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STABILITY OF QUADRATIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In this paper, some stability results concerning the functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+f(2x)-2f(x)$$

in intuitionistic fuzzy normed spaces are investigated. Also, the intuitionistic fuzzy continuity of the quadratic and additive functions is defined.

1. Introduction

The definition of stability in the case of homomorphisms between metric groups was suggested by a problem posed by S. M. Ulam [9] in 1940. Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\longmapsto G_2$ satisfies the inequality $d(h(x*y),h(x)\diamond h(y))<\delta$ for all $x,y\in G_1$, then there is a homomorphism $H:G_1\longmapsto G_2$ whit $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation.

In 1941, D. H. Hyers gave a first affirmative answer to the equation of Ulam for Banach space[5].

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Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It was also very useful application in various fields, e. g., population dynamics[1], computer programming[3], nonlinear operators[4], stability problem [6], etc.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t- norm if it satisfies the following conditions:
$a)$ $*$ is associative and commutative,
$b)$ $*$ is continuous,
$c)$ $a \times b \leq c \times d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$,
$d)$ $a * 1 = a$ for all $a \in [0,1]$.

Definition 1.2. A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t- conorm if it satisfies the following conditions:
$a')$ $\diamond$ is associative and commutative,
$b')$ $\diamond$ is continuous,
$c')$ $a \diamond b \leq c \times d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$,
$d')$ $a \diamond 0 = a$ for all $a \in [0,1]$.

Definition 1.3. The five- tuple $(X, \mu, \nu, *, \diamond)$ is said to be intuitionistic fuzzy normed spaces if X is a vector space, $*$ is continuous t- norm, $\diamond$ is a continuous t- conorm, and μ, ν are fuzzy sets on $X \times (0, +\infty)$ satisfying the following conditions, for every $x, y \in X$ and $s, t > 0$

$(i)\mu(x,t) + \nu(x,t) \leq 1$, $(ii)\mu(x,t) \geq 0$, $(iii)\mu(x,t) = 1$ if and only if $x = 0$, $(iv)\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ for each $\alpha \neq 0$, $(v)\mu(x,t) * \mu(y,s) \leq \mu(x + y, t + s)$, $(vi)\mu(x, .) : (0, +\infty) \to [0,1]$ is continuous, $(vii) \lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$, $(viii)\nu(x,t) < 1$, $(ix)\nu(x,t) = 0$ if and if $x = 0$, $(x)\nu(\alpha x, t) = \nu(x, t/|\alpha|)$ for each $\alpha \neq 0$, $(xi)\mu(x,t) \diamond \mu(y,s) \geq \nu(x + y, t + s)$, $(xii)\nu(x, ...) : (0, +\infty) \to [0,1]$ is continuous, $(xiii) \lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

For facility we represent the intuitionistic fuzzy normed space by IFNS.

The concepts convergence and Cauchy sequences in intuitionistic fuzzy normed space are in [8].

Let $(X, \mu, \nu, *, \diamond)$ be an INFS. Then, a sequences $X = (x_k)$ is said to be intuitionistic fuzzy Cauchy sequence if $\lim_{k\to\infty} \mu(x_k - x_k, t) = 1$ and $\lim_{k\to\infty} \nu(x_k - x_k, t) = 0$ for all $t > 0$ and $p = 1, 2, ...$

Let $(X, \mu, \nu, *, \diamond)$ be an INFS. Then a sequences $X = (x_k)$ is said to be intuitionistic fuzzy convergent to $L \in X$ if $\lim_{k\to\infty} \mu(x_k - L, t) = 1$ and $\lim \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write $x_k \to^{IF} L$ as $k \to \infty$.

Let $(X, \mu, \nu, *, \diamond)$ be an INFS. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intutitionistic fuzzy Cauchy in $(X, \mu, \nu, *, \diamond)$ is intiutionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$.

In [7], Mohiuddine determined some stability results concerning the Jensen functional equation $2f((x+y)/2) = f(x)+f(y)$ in intuitionistic fuzzy normed spaces.

