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GROWTH OF A POLYNOMIAL NOT VANISHING INSIDE A DISK

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Abstract. Recently, Aziz and Aliya [2] proved that if polynomial $P(z)$ of degree n does not vanish in the disk $|z| < k$ where $k > 0$, then for every $\beta \in \mathbb{C}$ and $0 \le r \le R \le k$,

$$\max_{|z|=1}|P(Rz) - \beta P(rz)| \le \left[(|\beta| + |1-\beta|)\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - |\beta|\right]\max_{|z|=r}|P(z)| - \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1\right]\min_{|z|=k}|P(z)|.$$

In this paper, a refinement of above inequality and other related results are obtained.

1. Introduction

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Then concerning the estimate of the maximum of $|P'(z)|$ on the unit circle $|z| = 1$, Serge Bernstien [4] proved that

$$\max_{|z|=1}\left|P'(z)\right| \le n \max_{|z|=1}|P(z)|. \tag{1.1}$$

The result is best possible and equality (1.1) holds for $P(z) = \rho z^n, \rho \neq 0$.

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Where as, for the estimate of $|P(z)|$ on a smaller circle $|z| = r$, where $0 < r \leq 1$, of a polynomial $P(z)$ in terms of its degree n and the maximum modulus on the unit circle, we have the following inequality due to Zarantonello and Varga [12].

$$\max_{|z|=r} |P(z)| \geq r^n \max_{|z|=1} |P(z)| . \tag{1.2}$$

The result is sharp and the extremal polynomial for (1.2) is $P(z) = \rho z^n, \rho \neq 0$.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then both the inequalities (1.1) and (1.2) can be sharpened and can be replaced by

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\max_{|z|=r} |P(z)| \geq \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |P(z)| , \tag{1.4}$$

respectively. Inequality (1.3) was conjectured by Erdös and later verified by Lax [6]. Where as, inequality (1.3) is due to Rivilin [11].

As an extension of (1.3), Malik [7] proved that if $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$ where $k \geq 1$, then

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)| . \tag{1.5}$$

As a generalization of (1.5), Qazi [8] proved if $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n which does not vanish in the disk $|z| < k$, $k \geq 1$ then

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{1+k^n \phi(\mu,k)} \max_{|z|=1} |P(z)| , \tag{1.6}$$

where

$$\phi(\mu,k) = \frac{k + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^\mu}{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}} \tag{1.7}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^n \leq 1, \quad 1 \leq \mu \leq n. \tag{1.8}$$

By using inequality (1.6), Qazi [8] also proved that if $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n which does not vanish in the disk $|z| < 1$, then for $0 \leq r \leq 1$

$$\max_{|z|=R\geq 1} |P(z)| \leq n \left(\frac{1+R^\mu}{1+r^\mu}\right)^{n/\mu} \max_{|z|=r} |P(z)| . \tag{1.9}$$

In litrature, there exists several extensions of these inequalities (e.g, see [1, 9, 10]).

Recently Aziz and Aliya [2] considered for a fixed μ, the class of polynomials

$$\mathcal{P}_{n,\mu} := \left\{ P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, \quad 1 \leq \mu \leq n \right\}$$

of degree at most n not vanishing in the disk $|z| < k$ where $k > 0$ and proved the following Theorem which provides an improvement as well as a generalization of the inequality (1.9).

Theorem 1.1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$ where $k > 0$, then for every $\beta \in \mathbb{C}$ and $0 \leq r \leq R \leq k$,*

$$\max_{|z|=1} \big|P(Rz) - \beta P(rz)\big| \leq \left[\big(|\beta| + |1-\beta|\big) \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right)^{\frac{n}{\mu}} - |\beta| \right] \max_{|z|=r} |P(z)| - \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right)^{\frac{n}{\mu}} - 1 \right] \min_{|z|=k} |P(z)|. \tag{1.10}$$

If we take $\beta = 0$ in (1.10), we obtain

$$\max_{|z|=R} \big|P(z)\big| \leq \left[\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right]^{\frac{n}{\mu}} \max_{|z|=r} |P(z)| - \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right)^{\frac{n}{\mu}} - 1 \right] \min_{|z|=k} |P(z)|.$$

Which clearly is an improvement as well as generalization of (1.9).

