Nonlinear Functional Analysis and Applications
Vol. 20, No. 3 (2015), pp. 465-478

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

STABILITY OF THE PERTURBED ISHIKAWA APPROXIMATION ALGORITHM FOR GNOVI IN ORDERED BANACH SPACE

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Abstract. A new class of generalized nonlinear ordered variational inequalities and equations are studied, and by using the B-restricted-accretive method of mapping $(A - \rho f - w)$ with constants α_1, α_2, an existence theorem and a perturbed algorithm for solving this kind generalized nonlinear ordered variational inequality(equation) is established and further, the stability and the convergence of iterative sequences generated by the algorithm is discussed in ordered Banach space, respectively. The results in the instrument are obtained in this field.

1. Introduction

Generalized nonlinear variational inequalities(ordered equation) have wide applications to many fields including, for example, mathematics, optimization,

[0]Received January 2, 2015. Revised May 5, 2015.
[0]2010 Mathematics Subject Classification: 49J40, 47H10, 47H17.
[0]Keywords: Generalized nonlinear ordered variational inequality, perturbed algorithm, stability of the perturbed algorithm, ordered Banach space, convergence.

control, nonlinear programming, and engineering sciences. In 1972, the number of solutions of nonlinear equations have been introduced and studied by Amann [7], and in recent years, the nonlinear mapping fixed point theory and application have been intensively studied in ordered Banach space(see [8], [11], [12]). The variational inclusion, which was introduced and studied by Hassouni and Moudafi [9], is a useful and important extension of the variational inequality. In recent years, monotonicity techniques were extended and applied because of their importance in the theory of variational inequalities, complementarity problems, and variational inclusions. From 2001 to 2006, Ding and Luo [5], Huang and Fang [10], Fang, Huang and Thompson [6], Lan, Cho and Verma [15] et al. introduced many concepts including, for example, generalized m-accretive mappings, generalized monotone mappings, maximal η-monotone mappings, (A, η)-accretive mappings, defined an associated resolvent operator and also studied some the algorithms for solving variational inclusion problems(variational inequalities) by the resolvent operator associated with those mappings.

On the other hand, recently, the author [16] have introduced and studied the approximation algorithm and the approximation solution for a class of generalized nonlinear ordered variational inequality and ordered equation in ordered Banach space. From 2006, Jin [13] studied the stability for strongly the nonlinear quasi-variational inclusion involving H-accretive operators, and Li studied some the stability problems of perturbed Ishikawa iterative algorithms for nonlinear mixed quasi-variational inclusions involving (A, η)-accretive mappings [18], and nonlinear random multi-valued mixed variational inclusions involving nonlinear random (A, η)-monotone mappings in Banach spaces by using the resolvent operator technique [19]. For details, we refer the reader to [1-38] and the references therein.

Inspired and motivated by recent research works in this field, a new class of generalized nonlinear ordered variational inequalities and equations are studied in ordered Banach space. By using the B-restricted-accretive method of mapping $(A - \rho f - w)$ with constants α_1, α_2, an existence theorem of solutions for this kind generalized nonlinear ordered variational inequality(equation) is established, a perturbed algorithm is suggested, the stability and the convergence of iterative sequences generated by the algorithm is discussed in ordered Banach space. In this field, the results are obtained in first.

2. Generalized nonlinear ordered variational inequality and preliminaries

Definition 2.1. Let X be a real Banach space with a norm $\|\cdot\|$, θ be a zero element in X. A nonempty closed convex subset $\mathbf{P}$ of X is said to be a cone if,

(i) for any $x \in \mathbf{P}$ and any $\lambda > 0$, $\lambda x \in \mathbf{P}$;
(ii) if $x \in \mathbf{P}$ and $-x \in \mathbf{P}$, then $x = \theta$.

Definition 2.2. Let $\mathbf{P}$ be a cone of X. Then $\mathbf{P}$ is said to be a normal cone if there exists a constant $N > 0$ such that for $\theta \leq x \leq y$, $\|x\| \leq N\|y\|$, in this case, N is called normal constant of $\mathbf{P}$.