2. Main Results

The quadratic functional equation is

$$\Phi_f(x,y) := f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-2f(2x)+2f(x) = 0,$$

where f is a mapping between liner spaces. It is easy to see that a mapping $f : X \to Y$ between liner spaces with $f(0) = 0$ satisfies the quadratic equation if only if it is additive[2].

We begin with a generalized Hyers-Ulam-Rassias type theorem in IFNS for the quadratic functional equation.

Theorem 2.1. Let X be linear space and f be a mapping from X to an intuitionisic fuzzy Banach space (Y,μ,ν) such that $f(0) = 0$. Suppose that ϕ is a function from X to an intuitionistic fuzzy normed space (Z,μ',ϕ') such that

$$\begin{aligned} \mu(\Phi_f(x,y), t+s) &\geq \mu'(\varphi(x),t) * \mu'(\varphi(y),s), \\ \nu(\Phi_f(x,y), t+s) &\leq \nu'(\varphi(x),t) \diamond \nu'(\varphi(y),s), \end{aligned} \tag{2.1}$$

for all $x, y \in X - \{0\}, t > 0$ and $s > 0$. If $\phi(2x) = \alpha\phi(x)$ for some real number α with $0 < |\alpha| < 2$, then there exists a unique additive mapping $T : X \to Y$ such that $T(x) = \lim_{n\to\infty} f(2^n x)/2^n$ and we have

$$\begin{aligned} \mu(T(x)-f(x),t) &\geq M(x, \frac{(2-\alpha)t}{4}), \\ \nu(T(x)-f(x),t) &\leq N(x, \frac{(2-\alpha)t}{4}), \end{aligned}$$

where

$M(x,t) = \mu'(\varphi(x),t/4) * \mu'(\varphi(x/2),t/4) * \mu'(\varphi(3/2x),t) * \mu'(\varphi(2x),t)$

and

$N(x,t) = \nu'(\varphi(x),t/4) * \nu'(\varphi(x/2),t/4) * \nu'(\varphi(3/2x),t) * \nu'(\varphi(2x),t)$.

Proof. Without loss of generality, we assume that $0 < \alpha < 2$. Putting $t = s$ in (2.1), we get

$$\begin{aligned} \mu(\Phi_f(x,y), 2t) &\geq \mu'(\varphi(x),t) * \mu'(\varphi(y),t), \\ \nu(\Phi_f(x,y), 2t) &\leq \nu'(\varphi(x),t) \diamond \nu'(\varphi(y),t), \end{aligned} \tag{2.2}$$

for all $x \in X$ and $t > 0$. By putting $y = x, y = 3x, y = 4x$ in (2.2), we get the inequalities

$$\begin{aligned} &\mu(f(3x) - f(4x) + 3f(x), 2t) \geq \mu'(\varphi(x), t) \\ &\nu(f(3x) - f(4x) + 3f(x), 2t) \leq \nu'(\varphi(x), t), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\mu(f(5x) - f(4x) - f(2x) + f(x), 2t) \geq \mu'(\varphi(x), t) * \mu'(\varphi(3x), t), \\ &\nu(f(5x) - f(4x) - f(2x) + f(x), 2t) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(3x), t), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\mu(f(6x) - f(5x) + f(3x) - 3f(2x) + 2f(x), 2t) \\ &\qquad \geq \mu'(\varphi(x), t) * \mu'(\varphi(4x), t), \\ &\nu(f(6x) - f(5x) + f(3x) - 3f(2x) + 2f(x), 2t) \\ &\qquad \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(4x), t). \end{aligned} \tag{2.5}$$

It follows from above inequalities

$$\begin{aligned} &\mu(f(6x) - f(4x) - f(2x), 6t) \geq \mu'(\varphi(x), t) * \mu'(\varphi(3x), t) * \mu'(\varphi(4x), t), \\ &\nu(f(6x) - f(4x) - f(2x), 6t) \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(3x), t) \diamond \nu'(\varphi(4x), t). \end{aligned} \tag{2.6}$$