They [2] also obtained the following result which not only extends and refines a result proved by Dewan and Bidkham [5] but, in particular, also includes a result due to Aziz and Shah [3] as a special case.

Theorem 1.2. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$ where $k > 0$, then for $\rho \geq 1$ and $0 \leq r \leq R \leq k$*

$$\max_{|z|=R} \big|P(\rho z) - P(z)\big| \leq \frac{R^\mu(\rho^n - 1)}{r^\mu + k^\mu} \left[\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right]^{\frac{n}{\mu}} \left\{ \max_{|z|=r} |P(z)| - m \right\}, \tag{1.11}$$

where $m = \min_{|z|=k} |P(z)|$.

2. Lemmas

For the proofs of our main theorems we need the following lemmas.

Lemma 2.1. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k, k > 0$, then*

$$|P(z)| \geq \min_{|z|=k}|P(z)| \quad for \quad |z| \leq k. \tag{2.1}$$

This Lemma is a simple consequence of Minimum modulus theorem and next lemma is implicit in [2].

Lemma 2.2. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k, k \geq 1$ and $0 \leq t \leq 1$, then*

$$\frac{|a_\mu|k^\mu}{|a_0| - tm} \leq \frac{n}{\mu}, \tag{2.2}$$

where $m = \min_{|z|=k}|P(z)|$.

Lemma 2.3. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| \leq k$ where $k \geq 1$, then for $0 \leq t \leq 1$ and $|z| = 1$,*

$$\left|P'(z)\right| \leq \frac{n}{1 + k^{\mu+1}\left\{\frac{1+\frac{\mu}{n}\frac{|a_\mu|k^{\mu-1}}{|a_0|-tm}}{1+\frac{\mu}{n}\frac{|a_\mu|k^{\mu+1}}{|a_0|-tm}}\right\}}\left\{\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right\}, \tag{2.3}$$

where $m = \min_{|z|=k}|P(z)|$.

The above Lemma is due to Aziz and Aliya [2].

Lemma 2.4. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| \leq k$ where $k > 0$, then for $0 < r \leq R \leq k$ and $0 \leq t \leq 1$,*

$$\exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}\rho^{\mu-1} + \rho^\mu}{\rho^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{\mu+1}\rho^\mu + k^{2\mu}\rho\right)}d\rho\right\} \leq \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}, \tag{2.4}$$

where $m = \min_{|z|=k}|P(z)|$.

Proof. Since $P(z) \neq 0$ in $|z| < k, k > 0$, the polynomial $F(z) = P(\rho z) \neq 0$ in $|z| < k/\rho$, $k/\rho \geq 1$, where $0 < \rho \leq k$. Hence applying Lemma 2.2 to $F(z)$, we get

$$\frac{|a_\mu|\rho^\mu}{|a_0| - tm'}\left(\frac{k}{\rho}\right)^\mu \leq \frac{n}{\mu}, \tag{2.5}$$

where $m' = \min_{|z|=k/\rho}|F(z)| = \min_{|z|=k/\rho}|P(\rho z)| = m$. Therefore, (2.5) becomes

$$\frac{\mu}{n}\frac{|a_\mu|\rho^\mu}{|a_0| - tm}k^\mu \leq 1,$$

which is equivalent to

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}\rho^{\mu-1}+\rho^\mu}{\rho^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{\mu+1}\rho^\mu+k^{2\mu}\rho\right)}\leq\frac{\rho^{\mu-1}}{\rho^\mu+k^\mu}. \tag{2.6}$$

