proposition has been obtained by H. Patrick et al [22], which ensures the existence of a blocking state for large enough yield limit.

Proposition 5.2. *If $\mathbf{f}\in L^\infty\left(\Omega\right)^n$ then*

$$g_{coag} = \sup_{\mathbf{v}\in\mathcal{V}\setminus\{0\}} \frac{\int_\Omega \mathbf{f}\cdot\mathbf{v}dx}{\int_\Omega\left|\varepsilon\left(\mathbf{v}\right)\right|\,dx} < +\infty, \tag{5.11}$$

and if $g\left(C_g\right)\geq g_{coag}$ a.e. $x\in\Omega$, then the blocking occurs, i.e. (5.9) *holds, which means in our case that the blood is coagulated in the domain Ω.*

We compare in the following statement the concentration for the non-coagulation phase and that of coagulation phase.

Theorem 5.3. (Maximum principle) *Let $\left(\mathbf{u},C\right)\in\mathcal{V}\times H_0^1\left(\Omega\right)$ be the solution of probelm P5 and C_g the unique solution of equation* (5.10)*. Suppose that $R\geq 0$ a.e. $x\in\Omega$, then*

$$C,\ C_g \geq 0, \tag{5.12}$$

$$\left\|\nabla C_g\right\|_{L^2(\Omega)^n} \geq \left\|\nabla C\right\|_{L^2(\Omega)^n}. \tag{5.13}$$

In addition, if the function R verifies the following assumption

$$\exists\, b\in L^\rho\left(\Omega\right)^n : R = -div\left(b\right),\ \ \rho > n, \tag{5.14}$$

then

$$C,\ C_g \in L^\infty\left(\Omega\right). \tag{5.15}$$

Proof. To prove (5.12) it is enough to choose $-C^-$ and $-C_g^-$ as test functions in equations (5.8) and (5.10), respectively, and proceeding as in the proof of theorem 4.7. For the proof of (5.13), we proceed as follows. Subtracting (5.10) from (5.8) and choosing $C_g - C$ as test function in the obtained equation, it follows via Lemma 2.1

$$\operatorname{ess\,inf}\left(\eta\right)\int_\Omega\left|\nabla\left(C_g - C\right)\right|^2 dx + E\left(C, C_g, \mathbf{u}\right) = 0.$$

Which asserts that

$$E\left(C, C_g, \mathbf{u}\right) \leq 0. \tag{5.16}$$

On the other hand, it follows by setting C_g and C as test functions in (5.8) and (5.10), respectively, and subtracting the two obtained equations

$$E\left(C, C_g, \mathbf{u}\right) = \int_\Omega R\left(C - C_g\right) dx. \tag{5.17}$$

Furthermore, by choosing C and C_g as test functions in (5.8) and (5.10), respectively, and subtracting the two obtained equations, the following equation holds

$$\operatorname{ess\,inf}\left(\eta\right) \int_\Omega \left(\left|\nabla C_g\right|^2 - \left|\nabla C\right|^2\right) dx = \int_\Omega R\left(C_g - C\right) dx. \tag{5.18}$$

Thus, (5.13) can be easily deduced from (5.16), (5.17), (5.18).

Now, we prove (5.15). To do this, let us consider for each $k \geq 0$, the cut functions T_k defined by

$$T_k = \left(x - k\right)^- - \left(x + k\right)^+ .$$

We now, see [26] that $\nabla T_k\left(C\right) = \mathbf{1}_{E_{C,k}} \nabla C \in H_0^1\left(\Omega\right)$ where $\mathbf{1}_{E_{C,k}}$ is the indicator function of the set $E_{C,k} = \left\{|C| \geq k\right\}$. We use $T_k\left(C_g\right)$ as test function in (5.10), keeping in mind (5.14), we can infer

$$\begin{aligned}
\int_\Omega b \cdot \mathbf{1}_{E_{C,k}} \nabla C_g dx &= \int_\Omega \eta \nabla C_g \cdot \left(\mathbf{1}_{E_{C,k}} \nabla C_g\right) dx \\
&= \int_\Omega \eta \left(\mathbf{1}_{E_{C,k}} \nabla C_g\right) \cdot \left(\mathbf{1}_{E_{C,k}} \nabla C_g\right) dx \\
&\geq \operatorname{ess\,inf}\left(\eta\right) \int_\Omega \left|\nabla T_k\left(C_g\right)\right|^2 dx.
\end{aligned}$$

Hence, we find using Hölder's inequality

$$\int_\Omega \left|\nabla T_k\left(C_g\right)\right|^2 dx \leq \frac{\|b\|_{L^\rho(\Omega)^n}}{\operatorname{ess\,inf}\left(\eta\right)} \left(\int_\Omega \left|\nabla T_k\left(C_g\right)\right|^{\rho'} dx\right)^{\frac{1}{\rho'}} . \tag{5.19}$$

