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AN OPERATOR PRESERVING INEQUALITIES BETWEEN A POLYNOMIAL AND ITS POLAR DERIVATIVES

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Abstract. Let $P(z)$ be a polynomial of degree n. In this paper we estimate the maximum and minimum modulli of the B-Operator of k^{th} polar derivative of $P(z)$ in terms of the modulus of $P(z)$ on the unit circle and there by obtain generalisations of some results recently proved by Bidkham and Mezerji. This interalia generalise the results earlier proved by Aziz and Shah, Shah and Liman and a conjecture of Erdös proved by Lax.

1. Introduction

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $P'(z)$ its derivative, then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| \tag{1.1}$$

and

$$\max_{|z|=R>1} |P(z)| \le R^n \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.1) is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial (For reference, see [6]), where as inequality (1.2) is a simple deduction from the maximum modulus principle [13, p.346]. Concerning the minimum modulus of a polynomial $P(z)$ and its

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derivative $P'(z)$, Aziz and Dawood [3] proved that, if $P(z)$ has all its zeros in $|z| \leq 1$,then

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\min_{|z|=R>1} |P(z)| \geq R^n \min_{|z|=1} |P(z)|. \tag{1.4}$$

For any complex number α, let $D_\alpha P(z)$ denote the polar differentiation of the polynomial $P(z)$ of degree n with respect to the point α, then $D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Concerning the inequalities for the polar derivative of $P(z)$, Aziz and Shah [5] proved :

Theorem 1.1. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then for any real or complex number α, with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \geq n|\alpha||z^{n-1}| \min_{|z|=1} |P(z)| \text{ for } |z| \geq 1. \tag{1.5}$$

Theorem 1.2. *If $P(z)$ is a polynomial of degree n which has no zero in the disk $|z| < 1$, then for every real or complex number α, with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \leq \frac{n}{2}\left\{(|\alpha||z^{n-1}|+1) \max_{|z|=1} |P(z)| - (|\alpha||z^{n-1}|-1) \min_{|z|=1} |P(z)|\right\} \tag{1.6}$$

for $|z| \geq 1$.

Like polar derivatives there are many other operators which are just as interesting (for reference see [14]). As an attempt to this characterisation, Shah and Liman [14] considered an operator B, which was earlier discussed by Rahman [12] and carries a polynomial $P(z)$ into

$$B[P(z)] := \lambda_o P(z) + \lambda_1 \left(\frac{nz}{2}\right) P'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!} \tag{1.7}$$

where $\lambda_o,\ \lambda_1,$ and λ_2 are such that

$$U(z) := \lambda_o + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2 \neq 0, \tag{1.8}$$

for $Re(z) \leq \frac{n}{4},\ C(n,r) = \frac{n!}{r!(n-r)!}$.

Concerning the inequalities for B-operator Shah and Liman [14, Theorem 1.1] infact proved:

Theorem 1.3. *If $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| > 1$, then*

$$|B[P(z)]| \geq |B[z^n]| \min_{|z|=1} |P(z)| \quad \text{for} \quad |z| \geq 1. \tag{1.9}$$

Theorem 1.4. *If $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < 1$, then*

$$|B[P(z)]| \leq \frac{1}{2}\left[\{|B[z^n]|+|\lambda_0|\} \max_{|z|=1} |P(z)| - \{|B[z^n]|-|\lambda_0|\} \min_{|z|=1} |P(z)|\right] \tag{1.10}$$

for $|z| \geq 1$.

Recently Bidkham and Mezerji combined the two operators D_α and B and obtained the following:

Theorem 1.5. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then for every complex number α, with $|\alpha| \geq 1$,*

$$|B[D_\alpha P(z)]| \geq n|\alpha||B[z^{n-1}]| \min_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.11}$$

Theorem 1.6. *If $P(z)$ is a polynomial of degree n such that $P(z)$ has no zeros in $|z| < 1$, then for every complex number α, with $|\alpha| \geq 1$,*

$$\begin{aligned} |B[D_\alpha P(z)]| \leq \frac{n}{2}\Bigg[&\{|\alpha||B[z^{n-1}]| + |\lambda_0|\} \max_{|z|=1} |P(z)| \\ &- \{|\alpha||B[z^{n-1}]| - |\lambda_0|\} \min_{|z|=1} |P(z)|\Bigg] \quad \text{for } |z| \geq 1. \end{aligned} \tag{1.12}$$

The aim of this paper is to prove some new results which generalises the recently proved inequalities of Bidkham and Mezerji [7] and provide some extensions of Bernstien type inequalities proved earlier.