Integrating both sides of (2.6) with respect to ρ from r to R where $0<r\leq R\leq k$, we get

$$\int_r^R\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}\rho^{\mu-1}+\rho^\mu}{\rho^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{\mu+1}\rho^\mu+k^{2\mu}\rho\right)}d\rho\leq\int_r^R\frac{\rho^{\mu-1}}{\rho^\mu+k^\mu}d\rho$$

or

$$n\int_r^R\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}\rho^{\mu-1}+\rho^\mu}{\rho^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{\mu+1}\rho^\mu+k^{2\mu}\rho\right)}d\rho\leq n\int_r^R\frac{\rho^{\mu-1}}{\rho^\mu+k^\mu}d\rho,$$

which implies

$$\begin{aligned}&\exp\left\{n\int_r^R\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}\rho^{\mu-1}+\rho^\mu}{\rho^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{\mu+1}\rho^\mu+k^{2\mu}\rho\right)}d\rho\right\}\\&\leq\exp\left\{\frac{n}{\mu}\int_r^R\frac{\mu\rho^{\mu-1}}{\rho^\mu+k^\mu}d\rho\right\}=\left(\frac{k^\mu+R^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}.\end{aligned}$$

This completes the proof of the Lemma 2.4. □

The next lemma is also implicit in [2, Theorem 1].

Lemma 2.5. *If $P\in\mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z|<k$ where $k\geq1$, then for every $R>r\geq1$, $0\leq t\leq1$ and $|z|=1$,*

$$\left|P(Rz)-P(rz)\right|\leq\left(\frac{R^n-1}{1+k^\mu}\right)\left(\max_{|z|=1}|P(z)|-t\min_{|z|=k}|P(z)|\right). \tag{2.7}$$

3. Main results

In this paper, we first establish an improvement of Theorem 1.1 by involving some of the coefficients of a polynomial. More precisely, we prove:

Theorem 3.1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$ where $k > 0$, then for every $\beta \in \mathbb{C}$, $0 \le r \le R \le k$ and $0 \le t \le 1$,*

$$\max_{|z|=1} |P(Rz) - \beta P(rz)| \le \Big[\Big(|\beta| + |1-\beta| \Big) \Lambda(R,r,k,\mu) - |\beta| \Big] \max_{|z|=r} |P(z)| - [\Lambda(R,r,k,\mu) - 1]\, tm, \tag{3.1}$$

where

$$\Lambda(R,r,k,\mu) = \exp\left\{ n \int_r^R \frac{\rho^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{\mu+1}\rho^{\mu-1}}{k^{\mu+1} + \rho^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{2\mu}\rho + k^{\mu+1}\rho^\mu\right)} d\rho \right\} \tag{3.2}$$

and $m = \min_{|z|=k} |P(z)|$.

Proof. Since $P(z)$ has no zero in $|z| < k, k > 0$, then for $0 < \rho \le k$, $F(z) = P(\rho z)$ has no zero in $|z| < k/\rho, k \ge \rho$. thus by applying Lemma 2.3 to $F(z)$, we obtain for $0 \le t \le 1$,

$$\max_{|z|=1} \left|F'(z)\right| \le \frac{n}{1+\left(\frac{k}{\rho}\right)^{\mu+1} \left\{ \frac{1+\frac{\mu}{n}\frac{|a_\mu|\rho^\mu}{|a_0|-tm}\left(\frac{k}{\rho}\right)^{\mu-1}}{1+\frac{\mu}{n}\frac{|a_\mu|\rho^\mu}{|a_0|-tm}\left(\frac{k}{\rho}\right)^{\mu+1}} \right\}} \left\{ \max_{|z|=1} |F(z)| - t \min_{|z|=k/\rho} |F(z)| \right\}.$$

This implies

$$\max_{|z|=1} \left|\rho P'(\rho z)\right| \le n \frac{\max_{|z|=1} |P(\rho z)| - t \min_{|z|=k/\rho} |P(\rho z)|}{1+\left(\frac{k}{\rho}\right)^{\mu+1} \left\{ \frac{1+\frac{\mu}{n}\frac{|a_\mu|\rho^\mu}{|a_0|-tm}\left(\frac{k}{\rho}\right)^{\mu-1}}{1+\frac{\mu}{n}\frac{|a_\mu|\rho^\mu}{|a_0|-tm}\left(\frac{k}{\rho}\right)^{\mu+1}} \right\}},$$

which is clearly equivalent to

$$\max_{|z|=\rho} |P'(z)| \le n \left\{ \frac{\rho^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{\mu+1}\rho^{\mu-1}}{k^{\mu+1} + \rho^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{2\mu}\rho + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{\mu+1}\rho^\mu} \right\} \times \left\{ \max_{|z|=\rho} |P(z)| - t \min_{|z|=k} |P(z)| \right\}. \tag{3.3}$$

