Nonlinear Functional Analysis and Applications
Vol. 20, No. 3 (2015), pp. 429-440

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm
Copyright © 2015 Kyungnam University Press

SOME FIXED POINT THEOREMS FOR GENERALIZED T-CONTRACTION MAPPING IN COMPLETE CONE METRIC SPACES

G. S. Saluja

Department of Mathematics
Govt. Nagarjuna P.G. College of Science
Raipur - 492001 (C.G.), India
e-mail: saluja1963@gmail.com

Abstract. The purpose of this paper is to establish some fixed point theorems for generalized T-contraction mapping in the framework of complete cone metric spaces. The results presented in this paper extend, generalize and unify several known results from the existing literature.

1. Introduction

The classical Banach's contraction principle which was published in 1922 is one of the most useful result in fixed point theory. In 1968, Kannan [15] established a fixed point theorem, extending Banach's contraction principle to mappings that need not be continuous. Kannan's theorem was followed by a lot of papers, devoted to obtaining fixed point theorems for various class of contractive type condition that do not require continuity of the corresponding mappings. One of them, actually a sort of dual of Kannan's fixed point theorem, is of Chatterjea [6]. Another important result on fixed points for contractive type mapping in the frame work of compact metric space is generally attributed to Edelstein [9].

Recently, Huang and Zhang [11] used the notion of cone metric spaces as a generalization of metric spaces. They have replaced the real numbers by an ordered Banach space. The authors described the convergence in cone metric spaces and introduced their completeness. Then they proved some

[0]Received December 17, 2014. Revised March 9, 2015.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Fixed point, generalized T-contraction mapping, complete cone metric space.

fixed point theorems for contractive single-valued mappings in such spaces. In their theorems cone is normal. For more fixed point results in cone metric spaces, see [1, 3, 13, 20, 21, 24]. In this paper we establish some fixed point theorems for T-contraction mapping in the framework of complete cone metric spaces.

The concepts of T-Banach contraction and T-contractive mappings were introduced by Beiranvand et al. [5] in 2009 and they extended Banach's contraction principle and Edelstein fixed point theorem. Followed by this, Moradi [16] introduced T-Kannan contractive mappings, extending in the way, the well-known Kannan fixed point theorem [15].

Recently, Morales and Rojas ([17], [18]) have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems for T-Kannan, T-Zamfirescu and T-weakly contraction mappings.

On the other hand, Subrahmanyam [22] obtained the fixed point of continuous Banach operator of type k in a complete metric space. Recently, Chen and Li [7] extended the concept of Banach operator of type k to Banach operator pair and proved various best approximation results using common fixed point theorems for f-nonexpansive mappings, where f is a self mapping of the subset M of a metric space X. Hussain [12], Al-thagafi and Shahzad [2] generalizing the results of Chen and Li [7], and proved various common fixed point theorems and invariant approximation results for generalized nonexpansive Banach operator pair mappings.

The new class of noncommuting mappings is different from the class of commuting mappings (viz. R-weakly commuting, R-subweakly commuting, compatible, weakly compatible, C_q-commuting etc.) existing in the literature so far. Hence the concept of Banach operator pair is of basic importance for study of common fixed points in best approximation.

The purpose of this paper is to prove fixed point theorem for generalized T-contraction mapping in the setting of cone metric spaces. If in addition, the pair of mappings is a Banach pair, then we have obtained a common fixed point. Our results generalize recent existing results in the literature of T-contraction mappings and cone metric space.

2. Definitions and preliminaries

Here we recall some definitions and other results that will be needed in the sequel.

Definition 2.1. Let (X, d) be a metric space and $T\colon X \to X$ be a mapping. A mapping T is said be a contraction mapping, if there exists a real number $k \in [0, 1)$ such that for all $x, y \in X$, $d(Tx, Ty) \leq k\, d(x, y)$.

The following new definition is recently introduced by Beiranvand et al. [5].

Definition 2.2. ([5]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-contraction, if there exists a real number $k \in [0, 1)$ such that for all $x, y \in X$, $d(TSx, TSy) \leq k\, d(Tx, Ty)$.

If we take $T = I$, the identity map, in the Definition 2.2, then we obtain the definition of Banach's contraction.

The following example shows that a T-contraction mapping need not be a contraction mapping.