Replacing x by $x/2$ in (2.6), we get

$$\begin{aligned} &\mu(f(3x) - f(2x) - f(x), 6t) \\ &\qquad \geq \mu'(\varphi(\frac{x}{2}), t) * \mu'(\varphi(\frac{3x}{2}), t) * \mu'(\varphi(2x), t), \\ &\nu(f(3x) - f(2x) - f(x), 6t) \\ &\qquad \leq \nu'(\varphi(\frac{x}{2}), t) \diamond \nu'(\varphi(\frac{3x}{2}), t) \diamond \nu'(\varphi(2x), t). \end{aligned} \tag{2.7}$$

Therefore, we infer from (2.3) and (2.7)

$$\begin{aligned} &\mu(f(2x) - 2f(x), 8t) \\ &\qquad \geq \mu'(\varphi(x), t) * \mu'(\varphi(\frac{x}{2}), t) * \mu'(\varphi(\frac{3x}{2}), t) * \mu'(\varphi(2x), t), \\ &\nu(f(2x) - 2f(x), 8t) \\ &\qquad \leq \nu'(\varphi(x), t) \diamond \nu'(\varphi(\frac{x}{2}), t) \diamond \nu'(\varphi(\frac{3x}{2}), t) \diamond \nu'(\varphi(2x), t). \end{aligned} \tag{2.8}$$

It follows that

$$\begin{aligned} &\mu(\frac{1}{2}f(2x) - f(x), t) \\ &\quad \geq \mu'(\varphi(x), \frac{t}{4}) * \mu'(\varphi(\frac{x}{2}), \frac{t}{4}) * \mu'(\varphi(\frac{3x}{2}), \frac{t}{4}) * \mu'(\varphi(2x), \frac{t}{4}), \\ &\nu(\frac{1}{2}f(2x) - f(x), t) \\ &\quad \leq \nu'(\varphi(x), \frac{t}{4}) \diamond \nu'(\varphi(\frac{x}{2}), \frac{t}{4}) \diamond \nu'(\varphi(\frac{3x}{2}), \frac{t}{4}) \diamond \nu'(\varphi(2x), \frac{t}{4}). \end{aligned} \tag{2.9}$$

Define

$$M(x,t) := \mu'(\varphi(x),\frac{t}{4}) * \mu'(\varphi(\frac{x}{2}),\frac{t}{4}) * \mu'(\varphi(\frac{3x}{2}),\frac{t}{4}) * \mu'(\varphi(2x),\frac{t}{4}),$$
$$N(x,t) := \nu'(\varphi(x),\frac{t}{4}) * \nu'(\varphi(\frac{x}{2}),\frac{t}{4}) * \nu'(\varphi(\frac{3x}{2}),\frac{t}{4}) * \nu'(\varphi(2x),\frac{t}{4}).$$

By our assumption

$$M(2x,t) = M(x,\frac{t}{\alpha}),\ N(2x,t) = N(x,\frac{t}{\alpha}).$$

Replacing x by $2^n x$ in (2.9), we have

$$\begin{aligned}
\mu(\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^n}f(2^n x), \frac{1}{2^n}\alpha^n t) &= \mu(f(2^{n+1}) - f(2^n x), \alpha^n t) \\
&\geq M(2^n x, \alpha^n t) = M(x,t), \\
\nu(\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^n}f(2^n x), \frac{1}{2^n}\alpha^n t) &= \nu(f(2^{n+1}) - f(2^n x), \alpha^n t) \\
&\leq N(2^n x, \alpha^n t) = N(x,t).
\end{aligned} \tag{2.10}$$