Lemma 2.3. ([16]) *Let $\boldsymbol{P}$ be a cone in X. For arbitrary $x, y \in X$, $x \leq y$ if and only if $x - y \in \boldsymbol{P}$. Then the relation $\leq$ in X is a partial ordered relation in X, where the Banach space X with an ordered relation $\leq$ defined by a normal cone $\boldsymbol{P}$ is called a ordered Banach space.*

Let $\Re$ be a real set, X be a real ordered Banach space with a norm $\|\cdot\|$, and θ be a zero in the X. Let $\mathbf{P}$ be a cone of X, and $\leq$ be a partial ordered relation defined by the cone $\mathbf{P}$. Let $A, g, f : X \to X$ be single-valued nonlinear ordered compression mappings and $range(g) \bigcap domA(\cdot) \neq \emptyset$. we consider the following problem:

For any $w \in x$ and any $\rho \in \Re$, find $x \in X$ such that

$$A(g(x)) - \rho f(x) \geq w. \tag{2.1}$$

The problem (2.1) is called a new class of generalized nonlinear ordered variational inequality($GNOVI$) in X.

Remark 2.4. For a suitable choice of the mappings ρ, ω, we can obtain several known results.

(i) If $\rho = 0$, and $\omega = \theta$, then problem (2.1) becomes the ordered variational inequality $A(g(x)) \geq \theta$, which was studied by Li [16].
(ii) If $\rho = -1$, and $\omega = \theta$, then problem (2.1) becomes the ordered variational inequality $f(x) + A(g(x)) \geq \theta$, which was studied by Li [17].

Let us recall some concepts and results.

Definition 2.5. ([35]) Let X be an ordered Banach space, $\mathbf{P}$ be a cone of X and $\leq$ be a partial ordered relation defined by the cone $\mathbf{P}$. If for $x, y \in X$, $x \leq y$(or $y \leq x$), then x and y is said to be comparison between each other(denoted by $x \propto y$ for $x \leq y$ and $y \leq x$).

Lemma 2.6. ([35]) *Let X be an ordered Banach space, $\boldsymbol{P}$ be a cone of X and $\leq$ be a partial ordered relation defined by the cone $\boldsymbol{P}$. Let for arbitrary $x, y \in X$, $lub\{x, y\}$ and $glb\{x, y\}$ denote the least upper bound of the set $\{x, y\}$ and the greatest lower bound of the set $\{x, y\}$ on the partial ordered relation $\leq$, respectively, Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, and we define the binary operators as followings:*

(i) $x \vee y = lub\{x, y\}$;
(ii) $x \wedge y = glb\{x, y\}$;
(iii) $x \oplus y = (x - y) \vee (y - x)$.

$\vee, \wedge$, and $\oplus$ is called OR, AND, and XOR operation, respectively. Then for arbitrary $x, y, w \in X$, the following statements hold:

(1) *if $x \leq y$, then $x \vee y = y$, $x \wedge y = x$;*
(2) *if x and y can be compared, then $\theta \leq x \oplus y$;*
(3) *$(x + w) \vee (y + w)$ exists and $(x + w) \vee (y + w) = (x \vee y) + w$;*
(4) *$(x \wedge y) = (x + y) - (x \vee y)$;*
(5) *if $\lambda \geq 0$, then $\lambda(x \vee y) = \lambda x \vee \lambda y$;*
(6) *if $\lambda \leq 0$, then $\lambda(x \wedge y) = \lambda x \vee \lambda y$;*
(7) *if $x \neq y$, then the converse holds for (5) and (6);*
(8) *if X is an ordered Banach space, and if for any $x, y \in X$, either $x \vee y$, and $x \wedge y$ exists, then X is a lattice;*
(9) *$(x + w) \wedge (y + w)$ exists and $(x + w) \wedge (y + w) = (x \wedge y) + w$;*
(10) *$(x \wedge y) = -(-x \vee -y)$;*
(11) *$(-x) \wedge (x) \leq \theta \leq (-x) \vee x$.*