Example 2.3. Let $X = [1, \infty)$ be with the usual metric. Define two mappings $T, S\colon X \to X$ as $Tx = \frac{1}{2x} + 2$ and $Sx = 3x$. Obviously, S is not contraction but S is T-contraction which is seen from the following:

$$|TSx - TSy| = \left|\frac{1}{6x} - \frac{1}{6y}\right| = \frac{1}{3}|Tx - Ty|.$$

Definition 2.4. ([5]) Let T be a self mapping of a metric space (X, d). Then

(1) the mapping T is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.
(2) the mapping T is said to be subsequentially convergent, if $\{y_n\}$ has a convergent subsequence whenever $\{Ty_n\}$ is convergent.

The following theorem has been proved by Beiranvand et al. [5].

Theorem 2.5. ([5]) *Let (X, d) be a complete metric space and $T\colon X \to X$ be a one-to-one, continuous and subsequentially convergent mapping. Then every T-contraction and continuous self mapping $f\colon X \to X$ has a unique fixed point in X. Also, if T is sequentially convergent, then for each $x_0 \in X$, the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.*

The following is the definition introduced by Subrahmanyam [22].

Definition 2.6. ([22]) Let T be a self mapping of a normed space X. Then T is called a Banach operator of type k if

$$\|T^2x - Tx\| \leq k\|Tx - x\|,$$

for some $k \geq 0$ and for all $x \in X$.

Extending the concept of Subrahmanyam [22], Chen and Li [7] introduced the following definition in the setting of normed linear space.

Definition 2.7. ([7]) Let T and f be two self mappings of a nonempty subset M of a normed linear space X. Then (T, f) is a Banach operator pair, if any one of the following conditions is satisfied:

(1) $T[F(f)] \subseteq F(f)$, i.e. $F(f)$ is T-invariant.
(2) $fTx = Tx$ for each $x \in F(f)$.
(3) $fTx = Tfx$ for each $x \in F(f)$.
(4) $\|Tfx - fx\| \leq k\|fx - x\|$ for some $k \geq 0$.

Definition 2.8. ([11]) Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

(c_1) P is closed, nonempty and $P \neq \{0\}$;
(c_2) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
(c_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int\, P$ (interior of P). If $int\, P \neq \emptyset$ then P is called a solid cone (see [23]).

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones [8].

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{imply} \quad \|x\| \leq K\,\|y\|, \tag{2.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{2.2}$$

The least positive number K satisfying (2.1) is called the normal constant of P. It is clear that $K \geq 1$.

Example 2.9. ([23]) Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (2.2) that P is a non-normal cone.

Definition 2.10. ([11, 25]) Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to E$ satisfies:

(d_1) $0 \le d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(d_3) $d(x,y) \le d(x,z) + d(z,y)$ $x, y, z \in X$.

Then d is called a cone metric [11] on X and (X, d) is called a cone metric space [11].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 2.11. ([11]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x \ge 0, y \ge 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \ge 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $K = 1$.

Example 2.12. ([20]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\ge1} \in E : x_n \ge 0, \text{for all}\, n\}$, (X, ρ) a metric space, and $d\colon X \times X \to E$ defined by $d(x,y) = \{\rho(x,y)/2^n\}_{n\ge1}$. Then (X, d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 2.13. ([11]) Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll \varepsilon$;
(ii) a convergent sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n > N$, $d(x_n, x) \ll \varepsilon$ for some fixed x in X;
(iii) a cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X;
(iv) a self mapping $T\colon X \to X$ is said to be continuous at a point $x \in X$, if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for every $\{x_n\}$ in X.

The following two lemmas of Huang and Zhang [11] will be required in the sequel.

Lemma 2.14. ([11]) *Let (X, d) be a cone metric space and P be a normal cone with normal constant K. A sequence $\{x_n\}$ in X converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.*

Lemma 2.15. ([11]) *Let (X, d) be a cone metric space and P be a normal cone with normal constant K. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

The following corollary of Rezapour [20] will be needed in the sequel.

Corollary 2.16. ([20]) *Let $a, b, c, u \in E$, the real Banach space,*

(i) *If $a \leq b$ and $b \ll c$, then $a \ll c$.*
(ii) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(iii) *If $0 \leq u \ll c$ for each $c \in int\, P$, then $u = 0$.*

Remark 2.17. ([14]) If $c \in int\, P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$, it follows that $a_n \ll c$.