Now, for $0 < r \le R \le k$ and $0 \le \theta < 2\pi$, we have

$$P\left(Re^{i\theta}\right) - P\left(re^{i\theta}\right) = \int_r^R e^{i\theta} P'\left(\rho e^{i\theta}\right) d\rho,$$

which gives

$$P\left(Re^{i\theta}\right) - \beta P\left(re^{i\theta}\right) = (1-\beta) P\left(re^{i\theta}\right) \int_r^R e^{i\theta} P'\left(\rho e^{i\theta}\right) d\rho,$$

where $\beta \in \mathbb{C}$. Hence for every $0 \leq \theta < 2\pi$ and $0 \leq r \leq R \leq k$,

$$\left|P\left(Re^{i\theta}\right) - \beta P\left(re^{i\theta}\right)\right| \leq |1-\beta| \left|P\left(re^{i\theta}\right)\right| + \int_r^R \left|P'\left(\rho e^{i\theta}\right)\right| d\rho,$$

from which it follows that

$$\max_{|z|=1} |P(Rz) - \beta P(rz)| \leq |1-\beta| \max_{|z|=1} |P(rz)| + \int_r^R \max_{|z|=1} \left|P'\left(\rho z\right)\right| d\rho. \tag{3.4}$$

Using (3.3) in (3.4), we get

$$\begin{aligned}
&\max_{|z|=1} |P(Rz) - \beta P(rz)| \\
&\leq |1-\beta| \max_{|z|=1} |P(rz)| \\
&\quad + n \int_r^R \frac{\rho^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{\mu+1}\rho^{\mu-1}}{k^{\mu+1} + \rho^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{2\mu}\rho + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{\mu+1}\rho^{\mu}} \\
&\quad \times \left\{ \max_{|z|=\rho} |P(z)| - t \min_{|z|=k} |P(z)| \right\} d\rho.
\end{aligned} \tag{3.5}$$

Now,

$$\begin{aligned}
\max_{|z|=\rho} |P(z)| &= \max_{|z|=1} |P(\rho z) - \beta P(rz) + \beta P(rz)| \\
&\leq \max_{|z|=1} |P(\rho z) - \beta P(rz)| + |\beta| \max_{|z|=1} |P(rz)| .
\end{aligned} \tag{3.6}$$

Also, the inequality (3.5) gives with the help of (3.6) that

$$\begin{aligned}
&\max_{|z|=1} |P(Rz) - \beta P(rz)| \\
&\leq |1-\beta| \max_{|z|=1} |P(rz)| \\
&\quad + n \int_r^R \frac{\rho^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{\mu+1}\rho^{\mu-1}}{k^{\mu+1} + \rho^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{2\mu}\rho + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm} k^{\mu+1}\rho^{\mu}} \\
&\quad \times \left\{ \max_{|z|=1} |P(\rho z) - \beta P(rz)| + |\beta| \max_{|z|=1} |P(rz)| - t \min_{|z|=k} |P(z)| \right\} d\rho.
\end{aligned} \tag{3.7}$$

If we denote right hand side of (3.7) by $\phi(R)$, then we have

$$\phi'(R) = n\left(\frac{R^{\mu}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu-1}}{k^{\mu+1}+R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{2\mu}R+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu}}\right) \times\left\{\max_{|z|=1}|P(Rz)-\beta P(rz)|+|\beta|\max_{|z|=1}|P(rz)|-t\min_{|z|=k}|P(z)|\right\}. \tag{3.8}$$

Also, (3.7) can be written as

$$\max_{|z|=1}|P(Rz)-\beta P(rz)| \leq \phi(R). \tag{3.9}$$

With the help of (3.9), the inequality (3.8) implies for $0 < r \leq R \leq k$ that

$$\phi'(R) - n\left(\frac{R^{\mu}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu-1}}{k^{\mu+1}+R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{2\mu}R+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu}}\right) \times\left\{\phi(R)+|\beta|\max_{|z|=1}|P(rz)|-t\min_{|z|=k}|P(z)|\right\} \leq 0. \tag{3.10}$$