Lemma 2.7. ([8]) *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist, $x - y \propto y - x$, and $\theta \leq (x - y) \vee (y - x)$.*

Lemma 2.8. ([30]) *If for any natural number n, $x \propto y_n$, and $y_n \to y^*(n \to \infty)$, then $x \propto y^*$.*

Lemma 2.9. ([17]) *Let X be an ordered Banach space, $\boldsymbol{P}$ be a cone of X and $\leq$ be a partial ordered relation defined by the cone $\boldsymbol{P}$. If for $x, y, z, w \in X$, they can be compared each other, then the following statements hold.*

(1) *$x \oplus y = y \oplus x$;*
(2) *$x \oplus x = \theta$;*
(3) *$\theta \leq x \oplus \theta$;*
(4) *if λ is a real number, then $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;*
(5) *if x, y and w can be comparative each other, then $(x \oplus y) \leq x \oplus w + w \oplus y$;*
(6) *let $(x + y) \vee (u + v)$ exist, and if $x \propto u, v$ and $y \propto u, v$, then $(x + y) \oplus (u + v) \leq (x \oplus u + y \oplus v) \wedge (x \oplus v + y \oplus u)$;*

(7) *if* x, y, z, w *can be compared with each other, then* $(x \wedge y) \oplus (z \wedge w) \leq ((x \oplus z) \vee (y \oplus w)) \wedge ((x \oplus w) \vee (y \oplus z))$.

Lemma 2.10. *Let X be an ordered Banach space, $\boldsymbol{P}$ be a cone of X and$\leq$ be a partial ordered relation defined by the cone $\boldsymbol{P}$. If $x \in \boldsymbol{P}$, then for real numbers a, b, we have*

$$(ax) \oplus (bx) = |a - b|x.$$

Proof. Let $x \in \mathbf{P}$. Then $\theta \leq x$. For real numbers a, b, we have

$$\begin{aligned} ax \oplus bx &= (ax - bx) \vee (bx - ax) \\ &= (a - b)x \vee (b - a)x \\ &= |a - b|x. \end{aligned}$$

□

Definition 2.11. Let X be a real ordered Banach space, $A, B : X \to X$ be two mappings.

(1) A is said to be comparison, if for each $x, y \in X$, $x \propto y$, then $A(x) \propto A(y)$, $x \propto A(x)$, and $y \propto A(y)$.
(2) A and B is said to be comparison each other, if for each $x \in X$, $A(x) \propto B(x)$ (denoted by $A \propto B$).

Obviously, if A is comparison, then $A \propto I$ (where, I is the identity mapping on X).

Definition 2.12. Let X be a real ordered Banach space and $\mathbf{P}$ be a normal cone with normal constant N in X. A mapping $A : X \to X$ is said to be β-order compression, if A is comparative, and there exists a constant $0 < \beta < 1$ such that

$$(A(x) \oplus A(y)) \leq \beta(x \oplus y).$$

Definition 2.13. Let X be a real ordered Banach space. Then $A : X \to X$ is said to be a restricted-accretive mapping, if A is comparative, and there exists two constants $0 < \alpha_1, \alpha_2 \leq 1$ such that for arbitrary $x, y \in X$,

$$(A(x) + I(x)) \oplus (A(y) + I(y)) \leq \alpha_1(A(x) \oplus A(y)) + \alpha_2(x \oplus y),$$

where I is the identity mapping on X.

Definition 2.14. Let X be a real ordered Banach space and $B : X \to X$ be a mapping. Then $A : X \to X$ is said to be a B-restricted-accretive mapping, if A, B and $A \wedge B : x \in X \to A(x) \wedge B(x) \in X$ all are comparative and they are comparison each other, and there exists two constants $0 < \alpha_1, \alpha_2 \le 1$ such that for arbitrary $x, y \in X$,

$$(A(x) \wedge B(x) + I(x)) \oplus (A(y) \wedge B(y) + I(y)) \\ \le \alpha_1((A(x) \wedge B(x)) \oplus (A(y) \wedge B(y))) + \alpha_2(x \oplus y),$$

where I is the identity mapping on X.

