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BALL CONVERGENCE THEOREMS FOR KING'S FOURTH-ORDER ITERATIVE METHODS UNDER WEAK CONDITIONS

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Abstract. We present a local convergence analysis for King's fourth-order iterative methods in order to approximate a solution of a nonlinear equation. We use hypotheses up to the first derivative in contrast to earlier studies such as [1, 7]–[27] using hypotheses up to the third derivative (or even higher). This way the applicability of these methods is extended under weaker hypotheses. Moreover the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples where earlier results cannot be used to solve equations but our results can be used are also presented in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq S \to S$ is a nonlinear function, D is a convex subset of S and S is $\mathbb{R}$ or $\mathbb{C}$. Newton-like methods are used for finding solution of (1.1), these methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure;

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while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [1]-[27].

Third order methods such as Euler's, Halley's, super Halley's, Chebyshev's [1]–[27] require the evaluation of the second derivative F'' at each step, which in general is very expensive. To overcome this difficulty, many third order methods have been introduced. In particular, J. Kou, Y.Li and X. Wang in [16] introduced iterative methods defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - \frac{F(x_n)}{F'(x_n) + \lambda_n F(x_n)}, \\ x_{n+1} &= x_n - \frac{2F(x_n)}{F'(y_n) + F'(x_n) + \mu_n F^2(x_n)} \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} z_n &= x_n - \frac{F(x_n)}{2(F'(x_n) + \lambda_n F(x_n))}, \\ x_{n+1} &= x_n - \frac{F(x_n)}{F'(z_n) + \mu_n F^2(x_n)}, \end{aligned} \tag{1.3}$$

where x_0 is an initial point and $\{\lambda_n\}, \{\mu_n\}$ are given bounded sequences in S. The third order of convergence was shown in [16] under the assumptions that there exists a single root $x^* \in D; F$ is three times differentiable;

$$\begin{aligned} sign(\lambda_n F(x_n)) &= sign(F'(x_n)), \\ sign(\mu_n) &= sign(F'(x_n) + F'(y_n)) \text{ (for method (1.2))} \end{aligned}$$

and

$$\begin{aligned} sign(\mu_n F(x_n)) &= sign(F'(x_n)), \\ sign(\mu_n) &= sign(F'(\frac{1}{2}(x_n + z_n)), \end{aligned}$$

for each $n = 0, 1, 2, \cdots$ (for method (1.3)), where

$$sign(t) = \begin{cases} 1, & t \geq 0, \\ -1, & t < 0, \end{cases}$$

is the *sign* function. Method (1.2) and Method (1.3) were introduced as alternatives to other iterative methods that do not converge to x^* if the derivative of the function is either zero or very small in the vicinity of the solution (see e.g. [1, 7, 14]-[16], [19, 26, 27]) King's family of iterative methods defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= y_n - A_n^{-1} F'(x_n)^{-1} F(y_n), \end{aligned} \tag{1.4}$$

where $A_n = \frac{F(x_n)+\alpha F(y_n)}{F(x_n)+(2+\alpha)F(y_n)}$ and α is a parameter, is a popular fourth order method [1, 3, 7]. This family includes the Ostrowski's method for $\beta = -2$ [2, 3, 21, 24]. Other single and multi-point methods can be found in [2, 3, 19, 24] and the references there in. The convergence of the preceding methods has been shown under hypotheses up to the third derivative (or even higher). These hypotheses restrict the applicability of the preceeding methods. As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ f'(1) = 3, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, function f''' is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative. This way we expand the applicability of method (1.4).

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of methods (1.4). The numerical examples are presented in the concluding Section 3.

2. Local convergence for method (1.4)

We present the local convergence analysis of method (1.4) in this section. Let $U(v,\rho), \bar{U}(v,\rho)$ the open and closed balls in S, respectively, with center $v \in S$ and of radius $\rho > 0$.

For the local convergence analysis that follows, we define some functions and parameters. Let $L_0 > 0, L > 0, M \geq 1$ and $\alpha > -\frac{5}{2}$ be given parameters. Let

$$r_1 = \frac{2}{2L_0 + L} < \frac{1}{L_0}.$$

Define functions on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} g_1(t) &= \frac{Lt}{2(1-L_0t)}, \\ g_0(t) &= |\alpha+3|^{-1}[\frac{L_0}{2}(1+|\alpha+2|g_1^2(t))t + |\alpha+2|(1+g_1(t))], \\ h_0(t) &= g_0(t) - 1, \\ g_2(t) &= \frac{2Mg_1(t)t}{|\alpha+3|(1-g_0(t))}, \end{aligned}$$

$$\begin{aligned} h_2(t) &= 2Mg_1(t)t + |\alpha+3|(g_0(t)-1), \\ g_3(t) &= (1 + \frac{M}{(1-L_0t)(1-g_2(t))})g_1(t) \end{aligned}$$