Multiplying the two sides of (3.10) by

$$\exp\left\{-n\int\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu-1}+R^{\mu}}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}(k^{\mu+1}R^{\mu}+k^{2\mu}R)}dR\right\},$$

we get

$$\frac{d}{dR}\left[\left\{\phi(R)+|\beta|\max_{|z|=1}|P(rz)|-t\min_{|z|=k}|P(z)|\right\} \times\exp\left\{-n\int\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu-1}+R^{\mu}}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}(k^{\mu+1}R^{\mu}+k^{2\mu}R)}dR\right\}\right] \leq 0, \tag{3.11}$$

for $0 < r \leq R \leq k$. Inequality (3.11) implies that the function

$$\psi(R) = \left\{\phi(R)+|\beta|\max_{|z|=1}|P(rz)|-t\min_{|z|=k}|P(z)|\right\} \times\exp\left\{-n\int\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu-1}+R^{\mu}}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}(k^{\mu+1}R^{\mu}+k^{2\mu}R)}dR\right\}$$

is a non-increasing function of R in $(0, k]$. Hence for $0 < r \leq R \leq k$, we have

$$\psi(R) \leq \psi(r),$$

that is,

$$\left\{\phi(R) + |\beta|\max_{|z|=1}|P(rz)| - t\min_{|z|=k}|P(z)|\right\}$$
$$\times \exp\left\{-n\int \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}R^{\mu-1} + R^{\mu}}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{\mu+1}R^{\mu} + k^{2\mu}R\right)}dR\right\}$$
$$\leq \left\{\phi(r) + |\beta|\max_{|z|=1}|P(rz)| - t\min_{|z|=k}|P(z)|\right\}$$
$$\times \exp\left\{-n\int \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}r^{\mu-1} + r^{\mu}}{r^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{\mu+1}r^{\mu} + k^{2\mu}r\right)}dr\right\}. \tag{3.12}$$

Since $\phi(R) \geq \max_{|z|=1}|P(Rz) - \beta P(rz)|$ and $\phi(r) = |1-\beta|\max_{|z|=1}|P(rz)|$, therefore, we have

$$\max_{|z|=1}|P(Rz) - \beta P(rz)| \leq \left[\Big(|\beta| + |1-\beta|\Big)\Lambda(R,r,k,\mu) - |\beta|\right]\max_{|z|=r}|P(z)|$$
$$- \left[\Lambda(R,r,k,\mu) - 1\right]tm, \tag{3.13}$$

where

$$\Lambda(R,r,k,\mu) = \exp\left\{n\int_r^R \frac{\rho^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}k^{\mu+1}\rho^{\mu-1}}{k^{\mu+1} + \rho^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-tm}\left(k^{2\mu}\rho + k^{\mu+1}\rho^{\mu}\right)}d\rho\right\}.$$

This completes the proof of Theorem 3.1. □

It is easy to verify that Theorem 3.1 provides a refinement of Theorem 1.1. To see this, we note that for every $r \leq k$, by Lemma 2.1, $\max_{|z|=r}|P(z)| \geq \min_{|z|=k}|P(z)|$ and $|\beta| + |1-\beta| \geq 1$, hence the function

$$S(x) = \left[\Big(|\beta| + |1-\beta|\Big)x - |\beta|\right]\max_{|z|=r}|P(z)| - [x-1]\,tm$$

is a non-decreasing function of x for every $\beta \in \mathbb{C}$ and $0 \leq t \leq 1$. If we combine this fact with Lemma 2.4, it is easy to conclude that Theorem 3.1 is an improvement of Theorem 1.1.

If we take $\beta = 1$ in (3.1), we obtain the following result.

Corollary 3.2. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$ where $k > 0$, then for $0 \leq r \leq R \leq k$ and $0 \leq t \leq 1$,*

$$\max_{|z|=1}|P(Rz) - P(rz)| \leq \left(\Lambda(R,r,k,\mu) - 1\right)\left\{\max_{|z|=r}|P(z)| - tm\right\}, \tag{3.14}$$

where $\Lambda(R,r,k,\mu)$ is given by (3.2) *and $m = \min_{|z|=k}|P(z)|$.*

By using triangle inequality, the following result follows immediately from Corollary 3.2.