Definition 2.15. Let S be a sel-fmap of X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\} \subset X$ and let $\varepsilon_n = \|u_{n+1} - h(S, u_n)\|$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S-stable or stable with respect to S.

Lemma 2.16. ([37]) *Let $\{\xi_n\}_{n=0}^{\infty}$ be a nonnegative real sequence and $\{\zeta_n\}_{n=0}^{\infty}$ be a real sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \zeta_n = \infty$. If there exists a positive integer n_1 such that*

$$\xi_{n+1} \le (1 - \zeta_n)\xi_n + \zeta_n \eta_n,$$

for all $n \ge n_1$, where $\eta_n \ge 0$ for all $n \ge 0$ and $\eta_n \to 0 (n \to \infty)$, then $\lim_{n\to\infty} \xi_n = 0$.

3. Existence of solution for generalized nonlinear ordered variational inequality

In this section, we will show the convergence of the approximation sequences for finding solution of the problem (2.1), and discuss the relation of between the first valued x_0 and the solution of the problem (2.1).

Theorem 3.1. *Let $\Re$ be a real set, X be a real ordered Banach space, $\boldsymbol{P}$ be a normal cone with the normal constant N in X, $\le$ be a partial ordered relation defined by the cone $\boldsymbol{P}$, $A, g, f : X \to X$ be continuous, where A is β-ordered compression, g is γ-ordered compression, and f is δ-ordered compression. If $(A - \rho f + w) + I$ is a restricted-accretive mapping with constants α_1, α_2 for $w \in X$ and $\rho \in \Re$, and for any $1 > \eta > 0$,*

$$\eta|\beta\gamma - |\rho|\delta| < \frac{1 - \alpha_2 N}{\alpha_1 N}, \tag{3.1}$$

then the equation

$$A(g(x)) - \rho f(x) = w (x \in X) \tag{3.2}$$

has a unique solution u^*.

Proof. This directly follows from the definition of the $\wedge$, and the condition that $A, g, B, A \wedge B, A(g(x)) - \rho f(x) - \omega : X \to X$ are comparison, respectively, and any two of them can compared each other. $\square$

We can have the following theorem from Theorem 3.1 and Lemma 3.2.

Theorem 3.2. *Let $\Re$ be a real set, X be a real ordered Banach space, $\boldsymbol{P}$ be a normal cone of X, $\leq$ be an order relation defined by the cone $\boldsymbol{P}$, A, g, f, B, $A(g(x))-\rho f(x)-\omega$, $(A(g(x))-\rho f(x)-\omega)\wedge B : X \to X$ be continuous, A, g, f, B, $A(g(x))-\rho f(x)-\omega$, $(A(g(x))-\rho f(x)-\omega)\wedge B : X \to X$ all be comparative and they are comparison each other, and A, B be β_i-ordered compression$(i = 1,2)$, g be γ-ordered compression and f be δ-ordered compression, respectively. If $A(g(x))-\rho f(x)-\omega$ is a B-restricted-accretive mapping with constants α_1, α_2 for $w \in X$ and $\rho \in \Re$, and for any $1 > \eta > 0$,*

$$\eta(|\beta_1\gamma - |\rho|\delta| \vee \beta_2) < \frac{1-\alpha_2}{N\alpha_1}, \tag{3.3}$$

then the generalized nonlinear ordered variational inequality (2.1) *has a unique solution u^*.*

Proof. Define $G : X \to X$ as follows:

$$G(u) = \eta(A(g(u)) - \rho f(u) - w) \wedge B(u) + I(u), \quad (\forall\, u \in X, \ \ \rho \in \Re). \tag{3.4}$$