2. Lemmas

The following lemmas are required for our investigation.

Lemma 2.1. *If all the zeros of an nth degree polynomial $P(z)$ lie in a circular region C and if none of the points $\alpha_1, \alpha_2, ..., \alpha_k$ lie in the region C, then each of the polar derivatives $D_{\alpha_1} ... D_{\alpha_k} P(z)$, $k = 1, 2, ..., n-1$ has all its zeros in C.*

The above lemma follows by repeated application of Laguerre's theroem [8, p.52].

The next Lemma which we require follows from Corollary 18.3 of [9, p.65] (see also[12, Lemma 14.5.7, p.540]).

Lemma 2.2. *If all the zeros of polynomial $P(z)$ of degree n lie in the circle $|z| \leq 1$, then all the zeros of polynomial $B[P(z)]$ also lie in the circle $|z| \leq 1$.*

On combining Lemma 2.1 and Lemma 2.2, we easily get the following.

Lemma 2.3. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \leq 1$, then for complex numbers $\alpha_1, \alpha_2, ...\alpha_k$ with $|\alpha_i| \geq 1$, $1 \leq i \leq k$, the polynomial $B[D_{\alpha_1}...D_{\alpha_k}P(z)]$ has all its zeros in $|z| \leq 1$.*

Lemma 2.4. *If the polynomial $P(z)$ of degree n has no zero in $|z| \leq 1$ then for real or complex numbers $\alpha_1, \alpha_2, ...\alpha_k$, with $|\alpha_i| \geq 1$, $1 \leq i \leq k$, we have*

$$|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| \leq |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| \tag{2.1}$$

for $|z| \geq 1$, where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

Proof. Since $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ and all the zeros of $P(z)$ lie in $|z| \geq 1$, therefore all the zeros of $Q(z)$ lie in $|z| \leq 1$ and $|P(z)| = |Q(z)|$ for $|z| = 1$. Hence $\frac{Q(z)}{P(z)}$ is analytic in $|z| \leq 1$ and $|\frac{Q(z)}{P(z)}| = 1$ for $|z| = 1$. By maximum modulus principle, $|Q(z)| \leq |P(z)|$ for $|z| \leq 1$. This gives $|P(z)| \leq |Q(z)|$ for $|z| \geq 1$. So by Rouche's theorem, for every β with $|\beta| > 1$, the polynomial $P(z) - \beta Q(z)$ has all its zeros in $|z| \leq 1$. Using Lemma 2.3, it follows that the polynomial

$$\begin{aligned} T(z) &= B[D_{\alpha_1}...D_{\alpha_k}(P(z) - \beta Q(z))] \\ &= B[D_{\alpha_1}...D_{\alpha_k}P(z)] - \beta[D_{\alpha_1}...D_{\alpha_k}Q(z)] \end{aligned} \tag{2.2}$$

has all its zeros in $|z| \leq 1$. This in particular gives

$$|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| \leq |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| \quad \text{for} \quad |z| \geq 1$$

and the proof is complete. □

Lemma 2.5. *If $P(z)$ is a polynomial of degree n then for complex numbers $\alpha_1, \alpha_2, ..., \alpha_k$ with $|\alpha_i| \geq 1, 1 \leq i \leq k$, we have for $|z| \geq 1$*

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| + |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| \\ &\leq n(n-1)...(n-k+1)\{|\alpha_1...\alpha_k||B[z^{n-k}] + |\lambda_o|\} \max_{|z|=1}|P(z)| \end{aligned} \tag{2.3}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

Proof. Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. If β is a complex number with $|\beta| > 1$, then by Rouche's theorem $P(z) - \beta M$ does not vanish in $|z| \leq 1$. Using Lemma 2.4, it follows that for $|z| \geq 1$

$$|B[D_{\alpha_1}...D_{\alpha_k}(P(z) - \beta M)]| \leq |B[D_{\alpha_1}...D_{\alpha_k}(Q(z) - \overline{\beta} M z^n)]|$$

where $Q(z) = z^n \overline{P(\frac{1}{\overline{z}})}$. This implies for $|z| \geq 1$,