Thus, for $n > m$, we have

$$\begin{aligned}
&\mu(\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x), \sum_{k=m}^{n-1}\frac{1}{2^k}\alpha^k t) \\
&\qquad = \mu(\sum_{k=m}^{n-1}(\frac{1}{2^k}f(2^k x) - \frac{1}{2^{k+1}}f(2^{k+1}x)), \sum_{k=m}^{n-1}\frac{1}{2^k}\alpha^k t) \\
&\qquad \geq \prod_{k=m}^{n-1}\mu(\frac{1}{2^k}f(2^k x) - \frac{1}{2^{k+1}}f(2^{k+1}x), \frac{1}{2^k}\alpha^k t) \\
&\qquad \geq M(x,t),
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\nu(\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x), \sum_{k=m}^{n-1}\frac{1}{2^k}\alpha^k t) \\
&\qquad = \nu(\sum_{k=m}^{n-1}(\frac{1}{2^k}f(2^k x) - \frac{1}{2^{k+1}}f(2^{k+1}x)), \sum_{k=m}^{n-1}\frac{1}{2^k}\alpha^k t) \\
&\qquad \leq \coprod_{k=m}^{n-1}\nu(\frac{1}{2^k}f(2^k x) - \frac{1}{2^{k+1}}f(2^{k+1}x), \frac{1}{2^k}\alpha^k t) \\
&\qquad \leq N(x,t),
\end{aligned} \tag{2.12}$$

where

$$\prod_{j=1}^{n} a_j = a_1 \star a_2 \star ... \star a_n,\ \coprod_{j=1}^{n} a_j = a_1 \diamond a_2 \diamond ... \diamond a_n.$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{n\to\infty} M(x,t) = 1$ and $\lim_{t\to\infty} N(x,t) = 0$, then there exists $t_0 > 0$ such that $M(x,t_0) > 1-\epsilon$ and $N(x,t_0) < \epsilon$. Since $\sum_{k=0}^{\infty} \alpha^k t_0/2^k < \infty$, then there exists $N \in \mathbb{N}$ such that $\sum_{k=m}^{n-1} \alpha^k t_0/2^k < \delta$, for all $n > m \geq N$. It follows that

$$\begin{aligned}\mu(\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x), \delta) &\geq \mu(\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x), \sum_{k=m}^{n-1}\frac{1}{2^k}\alpha^k t_0)\\ &\geq M(x,t_0) > 1-\epsilon,\end{aligned}$$

$$\begin{aligned}\nu(\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x), \delta) &\leq \nu(\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x), \sum_{k=m}^{n-1}\frac{1}{2^k}\alpha^k t_0)\\ &\leq N(x,t_0) < \epsilon.\end{aligned}$$

This shows that $\{\frac{1}{2^n}f(2^n x)\}$ is cauchy sequence in (Y,μ,ν). Since (Y,μ,ν) is complete, $\{\frac{1}{2^n}f(2^n x)\}$ converges to $T(x)$. Moreover, if we put $m = 0$ in (2.11) and(2.12), we get

$$\begin{aligned}\mu(\frac{1}{2^n}f(2^n x) - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{2^k}) \geq M(x,t),\\ \nu(\frac{1}{2^n}f(2^n x) - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{2^k}) \leq N(x,t).\end{aligned} \tag{2.13}$$

Now ,we will show that T is additive. Let $x, y \in X$. Then

$$\begin{aligned}&\mu(\Phi_T(x,y),t) \geq \mu(T(2x+y) - \frac{1}{2^n}f(2^n(2x+y)), \frac{t}{7})\\ &\star\, \mu(T(2x-y) - \frac{1}{2^n}f(2^n(2x-y)), \frac{t}{7}) \star \mu(\frac{1}{2^n}f(2^n(x+y)) - T(x+y), \frac{t}{7})\\ &\star\, \mu(\frac{1}{2^n}f(2^n(x-y) - T(x-y), \frac{t}{7}) \star \mu(\frac{2}{2^n}f(2^n(2x)) - 2T(2x), \frac{t}{7})\\ &\star\, \mu(2T(x) - \frac{2}{2^n}f(2^n x), \frac{t}{7}) \star \mu(\frac{1}{2^n}f(2^n(2x+y)) + \frac{1}{2^n}f(2^n(2x-y))\\ &-\frac{1}{2^n}f(2^n(x+y)) - \frac{1}{2^n}f(2^n(x-y)) - \frac{2}{2^n}f(2^n(2x)) + \frac{2}{2^n}f(2^n x), \frac{t}{7}),\end{aligned} \tag{2.14}$$

