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RESULTS ON n-TUPLED COINCIDENCE AND FIXED POINTS IN PARTIALLY ORDERED G-METRIC SPACES VIA SYMMETRIC (ϕ, ψ)-CONTRACTIONS

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Abstract. Recently, n-tupled fixed point theorems have been intensively studied by some authors in the framework of partially ordered G-metric spaces. In the present paper, some n-tupled coincidence as well as n-tupled fixed point results for a pair of symmetric (ϕ,ψ)-contractive mappings having mixed g-monotone property are established in the context of partially ordered complete G- metric spaces. Furthermore uniqueness of n-tupled common fixed points is presented. Our results improve the results of Karpinar et al. [8], Jain and Tas [7] and Mustafa [16]. In light of the comment given in Jain and Tas [7], our results also generalize the results of Choudhary and Maity [5], Nashine [18] and Mohiuddin et al. [15]. To substantiate the validity of our hypothesis, some examples are also presented herein.

1. Introduction

Bhaskar and Lakshmikantham [3] introduced the notion of a coupled fixed point and proved some coupled fixed point theorems in partially ordered complete metric spaces. Afterwards, Lakshmikantham and Ciric [13] extended

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these results by introducing mixed g-monotone property and coupled coincidence point and proved some coupled coincidence point and common coupled fixed point theorems in partially ordered complete metric spaces.

Recently, the notion of coupled fixed point is extended to higher dimensions by many authors(see [2],[4],[6],[9],[10],[11],[20]). In [2], Berinde and Borcut introduced the concept of tripled fixed point and obtained some related theorems. Afterwards, Karpinar [11] introduced the concept of quadruple fixed point and mixed monotone property of mapping $F: X^4 \to X$ and proved some related theorems. Most recently, Imdad et al. [6] introduced the concepts of n-tupled coincidence as well as n-tupled fixed point (for even n) and utilized to obtain n-tupled coincidence as well as n-tupled common fixed point theorems for nonlinear mappings satisfying ϕ-contractive condition in partially ordered complete metric spaces.

On the other hand, Mustafa and Sims [17] introduced a new structure of generalized metric space, called G-metric space. In which, a non-negative real number was assigned to every triplet of elements. Sadati et al. [19] proved some fixed point results for contractive mappings in generalized metric spaces endowed with the partial ordering, known as partially ordered G-metric spaces. Chaudhary and Maity [5] studied necessary conditions for the existence of a coupled fixed point in this space. Luong and Thuan [14] presented some coupled fixed point theorems for a mixed monotone mapping in a partially ordered G-metric space which are generalizations of the results of Bhaskar and Lakshmikantham [3] and provided an existence and uniqueness for a solution of a nonlinear integral equation. Afterwards, Karpinar et al. in [8] and Nashine [18] extended the results of [5] for a pair of commutative maps. Mustafa [16] established some quadruple coincidence and quadruple common fixed theorems in partially ordered G-metric spaces. Recently, Jain and Tas [7], generalized and enriched the result of Choudhary et al. [5], Nashine [18] and Mohiuddin et al. [15]. Most recently Abbas, Kim and Nazir in [1], proved some common fixed point theorems of mappings satisfying almost contractive condition in complete partially ordered G-metric space.

In this paper, we aim to establish the existence and uniqueness of n-tupled coincidence and n-tupled fixed point theorems for a pair of symmetric (ϕ,ψ)-contractive mappings having mixed g-monotone property in the perspective of partially ordered G-metric spaces equipped with a partial ordering in a virtually different and more natural way. Our result generalize and extend the results of Karpinar et al. [8], Jain and Tas [7] and Mustafa [16]. Some examples to show the validity of hypothesis of our main results are also presented.

2. Preliminaries

Here, we present some basic definitions and fundamental results, which will be useful to our article.

In 1984, Khan et al. [12] utilized the idea of altering distance function in metric fixed point theory which is indeed a control function that alters the distance between two points.

Definition 2.1. ([12]) Let Φ denote the class of all functions $\phi : [0, +\infty) \to [0, +\infty)$ with the following properties:

(i) ϕ is continuous and non-decreasing;
(ii) $\phi(t) > 0 \quad$ for all $\; t > 0$;
(iii) $\phi(a + b) < \phi(a) + \phi(b)$ for all $a, b \in [0, +\infty)$.

From (i) and (ii) it is clear that $\phi(t) = 0$ if and only if $t = 0$. Let Ψ denote the class of all functions $\psi : [0, +\infty) \to [0, +\infty)$ with the following properties:

(i) $\lim\limits_{t\to r} \psi(t) > 0 \quad$ for all $\quad r > 0$;
(ii) $\lim\limits_{t\to 0_+} \psi(t) = 0$.

We borrow the definition of n-tupled fixed point and n-tupled coincidence point from Imdad et al. [6].

Throughout the paper, we consider n to be an even integer.