In the sequel we assume that E is a real Banach space and that P is a normal solid cone in E, that is, normal cone with $intP \neq \emptyset$. The last assumption is necessary in order to obtain reasonable results connected with convergence and continuity. The partial ordering induced by the cone P will be denoted by $\leq$.

Generalized T-Contraction Mapping

Let X be a cone metric space and $S, T \colon X \to X$ be two mappings. Then S is called generalized T-contraction mapping if it satisfies the following condition:

$$\begin{aligned} d(TSx, TSy) \;\; &\leq \;\; a\, d(Tx, Ty) + b\,[d(Tx, TSx) + d(Ty, TSy)] \\ &\quad\;\; + c\,[d(Tx, TSy) + d(Ty, TSx)], \end{aligned} \tag{2.3}$$

for all $x, y \in X$ and $a, b, c \in [0, 1)$ are constants such that $a + 2b + 2c < 1$.

Remark 2.18. (1) If $T = I$ (the identity map), $b = c = 0$ and $a \in [0, 1)$, then (2.3) reduces to contraction mapping defined by Banach [4].
(2) If $T = I$ (the identity map), $a = c = 0$ and $b \in [0, 1/2)$, then (2.3) reduces to contraction mapping defined by Kannan [15].
(3) If $T = I$ (the identity map), $c = 0$ and $a, b \in [0, 1/2)$, then (2.3) reduces to contraction mapping defined by Fisher [10].
(4) If $T = I$ (the identity map), $a = b = 0$ and $c \in [0, 1/2)$, then (2.3) reduces to contraction mapping defined by Chatterjae [6].
(5) If $T = I$ (the identity map), $b = 0$ and $a, c \in [0, 1)$, then (2.3) reduces to contraction mapping defined by Reich [19].

3. Main results

In this section we shall prove some fixed point theorems for generalized T-contractive condition.

Theorem 3.1. *Let T and S be two continuous self mappings of a complete cone metric space (X, d). Assume that T is an injective mapping and P be a normal cone with normal constant K. If the mappings T and S satisfy generalized T-contractive condition (2.3) with $a+2b+2c<1$ for some $a, b, c \in [0,1)$. Then S has a unique fixed point in X. Moreover, if (T, S) is a Banach pair, then T and S have a unique common fixed point in X.*

Proof. Choose $x_0 \in X$. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = Sx_n$ for each $n = 0, 1, 2, \ldots, \infty$. Consider,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\leq a\, d(Tx_{n-1}, Tx_n) + b\,[d(Tx_{n-1}, TSx_{n-1}) + d(Tx_n, TSx_n)] \\ &\quad +c\,[d(Tx_{n-1}, TSx_n) + d(Tx_n, TSx_{n-1})] \\ &\leq a\, d(Tx_{n-1}, Tx_n) + b\,[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\ &\quad +c\,[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)] \\ &\leq a\, d(Tx_{n-1}, Tx_n) + b\,[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\ &\quad +c\,[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_n, Tx_n)] \\ &= (a+b+c)\, d(Tx_{n-1}, Tx_n) + (b+c)\, d(Tx_n, Tx_{n+1}) \end{aligned}$$

implies

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \frac{(a+b+c)}{(1-b-c)}\, d(Tx_{n-1}, Tx_n) \\ &= h\, d(Tx_{n-1}, Tx_n), \end{aligned} \tag{3.1}$$

where

$$h = \frac{(a+b+c)}{(1-b-c)}.$$

As $a + 2b + 2c < 1$ by the assumption of the theorem, we obtain that $h < 1$. Proceeding further, we have

$$d(Tx_n, Tx_{n+1}) \leq h^n\, d(Tx_0, Tx_1). \tag{3.2}$$

Next, we claim that $\{Tx_n\}$ is a Cauchy sequence. Consider m, $n \in N$ such that $m > n$, we have

$$\begin{aligned} d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \dots \\ &\quad + d(Tx_{m-1}, Tx_m) \\ &\leq (h^n + h^{n+1} + \dots + h^{m-1}) d(Tx_1, Tx_0) \\ &\leq \frac{h^n}{1-h}\, d(Tx_0, Tx_1). \end{aligned}$$