and by using (2.1), we have

$$\mu(\frac{1}{2^n}f(2^n(2x+y))+\frac{1}{2^n}f(2^n(2x-y))-\frac{1}{2^n}f(2^n(x+y))$$
$$-\frac{1}{2^n}f(2^n(x-y))-\frac{2}{2^n}f(2^n(2x))+\frac{2}{2^n}f(2^n x),\frac{t}{7})\geq \mu'(\varphi(2^n x),\frac{2^n t}{14}) \quad (2.15)$$
$$\star\,\mu'(\varphi(2^n y),\frac{2^n t}{14})=\mu'(\varphi(x),\frac{(\frac{2}{\alpha})^n t}{14})\star\mu'(\varphi(y),\frac{(\frac{2}{\alpha})^n t}{14}).$$

Letting $n\to\infty$ in (2.14) and (2.15), we get

$$\mu(T(2x+y)+T(2x-y)-T(x+y)-T(x-y)-2T(2x)+2T(x),t)=1,$$

similarly, we obtain

$$\nu(T(2x+y)+T(2x-y)-T(x+y)-T(x-y)-2T(2x)+2T(x),t)=0,$$

for all $x, y \in X$ and $t > 0$. This means that T satisfies the quadratic equation, so it is additive.

Now we approximate the difference between f and T in intuitionistic fuzzy sense. By (2.13), we have

$$\mu(T(x)-f(x),t)\geq \mu(T(x)-\frac{1}{2^n}f(2^n x),\frac{t}{2})\star\mu(\frac{1}{2^n}f(2^n x)-f(x),\frac{t}{2})$$
$$\geq M(x,\frac{t}{2\sum_{k=0}^{\infty}(\frac{\alpha}{2})^k})=M(x,\frac{(2-\alpha)t}{4}),$$

and

$$\nu(T(x)-f(x),t)\leq \nu(T(x)-\frac{1}{2^n}f(2^n x),\frac{t}{2})\star\nu(\frac{1}{2^n}f(2^n x)-f(x),\frac{t}{2})$$
$$\leq N(x,\frac{t}{2\sum_{k=0}^{\infty}(\frac{\alpha}{2})^k})=N(x,\frac{(2-\alpha)t}{4}),$$

for every $x \in X, t > 0$ and sufficiently large n. To prove the uniqueness of T, assume that $T^{'}$ be another additive mapping from X into Y, which satisfies the required inequality. Then

$$\mu(T(x)-T^{'}(x),t)\geq \mu(T(x)-f(x),\frac{t}{2})\star\mu(T^{'}(x)-f(x),\frac{t}{2})$$
$$\geq M(x,\frac{(2-\alpha)t}{8}),$$

and

$$\nu(T(x)-T'(x),t)\leq \nu(T(x)-f(x),\frac{t}{2})\diamond\nu(T'(x)-f(x),\frac{t}{2})$$
$$\leq N(x,\frac{(2-\alpha)t}{8}),$$

for all $x \in X, t > 0$. Therefore by the additivity of T and $T^{'}$, we have:

$$\mu(T(x) - T^{'}(x), t) = \mu(T(2^n x) - T^{'}(2^n x), 2^n t) \geq M(x, \frac{2^n(2-\alpha)}{8\alpha^n}),$$

and

$$\nu(T(x) - T^{'}(x), t) = \nu(T(2^n x) - T^{'}(2^n x), 2^n t) \leq N(x, \frac{2^n(2-\alpha)}{8\alpha^n}),$$

for all $x \in X, t > 0$ and $n \in \mathbb{N}$. Since $0 < \alpha < 2$ and $\lim_{n\to\infty}(\frac{2}{\alpha})^n = \infty$, we get

$$\lim_{n\to\infty} M(x, \frac{2^n(2-\alpha)}{8\alpha^n}) = 1, \ \lim_{n\to\infty} N(x, \frac{2^n(2-\alpha)}{8\alpha^n}) = 0.$$

Therefore $\mu(T(x)-T^{'}(x), t) = 1$ and $\nu(T(x)-T'(x), t) = 0$ for all $x \in X, t > 0$. Hence $T(x) = T'(x)$ for all $x \in X$. □

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