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)] - Mn(n-1)...(n-k+1)\beta\lambda_0| \\ &\leq |B[D_{\alpha_1}...D_{\alpha_k}Q(z)] - (\alpha_1...\alpha_k)Mn(n-1)...(n-k+1)\bar{\beta}B[z^{n-k}]|. \end{aligned} \tag{2.4}$$

Since $Q(z) - \overline{\beta} M z^n$ has all its zeros in $|z| < 1$, by Lemma 2.3, it follows that all the zeros of

$$\begin{aligned} &B[D_{\alpha_1}...D_{\alpha_k}(Q(z) - \overline{\beta} M z^n)] \\ &= B[D_{\alpha_1}...D_{\alpha_k}Q(z)] - n(n-1)...(n-k+1)\overline{\beta} M(\alpha_1...\alpha_k)B[z^{n-k}] \end{aligned}$$

lie in $|z| < 1$. This gives by the same argument as in the proof of Lemma 2.4

$$|B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| \leq n(n-1)...(n-k+1)|\alpha_1...\alpha_k||\beta||B[z^{n-k}]|M \tag{2.5}$$

for $|z| \geq 1$. Choosing argument of β in the right hand side of (2.4) which is possible by (2.5) such that

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}Q(z)] - n(n-1)...(n-k+1)\overline{\beta}(\alpha_1...\alpha_k)B[z^{n-k}]M| \\ &= n(n-1)...(n-k+1)|\beta||\alpha_1...\alpha_k||B[z^{n-k}]|M - |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| \end{aligned}$$

we obtain from (2.4)

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| - n(n-1)...(n-k+1)|\beta|||\lambda_0|M \\ &\leq n(n-1)...(n-k+1)|\beta||\alpha_1...\alpha_k||B[z^{n-k}]|M - |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| \end{aligned}$$

for $|z| \geq 1$. Equivalently

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| + |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| \\ &\leq n(n-1)...(n-k+1)|\beta|\{|\alpha_1...\alpha_k||B[z^{n-k}]| + |\lambda_0|\}M \end{aligned} \tag{2.6}$$

for $|z| \geq 1$.

Finally letting $|\beta| \to 1$, we get the desired result and the proof of Lemma 2.5 is complete. □

Remark 2.6. A result of Aziz [1] is a special case of Lemma 2.5 and is obtained by suitable choice of $\lambda_i,\ i = 0, 1, 2$. Also a result of Govil and Rahman[10] follows from inequality (2.6) with a special choice of λ_i and α_i.

3. Main results

Theorem 3.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le 1$, then for complex numbers $\alpha_1, \alpha_2, ..., \alpha_k$ with $|\alpha_k| \ge 1$, $k = 1, 2, ..., n$,*

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| \\ &\ge n(n-1)...(n-k+1)|\alpha_1...\alpha_k||B[z^{n-k}]| \min_{|z|=1}|P(z)| \end{aligned} \tag{3.1}$$

for $|z| \ge 1$. The result is sharp and equality holds for $P(z) = az^n$.

Proof. If $P(z)$ has a zero on $|z| = 1$, then $m = \min_{|z|=1}|P(z)| = 0$ and there is nothing to prove. Suppose that all the zeros of $P(z)$ lie in $|z| < 1$. Then $m > 0$ and so we have $m \le |P(z)|$ for $|z| = 1$. Therefore, for every complex number β with $|\beta| < 1$, we have $|m\beta z^n| < |P(z)|$ for $|z| = 1$. Applying Rouche's theorem, it follows that all the zeros of $P(z) - m\beta z^n$ lie in $|z| < 1$. Hence by Lemma 2.3, all the zeros of

$$\begin{aligned} H(z) :&= B[D_{\alpha_1}...D_{\alpha_k}(P(z) - m\beta z^n)] \\ &= B[D_{\alpha_1}...D_{\alpha_k}P(z)] - m\beta n(n-1)...(n-k+1)(\alpha_1...\alpha_k)B[z^{n-k}] \end{aligned} \tag{3.2}$$

also lie in $|z| < 1$. This gives for $|z| \ge 1$,

$$mn(n-1)\dots(n-k+1)|\alpha_1\dots\alpha_k||B[z^{n-k}]| \le |B[D_{\alpha_1}\dots D_{\alpha_k}P(z)]|. \tag{3.3}$$