For $u, v \in X$ and $u \propto v$, by using the restricted-accretivity and the B-ordered compression of A, Lemma 2.8(7), Lemma 2.9(6)(7) and the conditions, we have

$$\begin{aligned}
\theta &\leq G(u) \oplus G(v) \\
&\leq (\eta(A(g(u)) - \rho f(u) - w) \wedge B(u) + I(u)) \\
&\quad \oplus (\eta(A(g(v)) - \rho f(v) - w) \wedge B(v) + I(v)) \\
&\leq \alpha_1 \eta[A(g(u)) - \rho f(u) - w] \wedge B(u) \\
&\quad \oplus \eta[A(g(v)) - \rho f(v) - w] \wedge B(v)) + \alpha_2(u \oplus v) \\
&\leq \alpha_1(\eta[A(g(u)) - \rho f(u) - w] \wedge B(u) \\
&\quad \oplus \eta[A(g(v)) - \rho f(v) - w] \wedge B(v)) + \alpha_2(u \oplus v) \\
&\leq \alpha_1 \eta[((A(g(u)) - \rho f(u) - w) \\
&\quad \oplus (A(g(v)) - \rho f(v) - w)) \vee (B(u) \oplus B(v))] + \alpha_2(u \oplus v) \\
&\leq \alpha_1 \eta(|\beta_1\gamma - |\rho|\delta| \vee \beta_2)(u \oplus v) + \alpha_2(u \oplus v) \\
&\leq (\alpha_1 \eta(|\beta_1\gamma - |\rho|\delta| \vee \beta_2) + \alpha_2)(u \oplus v) \\
&\leq (\alpha_1 \eta|\beta\gamma - |\rho|\delta| \vee \beta_2 + \alpha_2) N(u \oplus v).
\end{aligned}$$

By (3.6) and Definition 2.2, we can obtain

$$\|G(u) \oplus G(v)\| \leq h\|u - v\|, \tag{3.5}$$

where $h = (\alpha_1\eta|\beta\gamma - |\rho|\delta| \vee \beta_2 + \alpha_2)N$. It follows from (3.4) that $0 < h < 1$, thus G is a contractive mapping. So there exists a unique point $u^* \in X$ such that

$$u^* = \eta(A(g(u^*)) - \rho f(u^*) - w) \wedge B(u^*) + u^*.$$

It follows that u^* is a solution of equation (3.2) from Theorem 3.1 and Lemma 3.2, and hence, u^* is a solution of the generalized nonlinear ordered variational inequality (2.1). This completes the proof. □

4. Stability of the Ishikawa approximation algorithm

Based on Theorem 3.3, we can develop a new Ishikawa iterative sequence for solving problem (2.1) as follows:

Algorithm 4.1. *Let $\Re$ be a real set, X be a real ordered Banach space, $\boldsymbol{P}$ be a normal cone with normal constant N in X, $\leq$ be a partial ordered relation defined by the cone $\boldsymbol{P}$. Let $\{\omega_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ be two sequences such that $\omega_n, \sigma_n \in [0,1]$ and $\sum_{n=0}^{\infty} \omega_n = \infty$. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences in X introduced to take into account possible inexact computation. For any given $x_0 \in X$, the perturbed Ishikawa type iterative sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$\begin{cases} x_{n+1} = (1-\omega_n)x_n + \omega_n a_n \\ \qquad\qquad +\omega_n[\eta(A(g(y_n)) - \rho f(y_n) + w) \wedge B(y_n) + I(y_n)], \\ y_n = (1-\sigma_n)x_n + \sigma_n b_n \\ \qquad\qquad +\sigma_n[\eta(A(g(y_n)) - \rho f(y_n) + w) \wedge B(y_n) + I(y_n)]. \end{cases} \tag{4.1}$$

Let $\{z_n\}_{n=0}^{\infty}$ be any sequence in X and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$\begin{cases} \varepsilon_n = \|z_{n+1} - [(1-\omega_n)z_n \\ \qquad\qquad +\omega_n(\eta(A(g(t_n)) - \rho f(t_n) + w) \wedge B(t_n) + I(t_n)) + \omega_n a_n]\|, \\ t_n = (1-\sigma_n)z_n + \omega_n b_n \\ \qquad\qquad +\sigma_n(\eta(A(g(z_n)) - \rho f(z_n) + w) \wedge B(z_n) + I(z_n)), \end{cases} \tag{4.2}$$

where $1 > \eta > 0$, $\rho \in \Re$ and $n = 0, 1, 2, \cdots$.