and

$$h_3(t) = g_3(t) - 1.$$

Then, we have that $g_1(r_1) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$. We also get by the choice of α that $h_0(0) = g_0(0) - 1 = \frac{|\alpha+2|}{|\alpha+3|} - 1 < 0$ and $h_0(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It follows from the Intermediate value theorem that function h_0 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_0 the smallest such zero. Moreover, we obtain that $h_2(0) = |\alpha+3|(g_0(0)-1) < 0$ and $h_2(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Hence, function h_2 has also zero in the interval $(0, \frac{1}{L_0})$. Denote by r_2 the smallest such zero. Furthermore, we also get that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Denote by r_3 the smallest zero of function h_3 in the interval $(0, \frac{1}{L_0})$. Set

$$r = \min\{r_i\}, \ i = 0, 1, 2, 3. \tag{2.1}$$

Then, we have that

$$0 \leq g_1(t) < 1, \tag{2.2}$$

$$0 \leq g_0(t)t < 1, \tag{2.3}$$

$$0 \leq g_2(t)t < 1 \tag{2.4}$$

and

$$0 \leq g_3(t) < 1 \quad \text{for each} \quad t \in [0, r). \tag{2.5}$$

Next, using the above notation we can show the local convergence result for method (1.4).

Theorem 2.1. *Let $F : D \subseteq S \to S$ be a differentiable function. Suppose that there exist $x^* \in D$, parameters $L_0 > 0, L > 0, M \geq 1$ and $\alpha > -\frac{5}{2}$ such that for each x, $y \in D$ the following hold*

$$F(x^*) = 0, \ F'(x^*) \neq 0,$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq L_0|x - x^*|, \tag{2.6}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \leq L|x - y|, \tag{2.7}$$

$$|F'(x^*)^{-1}F'(x)| \leq M, \tag{2.8}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.9}$$

where r is given by (2.1). Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.4) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$

and converges to x^*. *Moreover, the following estimates hold for each* $n = 0, 1, 2, \cdots$,

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r, \tag{2.10}$$

$$|A_n^{-1}| \leq \frac{1}{1 - g_2(|x_n - x^*|)} \tag{2.11}$$

and

$$|x_{n+1} - x^*| \leq g_3(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{2.12}$$

where the "g" functions are defined above Theorem 2.1. *Furthermore, suppose that there exists* $T \in [r, \frac{2}{L_0})$ *such that* $\bar{U}(x^*, T) \subset D$, *then the limit point* x^* *is the only solution of equation* $F(x) = 0$ *in* $\bar{U}(x^*, T)$.

Proof. By hypothesis $x_0 \in U(x^*, r)$, the definition of r_2 and (2.6) we get that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1. \tag{2.13}$$

It follows from (2.13) and the Banach Lemma on invertible functions [2, 3, 21, 24] that $F'(x_0)$ is invertible and

$$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1 - L_0|x_0 - x^*|} < \frac{1}{1 - L_0 r}. \tag{2.14}$$

Hence, y_0 is well defined by the first substep of method (1.4) for $n = 0$. Using (2.8), we get the estimate

$$\begin{aligned} |F'(x^*)^{-1}F(x_0)| &= |F'(x^*)^{-1}(F(x_0) - F(x^*))| \\ &= \left| \int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta \right| \\ &\leq M|x_0 - x^*|, \end{aligned} \tag{2.15}$$

since $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$ for each $\theta \in [0, 1]$. We also have that

$$y_0 - x^* = x_0 - x^* - \frac{F(x_0)}{F'(x_0)}. \tag{2.16}$$

Using (2.1), (2.3), (2.7) and (2.14) we get in turn that

$$\begin{aligned} |y_0 - x^*| &\leq |F'(x_0)^{-1}F'(x^*)| \left| \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) \right. \\ &\quad \left. -F'(x_0)]d\theta(x_0 - x^*) \right| \\ &\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} \\ &= g_1(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r, \end{aligned}$$

which shows (2.10) for $n = 0$.

Next, we shall show that A_0 is invertible. In view of (2.1), (2.2), (2.10) (for $n=0$) and (2.15) we get in turn since $x_0 \neq x^*$ that