From equation (2.1), it follows that

$$\|d(Tx_m, Tx_n)\| \leq \frac{h^n}{1-h} K \, \|d(Tx_0, Tx_1)\|. \tag{3.3}$$

Since $h \in (0,1)$, $h^n \to 0$ as $n \to \infty$. Therefore $\|d(Tx_m, Tx_n)\| \to 0$ as $m, n \to \infty$. Thus $\{Tx_n\}$ is a Cauchy sequence in X. As X is a complete cone metric space, there exists $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$. Since T is subsequentially convergent, $\{x_n\}$ has the convergent subsequence $\{x_m\}$ such that $\lim_{m\to\infty} x_m = u$. As T is continuous,

$$\lim_{m\to\infty} Tx_m = Tu. \tag{3.4}$$

By the uniqueness of the limit, $z = Tu$. Since S is continuous, $\lim_{m\to\infty} Sx_m = Su$. Again as T is continuous, $\lim_{m\to\infty} TSx_m = TSu$. Therefore

$$\lim_{m\to\infty} Tx_{m+1} = TSu. \tag{3.5}$$

Now consider,

$$\begin{aligned} d(TSu, Tu) &\leq d(TSu, Tx_m) + d(Tx_m, Tu) \\ &= d(TSu, TSx_{m-1}) + d(Tx_m, Tu) \\ &\leq a\, d(Tu, Tx_{m-1}) + b\, [d(Tu, TSu) + d(Tx_{m-1}, TSx_{m-1})] \\ &\quad + c\, [d(Tu, TSx_{m-1}) + d(Tx_{m-1}, TSu)] + d(Tx_m, Tu) \\ &= a\, d(Tu, Tx_{m-1}) + b\, [d(Tu, TSu) + d(Tx_{m-1}, Tx_m)] \\ &\quad + c\, [d(Tu, Tx_m) + d(Tx_{m-1}, TSu)] + d(Tx_m, Tu) \\ &\leq \frac{a}{1-b}\, d(Tu, Tx_{m-1}) + \frac{b}{1-b}\, d(Tx_{m-1}, Tx_m) \\ &\quad + \frac{1+c}{1-b}\, d(Tx_m, Tu) + \frac{c}{1-b} d(Tx_{m-1}, TSu) \\ &\leq \frac{a}{1-b}\, [d(Tu, Tx_m) + d(Tx_m, Tx_{m-1}] + \frac{b}{1-b}\, d(Tx_{m-1}, Tx_m) \\ &\quad + \frac{1+c}{1-b}\, d(Tx_m, Tu) + \frac{c}{1-b} [d(Tx_{m-1}, Tx_m) \\ &\quad + d(Tx_m, Tu) + d(Tu, TSu)] \end{aligned}$$

implies

$$\begin{aligned}\Big(1-\frac{c}{1-b}\Big)d(Tu,TSu) &\leq \Big(\frac{1+a+2c}{1-b}\Big)d(Tu,Tx_m)\\ &\quad+\Big(\frac{a+b+c}{1-b}\Big)d(Tx_{m-1},Tx_m).\end{aligned}$$

Therefore,

$$\begin{aligned}d(Tu,TSu) &\leq \Big(\frac{1+a+2c}{1-b-c}\Big)d(Tu,Tx_m)\\ &\quad+\Big(\frac{a+b+c}{1-b-c}\Big)d(Tx_{m-1},Tx_m). \qquad (3.6)\end{aligned}$$

Let $0 \ll \varepsilon$ be arbitrary. By equation (3.4), $d(Tu,Tx_m) \ll \frac{\varepsilon(1-b-c)}{2(1+a+2c)}$. Similarly by equation (3.5), $d(Tx_m,Tx_{m-1}) \ll \frac{\varepsilon(1-b-c)}{2(a+b+c)}$. Then, equation (3.6) becomes

$$d(Tu,TSu) \ll \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus $d(Tu,TSu) \ll \varepsilon$ for each $\varepsilon \in int\,P$. Now, using Corollary 2.16(iii), it follows that $d(Tu,TSu)=\mathbf{0}$ which implies that $Tu=TSu$. As T is injective, $u=Su$. This shows that u is a fixed point of S.