Remark 4.2. For a suitable choice of the mappings $A, g, f, B, \sigma_n, \omega_n$ and space X, then the Algorithm 4.1 can be degenerated to a known algorithms which due to [16].

Theorem 4.3. *Let $\Re, X, A, g, f, B$ be the same as in Theorem* 3.3*,* $\{\omega_n\}_{n=0}^{\infty}$ *and* $\{\sigma_n\}_{n=0}^{\infty}$ *be two sequences such that* $\omega_n, \sigma_n \in [0,1]$ *and* $\sum\limits_{n=0}^{\infty} \omega_n = \infty$*. Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be two sequences in X introduced to take into account possible inexact computation. If*

$$\eta(|\beta_1\gamma - |\rho|\delta| \vee \beta_2) < \frac{1-\alpha_2}{2N\alpha_1}, \tag{4.3}$$

then we have the following statements.

(i) *If* $\lim\limits_{n\to\infty} \|a_n \vee -a_n\| = \lim\limits_{n\to\infty} \|b_n \vee -b_n\| = 0$*, then the sequence* $\{x_n\}$ *generated by* (4.1) *converges strongly to* $x^* \in X$*, and* x^* *is a unique solution of problem* (2.1)*.*

(ii) *Moreover, for* $0 < \varphi \leq \omega_n$*,* $\lim\limits_{n\to\infty} z_n = x^*$ *if and only if* $\lim\limits_{n\to\infty} \varepsilon_n = 0$*, where* ε_n *is defined by* (4.2)*, that's the sequence* $\{x_n\}$ *generated by* (4.1) *is S-stable.*

Proof. (i) Let $\Re, X, A, g, f, B$ be the same as in Theorem 3.3. Let x^* be a unique solution of problem (2.1). Then we have

$$\begin{aligned} x^* &= (1-\omega_n)x^* + \omega_n[\eta(A(g(x^*)) - f(x^*) - w) \wedge B(x^*) + x^*] \\ &= (1-\sigma_n)x^* + \sigma_n[\eta(A(g(x^*) - f(x^*)) \wedge B(x^*) + x^*]. \end{aligned} \tag{4.4}$$

From (4.1), (4.4), (3.6), (3.7) and Lemma 2.9(6), it follows that

$$\begin{aligned} \theta &\leq x_{n+1} \oplus x^* \\ &\leq (1-\omega_n)(x_n \oplus x^*) + \omega_n(G(y_n) \oplus G(x^*)) + \omega_n(a_n \oplus \theta). \end{aligned}$$

Therefore,

$$x_{n+1} \oplus x^* \leq (1-\omega_n)(x_n \oplus x^*) + h\omega_n(y_n \oplus x^*) + \omega_n(a_n \oplus \theta), \tag{4.5}$$

where

$$h = (\alpha_1\eta(|\beta_1\gamma - |\rho|\delta| \vee \beta_2) + \alpha_2). \tag{4.6}$$

Similarly, we can prove that

$$\begin{aligned} \theta &\leq y_n \oplus x^* \\ &\leq (1-\sigma_n)(x_n \oplus x^*) + \sigma_n(G(x_n) \oplus G(x^*)). \end{aligned}$$

Therefore,

$$y_n \oplus x^* \leq (1-\sigma_n)(x_n \oplus x^*) + \sigma_n h(x_n \oplus x^*) + \sigma_n(b_n \oplus \theta). \tag{4.7}$$

If follows from (4.5), (4.6), (4.7) that

$$\begin{aligned}\theta &\le x_{n+1} \oplus x^* \\ &\le (1-\omega_n)(x_n \oplus x^*) + \omega_n(G(y_n) \oplus G(x^*)) + \omega_n(a_n \oplus \theta) \\ &\le (1-\omega_n)(x_n \oplus x^*) + h\omega_n((1-\sigma_n)(x_n \oplus x^*) \\ &\quad + \sigma_n h(x_n \oplus x^*) + \sigma_n(b_n \oplus \theta)) + \omega_n(a_n \oplus \theta) \\ &\le (1-\omega_n)(x_n \oplus x^*) + h((1-\sigma_n)(x_n \oplus x^*) \\ &\quad + \sigma_n h(x_n \oplus x^*) + \sigma_n(b_n \oplus \theta)) + \omega_n(a_n \oplus \theta).\end{aligned}$$