$$\begin{aligned}
& |((\alpha+3)F'(x^*)(x_0-x^*))^{-1}(F(x_0)+(2+\alpha)F(y_0)-(\alpha+3)F'(x^*)(x_0-x^*)| \\
\leq\ & (|\alpha+3||x_0-x^*|)^{-1}\Big[|F'(x^*)^{-1}(F(x_0)-F(x^*)-F'(x^*)(x_0-x^*)| \\
& +|\alpha+2||F'(x^*)^{-1}(F(y_0)-F(x^*)-F'(x^*)(y_0-x^*)| \\
& +|\alpha+2|(|x_0-x^*|+|y_0-x^*|)\Big] \\
\leq\ & (|\alpha+3||x_0-x^*|)^{-1}\Big[\frac{L_0}{2}(|x_0-x^*|^2+|\alpha+2||y_0-x^*|^2) \\
& +|\alpha+2|(1+g_1(|x_0-x^*|))|x_0-x^*|\Big] \\
\leq\ & |\alpha+3|^{-1}\Big[\frac{L_0}{2}(1+|\alpha+2|g_1^2(|x_0-x^*|))|x_0-x^*| \\
& +|\alpha+2|(1+g_1(|x_0-x^*|))\Big] \\
=\ & g_0(|x_0-x^*|) < g_0(r) < 1. \qquad (2.17)
\end{aligned}$$

It follows from (2.17) that $F(x_0)+(\alpha+2)F(y_0)$ is invertible and

$$|(F(x_0)+(\alpha+2)F(y_0))^{-1}F'(x^*)| \leq \frac{1}{|\alpha+3||x_0-x^*|(1-g_0(|x_0-x^*|))}. \qquad (2.18)$$

Then, using (2.1), (2.4), (2.15) (for $n=0$) and (2.18), we get that

$$\begin{aligned}
|A_0-I| &= |2(F(x_0)+(\alpha+2)F(y_0))^{-1}F'(x^*)(F'(x^*)^{-1}F(y_0))| \\
&\leq \frac{2M|y_0-x^*|}{|\alpha+3||x_0-x^*|(1-g_0(|x_0-x^*|))} \\
&\leq \frac{2Mg_1(|x_0-x^*|)|x_0-x^*|}{|\alpha+3|(1-g_0(|x_0-x^*|))} \\
&= g_2(|x_0-x^*|) < g_2(r) < 1. \qquad (2.19)
\end{aligned}$$

It follows from (2.19) that A_0 is invertible and (2.11) satisfied for $n=0$. Then, using the second substep of method (1.4) for $n=0$, (2.1), (2.5), (2.10), (2.11)(for $n = 0$), (2.14) and (2.15) (for $y_0 = x_0$) we get in turn that

$$\begin{aligned}
|x_1 - x^*| &\leq |y_0 - x^*| + |A_0^{-1}||F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}F(y_0)| \\
&\leq |y_0 - x^*| + \frac{M|y_0 - x^*|}{(1 - L_0|x_0 - x^*|)(1 - g_2(|x_0 - x^*|))} \\
&= \left[1 + \frac{M}{(1 - L_0|x_0 - x^*|)(1 - g_2(|x_0 - x^*|))}\right]|y_0 - x^*| \\
&\leq \left[1 + \frac{M}{(1 - L_0|x_0 - x^*|)(1 - g_2(|x_0 - x^*|))}\right]g_1(|x_0 - x^*|)|x_0 - x^*| \\
&= g_3(|x_0 - x^*|)|x_0 - x^*| \\
&< g_3(r)|x_0 - x^*| \ < \ |x_0 - x^*| \ < \ r,
\end{aligned}$$

which shows (2.12) for $n = 0$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at estimate (2.10)–(2.12). Using the estimate $|x_{k+1} - x^*| < |x_k - x^*| < r$, we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.6) we get that

$$\begin{aligned}
|F'(x^*)^{-1}(Q - F'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\
&\leq \int_0^1 (1-\theta)|x^* - y^*|d\theta \ \leq \ \frac{L_0}{2}T \ < \ 1. \quad (2.20)
\end{aligned}$$

It follows from (2.20) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. □

Remark 2.2. (1) In view of (2.6) and the estimate

$$\begin{aligned}
|F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I| \\
&\leq 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq 1 + L_0|x - x^*|
\end{aligned}$$

condition (2.8) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t.$$

(2) The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x)),$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

(3) It is worth noticing that method (1.4) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [1, 7, 14, 15, 16, 19, 26, 27]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

3. Numerical examples

We present two numerical examples in this section.

Example 3.1. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 101.5578008, \alpha = 1$. The parameters are given in Table 1.

$r_0 = 0.0175$
$r_1 = 0.0045$
$r_2 = 0.0945$
$r_3 = 0.0001$
$r = 0.0001$
$\xi_1 = 0.9997$
$\xi = 0.9994$

Table 1

Example 3.2. Let $D = [-1, 1]$. Define function f of D by

$$f(x) = e^x - 1. \tag{3.1}$$

Using (3.1) and $x^* = 0$, we get that $L_0 = e - 1 < L = M = e, \alpha = 1$. The parameters are given in Table 2.

$r_0 = 0.6597$
$r_1 = 0.3249$
$r_2 = 0.3413$
$r_3 = 0.1332$
$r = 0.1332$
$\xi_1 = 0.9932$
$\xi = 0.9932$

Table 2

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