Corollary 3.3. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$ where $k > 0$, then for $0 \le r \le R \le k$ and $0 \le t \le 1$,*

$$\max_{|z|=1} |P(Rz)| \le \Lambda(R,r,k,\mu) \max_{|z|=r} |P(z)| - (\Lambda(R,r,k,\mu) - 1)\, tm, \tag{3.15}$$

where $\Lambda(R,r,k,\mu)$ is given by (3.2) and $m = \min_{|z|=k}|P(z)|$.

Next, as an improvement of Theorem 1.2, we present the following theorem.

Theorem 3.4. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$ where $k > 0$, then for $\rho > 1$, $0 \le r \le R \le k$ and $0 \le t \le 1$,*

$$\max_{|z|=R} \big|P(\rho z) - P(z)\big| \le \frac{R^\mu(\rho^n - 1)}{r^\mu + k^\mu} \Lambda(R,r,k,\mu) \left\{ \max_{|z|=r} |P(z)| - tm \right\}, \tag{3.16}$$

where $\Lambda(R,r,k,\mu)$ is given by (3.2) and $m = \min_{|z|=k}|P(z)|$.

Proof. By hypothesis $P \in \mathcal{P}_{n,\mu}$ and $P(z) \ne 0$ for $|z| < k$, where $k > 0$, therefore the polynomial $F(z) = P(Rz)$ does not vanish in $|z| < \frac{k}{R}, R > 0$ and $F \in \mathcal{P}_{n,\mu}$. Hence for $0 < R \le k$ and $0 \le t \le 1$, it follows by using Lemma 2.5 (with k replaced by $\frac{k}{R} \ge 1$) that for every $\rho \ge 1$,

$$\max_{|z|=1} \big|F(\rho z) - F(z)\big| \le \frac{(\rho^n - 1)}{1 + \left(\frac{k}{R}\right)^\mu} \left\{ \max_{|z|=1} \big|F(z)\big| - t \min_{|z|=\frac{k}{R}} \big|F(z)\big| \right\}. \tag{3.17}$$

Replacing $F(z)$ by $P(Rz)$ and noting that

$$\max_{|z|=1} \big|F(z)\big| = \max_{|z|=1} \big|P(Rz)\big| = \max_{|z|=R} \big|P(z)\big|$$

and

$$\min_{|z|=\frac{k}{R}} \big|F(z)\big| = \min_{|z|=\frac{k}{R}} \big|P(Rz)\big| = \min_{|z|=k} \big|P(z)\big|,$$

from (3.17) it follows that

$$\max_{|z|=1} \big|P(R\rho z) - P(Rz)\big| \le \frac{R^\mu(\rho^n - 1)}{R^\mu + k^\mu} \left\{ \max_{|z|=R} \big|P(z)\big| - t \min_{|z|=k} \big|P(z)\big| \right\}, \tag{3.18}$$

for $\rho \ge 1$ and $0 < R \le k$. Now if $0 \le r \le R \le k$, then by inequality (3.15), we have

$$\max_{|z|=1} |P(Rz)| \le \Lambda(R,r,k,\mu) \left\{ \max_{|z|=r} |P(z)| - tm \right\} + tm, \tag{3.19}$$

where $\Lambda(R, r, k, \mu)$ is given by (3.2) and $m = \min|z| = k|P(z)|$. Using (3.19) in (3.18), we obtain

$$\max_{|z|=R}\left|P(\rho z) - P(z)\right| \leq \frac{R^\mu(\rho^n - 1)}{r^\mu + k^\mu}\Lambda(R, r, k, \mu)\left\{\max_{|z|=r}|P(z)| - tm\right\},$$

for $0 \leq r \leq R \leq k$, $0 \leq t \leq 1$ and $\rho \geq 1$, which is (3.16) and this completes the proof of Theorem 3.4. □

Again, by using Lemma 2.4, it can be easily verified that Theorem 3.4 is an improvement of Theorem 1.2.

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