To Prove Uniqueness: If v is another fixed point of S, then $v=Sv$.

$$\begin{aligned}d(Tu,Tv)=d(TSu,TSv) &\leq a\,d(Tu,Tv)+b\,[d(Tu,TSu)+d(Tv,TSv)]\\ &\quad+c\,[d(Tu,TSv)+d(Tv,TSu)]\\ &= (a+2c)\,d(Tu,Tv)\\ &\leq (a+2b+2c)\,d(Tu,Tv)\\ &< d(Tu,Tv) \text{ as } a+2b+2c<1,\end{aligned}$$

a contradiction. Hence $d(Tu,Tv)=0$ which implies $Tu=Tv$. As T is injective, $u=v$ is the unique common fixed point of S.

As per assumption of the theorem (T,S) is a Banach pair, T and S commutes at the fixed point of S which implies that $TSu=STu$ for $u \in F(S)$, that is, $Tu=STu$ which implies that Tu is another fixed point of S. By uniqueness of fixed point of S, $u=Tu$. Hence $u=Su=Tu$ is the unique common fixed point of S and T in X. This completes the proof. □

The following corollary extends the main result of Beiranvand et al. [5] to cone metric space.

Corollary 3.2. *Let T and S be two continuous self mappings of a complete cone metric space (X,d). Assume that T be injective and P be a normal cone* *with normal constant K. If the mappings T and S satisfy*

$$d(TSx, TSy) \leq k\, d(Tx, Ty)$$

for all $x, y \in X$, for some $k < 1$, then S has a unique fixed point in X.

Proof. The proof of this Corollary follows by taking $a = k$, $b = c = 0$ in Theorem 3.1. Then $k = a \leq a + 2b + 2c < 1$. This completes the proof. □

The following Corollaries are Theorems 1, 3, 4 and Theorems 2.3, 2.6, 2.7 of Huang and Zhang [11] and Rezapour and Hamlbarani [21] respectively in the setup of cone metric space.

Corollary 3.3. ([11, Theorem 1], [21, Theorem 2.3]) *Let (X, d) be a complete cone metric space. Assume that P is a normal cone with normal constant K. If the mapping $S\colon X \to X$ satisfies the condition*

$$d(Sx, Sy) \leq k\, d(x, y),$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then S has a unique fixed point in X.

Proof. The proof of this Corollary follows by taking $T = I$ (the identity map), $a = k$ and $b = c = 0$ in Theorem 3.1. Then $k = a \leq a + 2b + 2c < 1$. This completes the proof. □

Corollary 3.4. ([11, Theorem 3], [21, Theorem 2.6]) *Let (X, d) be a complete cone metric space. Assume that P is a normal cone with normal constant K. If the mapping $S\colon X \to X$ satisfies the condition*

$$d(Sx, Sy) \leq k\, [d(x, Sx) + d(y, Sy)],$$

for all $x, y \in X$, where $k \in [0, 1/2)$ is a constant. Then S has a unique fixed point in X.

Proof. The proof of this Corollary follows by taking $T = I$ (the identity map), $b = k$ and $a = c = 0$ in Theorem 3.1. This completes the proof. □

Corollary 3.5. ([11, Theorem 4], [21, Theorem 2.7]) *Let (X, d) be a complete cone metric space. Assume that P is a normal cone with normal constant K. If the mapping $S\colon X \to X$ satisfies the condition*

$$d(Sx, Sy) \leq k\, [d(x, Sy) + d(y, Sx)],$$

for all $x, y \in X$, where $k \in [0, 1/2)$ is a constant. Then S has a unique fixed point in X.

Proof. The proof of this Corollary follows by taking $T = I$ (the identity map), $c = k$ and $b = c = 0$ in Theorem 3.1. This completes the proof. □

The following examples show that we cannot omit subsequentially convergence hypothesis of the function T in Theorem 3.1.

Example 3.6. Let us consider $\mathbb{R}$ with the usual metric defined by $d(x, y) = |x - y|$. Let $T, S\colon \mathbb{R} \to \mathbb{R}$ be two functions defined $Tx = e^{-x}$ and $Sx = x + 1$. As we see S is a T contraction but T is not subsequentially convergent, because $T(n) \to 0$ as $n \to \infty$ but the sequence (n) has not any convergent subsequence and S has not a fixed point.