Moreover, Hölder's inequality leads also to

$$\begin{aligned}
\int_\Omega \left|\nabla T_k\left(C_g\right)\right|^{\rho'} dx &= \int_\Omega \mathbf{1}_{E_k} \left|\nabla T_k\left(C_g\right)\right|^{\rho'} dx \\
&\leq \left(\int_\Omega \left|\nabla T_k\left(C_g\right)\right|^2 dx\right)^{\frac{\rho'}{2}} \left|E_{C_g,k}\right|^{\frac{2-\rho'}{2}} ,
\end{aligned} \tag{5.20}$$

where $|.|$ denotes the Lebesgue measure. Combining now (5.19) and (5.20), we can infer

$$\left\|\nabla T_k\left(C_g\right)\right\|_{L^2(\Omega)} \leq \frac{\|b\|_{L^\rho(\Omega)^n}}{\operatorname{ess\,inf}\left(\eta\right)} \left|E_{C_g,k}\right|^{\frac{2-\rho'}{2\rho'}} . \tag{5.21}$$

Let us choose $h \geq k$. We remark that in the set $E_{C_g,h}$, $|T_k(C_g)| \geq h-k$. Thus

$$\begin{aligned}\int_\Omega |T_k(C_g)|^{\frac{n}{n-1}}\,dx &\geq \int_{E_{C_g,h}} (h-k)^{\frac{n}{n-1}}\,dx \\ &= (h-k)^{\frac{n}{n-1}} \left|E_{C_g,h}\right|.\end{aligned} \tag{5.22}$$

Furthermore, the Sobolev imbedding $W_0^{1,1}(\Omega) \subset L^{\frac{n}{n-1}}(\Omega)$ gives

$$\begin{aligned}\int_\Omega |T_k(C_g)|^{\frac{n}{n-1}}\,dx &= \|T_k(C_g)\|_{L^{\frac{n}{n-1}}(\Omega)}^{\frac{n}{n-1}} \\ &\leq c\,\|T_k(C_g)\|_{W_0^{1,1}(\Omega)}^{\frac{n}{n-1}} \\ &= c\left(\int_\Omega |\nabla T_k(C_g)|\,dx\right)^{\frac{n}{n-1}}.\end{aligned} \tag{5.23}$$

(5.22) and (5.23) permit us to find, using again (5.21) and Hölder's inequality

$$\begin{aligned}(h-k)\left|E_{C_g,h}\right|^{\frac{n-1}{n}} &\leq c\int_\Omega |\nabla T_k(C_g)|\,\mathbf{1}_{E_{C,k}}dx \\ &\leq c\,\|\nabla T_k(C_g)\|_{L^2(\Omega)}\left|E_{C_g,k}\right|^{\frac{1}{2}} \\ &\leq c\frac{\|b\|_{L^\rho(\Omega)^n}}{\operatorname{ess\,inf}(\eta)}\left|E_{C_g,k}\right|^{\frac{2-\rho'}{2\rho'}+\frac{1}{2}}.\end{aligned}$$

We obtain finally

$$\left|E_{C_g,h}\right| \leq \frac{M}{(h-k)^{\lambda_2}}\left|E_{C_g,k}\right|^{\lambda_1},$$

where

$$\lambda_1 = \frac{n}{\rho'(n-1)} > 1, \quad \lambda_2 = \frac{n}{n-1} \quad \text{and} \quad M = \left(c\frac{\|b\|_{L^\rho(\Omega)^n}}{\operatorname{ess\,inf}(\eta)}\right)^{\frac{n}{n-1}}.$$

Then, if there exists h_0 such that $\left|E_{C_g,h_0}\right| = 0$, inequality (5.22) gives $|C_g| \leq h_0$ a.e. $x \in \Omega$, which permits as to deduce that $C_g \in L^\infty(\Omega)$.

For the existence of h_0, we recall following result due to Droniu and Imbert [12].

Lemma 5.4. *Let $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a decreasing function such that there exists $\lambda_1 > 1$ and $\lambda_2 > 0$ satisfying, for every $h > k$*

$$\Phi(h) \leq \frac{M}{(h-k)^{\lambda_2}}\Phi(k)^{\lambda_1}.$$

Then, there exists a constant c_0 *such that*

$$\Phi\left(c_0\left(2M\right)^{\frac{1}{\lambda_2}}\Phi\left(0\right)^{\frac{\lambda_1}{\lambda_2-1}}\right)=0.$$

Now, to achieve the proof of the theorem it remains to verify that $C\in L^\infty(\Omega)$. To this end, it is enough to choose $T_k(C)$ as test function in (5.8) and remark, using the definition of $\mathbf{1}_{E_{C_g,k}}$ and Lemma 2.1, that

$$E\left(C,T_k\left(C\right),\mathbf{u}\right)=\int_\Omega C\mathbf{1}_{E_{C_g,k}}\nabla\left(C\right)\cdot\mathbf{u}dx=0.$$

□

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