Therefore,

$$\begin{aligned}x_{n+1} \oplus x^* \le\ & (1-\omega_n(1-2h))(x_n \oplus x^*) \\ & + \omega_n(h\sigma_n(b_n \oplus \theta) + (a_n \oplus \theta)).\end{aligned} \tag{4.8}$$

By the assumption (4.3), we have $0 < 1-2h < 1$. It follows from (4.8), Definition 2.2, and $\|a_n \oplus \theta\| = \|a_n \vee -a_n\|$ and $\|b_n \oplus \theta\| = \|b_n \vee -b_n\|$ that

$$\begin{aligned}\|x_{n+1} - x^*\| \le\ & (1-\omega_n(1-2h))N\|x_n - x^*\| \\ & + \omega_n(1-2h)N\left(\frac{h\|b_n \vee -b_n\| + \|a_n \vee -a_n\|}{1-2h}\right).\end{aligned} \tag{4.9}$$

Letting

$$\begin{aligned}\xi_n &= \|x_n - x^*\|, \\ \zeta_n &= \omega_n(1-2h)N, \\ \chi_n &= \frac{h\|b_n \vee -b_n\| + \|a_n \vee -a_n\|}{1-2h},\end{aligned} \tag{4.10}$$

then (4.10) can be written as

$$\xi_{n+1} \le (1-\zeta_n)\xi_n + \zeta_n \chi_n. \tag{4.11}$$

It follows from Lemma 2.16 and $\lim\limits_{n\to\infty} \|a_n \vee -a_n\| = \lim\limits_{n\to\infty} \|b_n \vee -b_n\| = 0$ that $\xi_n \to 0 (n \to \infty)$, and so $\{x_n\}$ converges strongly to the unique solution x^* of problem (2.1).

Now we prove (ii). By (4.2), Lemma 2.8(5) and the proof of inequality (4.8), we obtain

$$
\begin{aligned}
\theta &\le z_{n+1} \oplus x^* \\
&\le z_{n+1} \oplus [(1-\omega_n)z_n+\omega_n(\eta((A(g(t_n))-\rho f(t_n))-w) \wedge B(t_n)+I(t_n))+\omega_n a_n] \\
&\quad + [(1-\omega_n)z_n+\omega_n(\eta((A(g(t_n))-\rho f(t_n))-w) \wedge B(t_n)+I(t_n))+\omega_n a_n] \oplus x^* \\
&\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n G(t_n) + \omega_n a_n)) \\
&\quad + ((1-\omega_n)z_n + \omega_n G(t_n) + \omega_n a_n) \oplus ((1-\omega_n)x^* + \omega_n G(x^*)) \\
&\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n G(t_n) + \omega_n a_n)) \\
&\quad + (1-\omega_n)(z_n \oplus x^*) + \omega_n G(t_n) \oplus G(x^*) + \omega_n(a_n \oplus \theta) \\
&\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n G(t_n) + \omega_n a_n)) \\
&\quad + (1-\omega_n)(z_n \oplus x^*) + \omega_n G(t_n) \oplus G(x^*) + \omega_n(a_n \oplus \theta) \\
&\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n G(t_n) + \omega_n a_n)) \\
&\quad + (1-\omega_n)(z_n \oplus x^*) + \omega_n h(t_n \oplus x^*) + \omega_n(a_n \oplus \theta) \\
&\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n G(t_n) + \omega_n a_n)) \\
&\quad + (1-\omega_n(1-2h))(z_n \oplus x^*) + \omega_n(h\sigma_n(b_n \oplus \theta) + (a_n \oplus \theta)).
\end{aligned} \tag{4.12}
$$