If the inequality (3.3) is not true, then there is a point $z = z_o$ with $|z_o| \ge 1$, such that

$$mn(n-1)\dots(n-k+1)|\alpha_1\dots\alpha_k||B[z_o^{n-k}]| > |B[D_{\alpha_1}\dots D_{\alpha_k}P(z_o)]|.$$

We take

$$\beta = \frac{B[D_{\alpha_1}\dots D_{\alpha_k}P(z_o)]}{mn(n-1)\dots(n-k+1)\alpha_1\dots\alpha_k B[z_o^{n-k}]},$$

so that $|\beta| < 1$ and for this value of β, we have from (3.2) $H(z_o) = 0$ for $|z_o| \ge 1$. This is contradiction to the fact that all the zeros of $H(z)$ lie in $|z| < 1$. Thus

$$mn(n-1)\dots(n-k+1)|\alpha_1\dots\alpha_k||B[z^{n-k}]| \le |B[D_{\alpha_1}\dots D_{\alpha_k}P(z)]|,$$

for $|z| \ge 1$. This completes the proof of Theorem 3.1. □

Now taking $\alpha_1 = \alpha_2 = ... = \alpha_k = \alpha$ in Theorem 3.1, we have

Corollary 3.2. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z| \le 1$, then for every complex number α with $|\alpha| > 1$, we have*

$$|B[D_\alpha^k P(z)]| \ge n(n-1)...(n-k+1)|\alpha^k||B[z^{n-k}]| \min_{|z|=1}|P(z)| \tag{3.4}$$

for $|z| \ge 1$.

Dividing both sides by $|\alpha^k|$ and noting that $\lim_{\alpha\to\infty}\frac{D_\alpha^k P(z)}{\alpha^k} = P^k(z)$, we get the following result:

Corollary 3.3. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z| \le 1$, then for every complex number α, with $|\alpha| > 1$, we have for $|z| \ge 1$,*

$$|B[P^k(z)]| \ge n(n-1)...(n-k+1)|B[z^{n-k}]| \min_{|z|=1} |P(z)|. \tag{3.5}$$

For $k = 1$, this reduces to a result due to Shah and Liman [14, Theorem 1.1].

Again for $k = 1$, we have from (3.1) for $|z| \ge 1$,

$$|B[D_\alpha P(z)]| \ge n|\alpha||B[z^{n-1}]| \min_{|z|=1} |P(z)|. \tag{3.6}$$

Substituting for $D_\alpha P(z)$ in (3.6), we get

$$|B[nP(z) + (\alpha - z)P'(z)]| \ge n|B[\alpha z^{n-1}]| \min_{|z|=1} |P(z)| \text{ for } |z| \ge 1.$$

For $z = \alpha$, this gives

$$|B[P(z)]| \ge |B[z^n]| \min_{|z|=1} |P(z)| \text{ for } |z| \ge 1. \tag{3.7}$$

Which is the result due to Shah and Liman [14, Theorem 1.1].

Next, we have the following:

Theorem 3.4. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| \le 1$, then for complex numbers α_k with $|\alpha_k| \ge 1$, $k = 1, 2, ..., n$,*

$$\begin{aligned}&|B[D_{\alpha_1}...D_{\alpha_k}P(z)]|\\ &\le \frac{n(n-1)...(n-k+1)}{2}\left\{|\alpha_1...\alpha_k||B[z^{n-k}|] + |\lambda_o|\right\} \max_{|z|=1} |P(z)|,\end{aligned} \tag{3.8}$$

for $|z| \ge 1$. The result is sharp and equality holds for polynomials whose zeros lie on the unit disc.