Example 3.7. Let $M = [1, 10^4]$ with the usual metric defined by $d(x, y) = |x - y|$. Consider the mappings $T, S\colon M \to M$ defined by $Sx = e^{\frac{1}{10^4}(x-1)}$ and $Tx = x^{1/2}$. One can show that

(1) $d(Sx, Sy) \le \frac{18}{10} d(x, y)$, that is, S is not a Banach contraction on M.
(2) $d(TSx, TSy) \le \frac{4}{10} d(Tx, Ty)$, that is, S is a T-contraction for $k \in [0, 4/10)$.

Since the mapping T is non-decreasing, continuous and injective, then it is sequentially convergent on M. Thus, from Theorem 3.1, we have that $u = 1$ is the unique fixed point of S on M.

Remark 3.8. Our results generalize recent existing results in the literature of T-contraction mappings and cone metric space.

4. Concluding remarks

The generalized T contraction mapping include T-contraction introduced by [5], the Banach contraction, the Kannan contraction, the Chatterjea contraction and the Fisher contraction mappings. Thus the results presented in this paper extend, generalize and unify several results from the existing literature (see, for example, [5, 11, 21] and many others).

References

[1] M. Abbas and B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett., **22**(4) (2009), 511–515.
[2] M.A. Al-Thagafi and N. Shahzad, *Banach operator pairs, common fixed points, invariant approximations and ∗-nonexpansive mappings*, Nonlinear Anal., **69**(8) (2008), 2733–2739.
[3] M. Asadi, H. Soleimani and S.M. Vaezpour, *An order on subsets of cone metric spaces and fixed points of set valued contractions*, Fixed Point Theory Appl., **2009** (2009), Article ID 723203, 8 pages.
[4] S. Banach, *Surles operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math., **3** (1922), 133–181.

[5] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, *Two fixed point theorems for spcial mappings*, arXiv:0903.1504v1 [math.FA].
[6] S.K. Chatterjae, *Fixed point theorems compactes*, Rend. Acad. Bulgare Sci., **25** (1972), 727–730.
[7] J. Chen and Z. Li, *Common fixed points for Banach operator pairs in best approximation*, J. Math. Anal. Appl., **336** (2007), 1460–1475.
[8] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, (1985).
[9] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc., **12** (1961), 7–10.
[10] B. Fisher, *A fixed point theorem for compact metric space*, Publ. Inst. Math., **25** (1976), 193–194.
[11] L.-G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332**(2) (2007), 1468–1476.
[12] N. Hussain, *Common fixed points in best approximation for Banach operator pairs with ciric type I-contractions*, J. Math. Anal. Appl., **338**(2) (2008), 1351–1363.
[13] D. Ilic and V. Rakocevic, *Quasi-contraction on a cone metric space*, Appl. Math. Lett., **22**(5) (2009), 728-731.
[14] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory Appl., **2009** (2009), 1–13.
[15] R. Kannan, *Some results on fixed point theorems*, Bull. Calcutta Math. Soc., **10** (1968), 71–76.
[16] S. Moradi, *Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depended on another function*, arXiv:0903.1577v1 [math.FA].
[17] J. Morales and E. Rojas, *Cone metric spaces and fixed point theorems of T-Kannan contractive mappings*, Int. J. Math. Anal., **4**(4) (2010), 175–184.
[18] J. Morales and E. Rojas, *T-Zamfirescu and T-weak contraction mappings on cone metric spaces*, arXiv:0909.1255v1 [math.FA].
[19] S. Reich, *Some remarks concerning contraction mapping*, Canada Math. Bull., **14** (1971), 121–124.
[20] Sh. Rezapour, *A review on topological properties of cone metric spaces*, in Proceedings of the International Conference on Analysis, Topology and Appl. (ATA 08), Vrinjacka Banja, Serbia, May-June (2008).
[21] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl., **345**(2) (2008), 719–724.
[22] P.V. Subrahmanyam, *Remarks on some fixed point related to Banach's contraction principle*, J. Math. Phy. Sci., **8** (1974), 445-457. Erratum: J. Math. Phy. Sci. 9(1975), 195.
[23] J. Vandergraft, *Newton method for convex operators in partially ordered spaces*, SIAM J. Numer. Anal., **4** (1967), 406–432.
[24] D. Wardowski, *Endpoints and fixed points of set-valued contractions in cone metric spaces*, Nonlinear Anal. TMA, **71**(1-2) (2009), 512–516.
[25] P.P. Zabrejko, *K-metric and K-normed linear spaces*: survey, Collectanea Mathematica, **48**(4-6) (1997), 825–859.