As the proof of inequality (4.9), and $a_n \oplus \theta = a_n \vee -a_n$ and $b_n \oplus \theta = b_n \vee -b_n$, we have

$$
\begin{aligned}
\|z_{n+1} - x^*\| \le N\|z_{n+1} - [(1-\omega_n)z_n + \omega_n G(t_n) + \omega_n a_n\| \\
+ N(1-\omega_n(1-2h))\|z_n - x^*\| \\
+ N\omega_n(h\sigma_n\|b_n \vee -b_n\| + \|a_n \vee -a_n\|).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|z_{n+1} - x^*\| \le N\varepsilon_n + N(1-\omega_n(1-2h))\|z_n - x^*\| \\
+ N\omega_n(h\|b_n \vee -b_n\| + \|a_n \vee -a_n\|).
\end{aligned} \tag{4.13}
$$

Since $0 < \varphi \le \omega_n$, by (4.13) and Definition 2.2, we have

$$
\begin{aligned}
&\|z_{n+1} - x^*\| \\
&\le [1-\omega_n(1-2h)]N\|z_n - x^*\| \\
&\quad + (1-2h)\omega_n N\left(\frac{h\|b_n \vee -b_n\| + \|a_n \vee -a_n\|}{1-2h} + \frac{\varepsilon_n}{\varphi(1-2h)}\right).
\end{aligned} \tag{4.14}
$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$, we have $\lim_{n\to\infty} z_n = x^*$ for $\sum_{n=0}^{\infty} \omega_n = \infty$, Lemma 2.16 and $\lim_{n\to\infty} \|a_n \vee -a_n\| = \lim_{n\to\infty} \|b_n \vee -b_n\| = 0$.

Conversely, if $\lim_{n\to\infty} z_n = x^*$, then by (4.4) and

$$
\lim_{n\to\infty} \|a_n \vee -a_n\| = \lim_{n\to\infty} \|b_n \vee -b_n\| = 0,
$$

we get

$$\begin{aligned}\theta &\le z_{n+1}\oplus[(1-\omega_n)z_n+\omega_n(\eta(A(g(t_n))-\rho f(t_n)-w)\wedge B(t_n)+I(t_n))+\omega_n a_n]\\ &\le (z_{n+1}\oplus x^*)\\ &\quad + [(1-\omega_n)z_n+\omega_n(\eta(A(g(t_n))-\rho f(t_n)-w)\wedge B(t_n)+I(t_n))+\omega_n a_n]\oplus x^*\\ &\le (z_{n+1}\oplus x^*)+[(1-\omega_n)(z_n\oplus x^*)+\omega_n(G(t_n)\oplus G(x^*))+\omega_n a_n]\\ &\le (z_{n+1}\oplus x^*)+(1-\omega_n(1-2h))(z_n\oplus x^*)+\omega_n(h\sigma_n(b_n\oplus\theta)+(a_n\oplus\theta)).\end{aligned}$$

It follows that the following result from (4.2) and Definition 2.2

$$\begin{aligned}&\|\varepsilon_n\|\\ &\le \|z_{n+1}-[(1-\omega_n)z_n+\omega_n(\eta(A(g(t_n))-\rho f(t_n)-w)\wedge B(t_n)+I(t_n))+\omega_n a_n]\|\\ &\le N\|z_{n+1}\oplus x^*\|+N(1-\omega_n(1-2h))\|z_n-x^*\|\\ &\quad +\omega_n(h\|b_n\oplus\theta\|+\|a_n\oplus\theta\|)\\ &\le N\|z_{n+1}\oplus x^*\|+N(1-\omega_n(1-2h))\|z_n-x^*\|\\ &\quad +\omega_n(h\|b_n\vee -b_n\|+\|a_n\vee -a_n\|).\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\|\varepsilon_n\|=0. \tag{4.15}$$

The sequence $\{x_n\}$ generated by (4.1) is S-stable. This completes the proof. □

Remark 4.4. For a suitable choice of the mappings A, g, f, B, we can obtain the well-known results [16] as special cases of Theorem 3.1–3.3.

Acknowledgments: The authors acknowledgment support of the Educational Science Foundation of Chongqing, Chongqing(KJ1400426). Authors are thankful to referee for his valuable suggestions.

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