Proof. Combining Lemma 2.4 and Lemma 2.5, we have, for $|z| \ge 1$,

$$\begin{aligned}&2|B[D_{\alpha_1}...D_{\alpha_k}P(z)]|\\ &\le |B[D_{\alpha_1}...D_{\alpha_k}P(z)]| + |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]|\\ &\le n(n-1)...(n-k+1)\left\{|\alpha_1...\alpha_k||B[z^{n-k}]| + |\lambda_0|\right\} \max_{|z|=1} |P(z)|.\end{aligned}$$

From this proof of Theorem 3.4 follows.
Substituting for $B[D_{\alpha_1}...D_{\alpha_k}P(z)]$ in (3.8), we have for $|z| \ge 1$,

$$\left|\lambda_o D_{\alpha_1}...D_{\alpha_k}P(z)+\lambda_1(\frac{mz}{2})D_{\alpha_1}...D_{\alpha_k}P'(z)+\lambda_2(\frac{mz}{2})^2\frac{D_{\alpha_1}...D_{\alpha_k}P''(z)}{2!}\right|$$
$$\leq \frac{n(n-1)...(n-k+1)}{2}\left\{|\alpha_1...\alpha_k|\left|\lambda_o z^{n-k}+\lambda_1\left(\frac{(n-k)z}{2}\right)(n-k)z^{n-k-1}\right.\right.$$
$$\left.\left.+\lambda_2\left(\frac{(n-k)z}{2}\right)^2\frac{(n-k)(n-k-1)}{2!}z^{n-k-2}\right|+|\lambda_o|\right\}\max_{|z|=1}|P(z)|, \tag{3.9}$$

where $0 \leq m \leq (n-1)$, λ_o, λ_1 and λ_2 are such that all the zeros of $U(z)$ defined by (1.8) lie in the half plane $Re(z) \leq \frac{m}{4}$.
Result of Bidkham and Mezerji is a special case of Theorem 3.4 when $k = 1$. Also the result of Shah and Liman [14, Theorem 1.2] follows from Theorem 3.4, when we take $k = 1$ and $\alpha = z$. Taking $\alpha_1 = \alpha_2 = ... = \alpha_k = \alpha$, we have for $|z| \geq 1$

$$|B[D_\alpha^k P(z)]|$$
$$\leq \frac{n(n-1)...(n-k+1)}{2}\left\{|\alpha^k||B[z^{n-k}]|+|\lambda_o|\right\}\max_{|z|=1}|P(z)|. \tag{3.10}$$

□

Dividing both sides of the above inequality by $|\alpha^k|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 3.5. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \geq 1$, then for every complex number α with $|\alpha| \geq 1$,*

$$|B[P^k(z)]| \leq \frac{n(n-1)...(n-k+1)}{2}|B[z^{n-k}]| \max_{|z|=1}|P(z)| \tag{3.11}$$

for $|z| \geq 1$. In particular for $k = 1$, we have

$$|B[P'(z)]| \leq \frac{n}{2}\{|B[z^{n-1}]|\} \max_{|z|=1}|P(z)|, \tag{3.12}$$

a result earlier proved by Shah and Liman [14]*. Again if we take $k = 1$ and $z = \alpha$ in inequality* (3.9)*, we get for $|z| \geq 1$,*

$$|B[P(z)]| \leq \frac{1}{2}[|B[z^n]|+|\lambda_0|] \max_{|z|=1}|P(z)|. \tag{3.13}$$

Lastly, we prove:

Theorem 3.6. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| \leq 1$, then for every α_k with $|\alpha_k| \geq 1$, $k = 1, 2, ..., n$,*

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| \\ &\leq \frac{n(n-1)...(n-k+1)}{2}\Bigg[\{|\alpha_1...\alpha_k||B[z^{n-k}]| + |\lambda_o|\} \max_{|z|=1}|P(z)| \\ &\quad - \{|\alpha_1...\alpha_k||B[z^{n-k}]| - |\lambda_o|\} \min_{|z|=1}|P(z)|\Bigg] \end{aligned} \tag{3.14}$$

for $|z| \geq 1$. The result is sharp and equality holds for polynomials whose zeros lie on the unit disc.

Proof. If $P(z)$ has a zero on $|z| = 1$, then $m = \min_{|z|=1}|P(z)| = 0$ and Theorem 3.6 reduces to Theorem 3.4. We now suppose that all the zeros of $P(z)$ lie in $|z| > 1$, so that $m > 0$. Also

$$m \leq |P(z)| \text{ for } |z| = 1. \tag{3.15}$$

It follows by Rouche's theorem that for every complex number λ with $|\lambda| < 1$, the polynomial $H(z) = P(z) - \lambda m$ does not vanish in $|z| < 1$. We note that $H(z)$ has no zero on $|z| = 1$. Because if for some $z = z_0$ with $|z_0| = 1$,

$$H(z_0) = P(z_0) - \lambda m = 0,$$

then $|P(z_0)| = m|\lambda| < m$, which is a contradiction to (3.15).
Now, if we let

$$G(z) = z^n\overline{H(\frac{1}{\bar{z}})} = z^n\overline{P(\frac{1}{\bar{z}})} - \bar{\lambda}mz^n = Q(z) - \bar{\lambda}mz^n,$$

then all the zeros of $G(z)$ lie in $|z| < 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$. So for every β with $|\beta| > 1$, applying Rouche's theorem again, it follows that all the zeros of $H(z) - \beta G(z)$ lie in $|z| < 1$. Using Lemma 2.3, we see that for every α_k with $|\alpha_k| \geq 1$, $k = 1, 2, \ldots, n$, the polynomial $B[D_{\alpha_1}...D_{\alpha_k}(H(z) - \beta G(z))]$ has all its zeros in $|z| \leq 1$. This gives by the same argument as above

$$B[D_{\alpha_1}...D_{\alpha_k}H(z)] \leq B[D_{\alpha_1}...D_{\alpha_k}G(z)],$$

for $|z| > 1$. Substituting for $H(z)$ and $G(z)$ we get for $|z| \geq 1$,

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)] - mn(n-1)...(n-k+1)\lambda\lambda_0| \\ &\leq |B[D_{\alpha_1}...D_{\alpha_k}Q(z)] - \lambda(\alpha_1...\alpha_k)mn(n-1)...(n-k+1)B[z^{n-k}]|. \end{aligned}$$

Choosing the argument of λ in the right hand side of the above inequality suitability, which is possible by Theorem 3.1, and making $|\lambda| \to 1$, we get for $|z| \geq 1$,

$$\begin{aligned} &|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| - mn(n-1)...(n-k+1)|\lambda_0| \\ &\leq |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]| - |(\alpha_1...\alpha_k)|mn(n-1)...(n-k+1)|B[z^{n-k}]|. \end{aligned}$$

Equivalently,

$$\begin{aligned}&|B[D_{\alpha_1}...D_{\alpha_k}P(z)]|\\&\le |B[D_{\alpha_1}...D_{\alpha_k}Q(z)]|\\&\quad -|(\alpha_1...\alpha_k)|mn(n-1)...(n-k+1)\{|B[z^{n-k}]|-|\lambda_0|\},\end{aligned} \tag{3.16}$$

for $|z|\ge 1$. By using Lemma 2.4 we have from inequality (3.16)

$$\begin{aligned}2|B[D_{\alpha_1}...D_{\alpha_k}P(z)]| &\le |B[D_{\alpha_1}...D_{\alpha_k}P(z)]|+|B[D_{\alpha_1}...D_{\alpha_k}Q(z)]|\\ &\le n(n-1)...(n-k+1)\Bigg[\{|\alpha_1...\alpha_k||B[z^{n-k}]|+|\lambda_0|\}\max_{|z|=1}|P(z)|\\ &\quad -\{|\alpha_1...\alpha_k||B[z^{n-k}]|-|\lambda_0|\}\min_{|z|=1}|P(z)|\Bigg].\end{aligned}$$

From this, proof of Theorem 3.6 follows completely. □

Remark 3.7. If we take $k=1$ and then put $\alpha=z$ in Theorem 3.6, we get a result of Shah and Liman [14, 1.3].

Next, if we put $\alpha_1=\alpha_2=...=\alpha_k=\alpha$ in Theorem 3.6, we get the following:

Corollary 3.8. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then for every α with $|\alpha|\ge 1$,*

$$\begin{aligned}&|B[D_\alpha^k P(z)]|\\&\le \frac{n(n-1)...(n-k+1)}{2}\Bigg[\{|B[\alpha^k z^{n-k}]|+|\lambda_o|\}\max_{|z|=1}|P(z)|\\&\quad -\{|B[\alpha^k z^{n-k}]|-|\lambda_o|\}\min_{|z|=1}|P(z)|\Bigg]\end{aligned} \tag{3.17}$$

for $|z|\ge 1$.

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