Definition 2.2. ([6]) An element $(x^1, x^2, \ldots, x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, \ldots, x^n) = x^1, \\ F(x^2, x^3, \ldots, x^n, x^1) = x^2, \\ F(x^3, \ldots, x^n, x^1, x^2) = x^3, \\ \vdots \\ F(x^n, x^1, x^2, \ldots, x^{n-1}) = x^n. \end{cases}$$

In the following example we establish n-tupled fixed point.

Example 2.3. Let $X = \mathbb{R}$. Then $(X, \leq)$ is a partially ordered set with usual ordering. Let $F : X^n \to X$ be a mapping defined by $F(x^1, x^2, x^3, \ldots, x^n) = \sin x^1 + \sin x^2 + \sin x^3 + \ldots + \sin x^n$, for all $(x^1, x^2, x^3, \ldots, x^n) \in X$. Then $(0, 0, 0, \ldots, 0)$ is a unique n-tupled fixed point of F.

Definition 2.4. ([6]) An element $(x^1, x^2, \dots, x^n) \in X^n$ is called an n-tupled coincidence point of the mapping $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, \dots, x^n) = gx^1, \\ F(x^2, x^3, \dots, x^n, x^1) = gx^2, \\ F(x^3, \dots, x^n, x^1, x^2) = gx^3, \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = gx^n. \end{cases}$$

In following example establishes n-tupled coincidence point.

Example 2.5. Let $X = \mathbb{R}$. Then $(X, \leq)$ is a partially ordered set with usual ordering. Let $F : X^n \to X$ and $g : X \to X$ be two mappings defined by $F(x^1, x^2, x^3, \dots, x^n) = \frac{x^{1^2} + x^{2^2} + x^{3^2} + \dots + x^{n^2}}{2n^2}$, for all $(x^1, x^2, x^3, \dots, x^n) \in X$, and $gx = x^2$. Then (0,0,0,…,0) is a unique n-tupled coincidence point of F.

Definition 2.6. Let (X, G) be a G-metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. We say that, F and g are symmetric (ϕ, ψ)-contractive mappings on X if there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\begin{aligned} &\phi((G(F(x^1, x^2, \dots, x^n), F(y^1, y^2, \dots, y^n), F(z^1, z^2, \dots, z^n)) \\ &\quad + G(F(x^2, \dots, x^n, x^1), F(y^2, \dots, y^n, y^1), F(z^2, \dots, z^n, z^1)) + \dots \\ &\quad + G(F(x^n, n^1, \dots, x^{n-1}), F(y^n, y^1, \dots, y^{n-1}), F(z^n, z^1, \dots, z^{n-1}))) \cdot n^{-1}) \\ &\leq \phi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \dots + G(gx^n, gy^n, gz^n)}{n}\Big) \\ &\quad - \psi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \dots + G(gx^n, gy^n, gz^n)}{n}\Big), \end{aligned}$$

for all $(x^1, x^2, \dots, x^n), (y^1, y^2, \dots, y^n), (z^1, z^2, \dots, z^n) \in X$.

Definition 2.7. Mappings $F : X^n \to X$ and $g : X \to X$ are commutative if

$$g(F(x^1, x^2, x^3, \dots, x^n)) = F(gx^1, gx^2, gx^3, \dots, gx^n),$$

for all $(x^1, x^2, x^3, \dots, x^n) \in X$.

Definition 2.8. Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space. Then $(X, G, \preceq)$ is called regular if the following conditions hold:

(i) If a non-decreasing sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ then $x_n \preceq x, \ \forall n \in \mathbb{N}$.

(ii) If a non-increasing sequence $\{y_n\} \subseteq X$ such that $y_n \to y$ then $y \preceq y_n, \ \forall n \in \mathbb{N}$.

For the rest of the definitions and other notions utilized in our paper one can refer to Imdad [6] and Mustafa et al. [17].

3. Main results

Our main result runs as follows:

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space such that (X, G) is G-complete. Let $F : X^n \to X$ and $g : X \to X$ be two symmetric (ϕ, ψ)-contractive mappings on X, that is,*

$$\begin{aligned}
&\phi((G(F(x^1, x^2, \dots, x^n), F(y^1, y^2, \dots, y^n), F(z^1, z^2, \dots, z^n)) \\
&\quad + G(F(x^2, \dots, x^n, x^1), F(y^2, \dots, y^n, y^1), F(z^2, \dots, z^n, z^1)) + \dots \\
&\quad + G(F(x^n, x^1, \dots, x^{n-1}), F(y^n, y^1, \dots, y^{n-1}), F(z^n, z^1, \dots, z^{n-1}))) \cdot n^{-1}) \\
&\leq \phi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \dots + G(gx^n, gy^n, gz^n)}{n}\Big) \\
&\quad - \psi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \dots + G(gx^n, gy^n, gz^n)}{n}\Big),
\end{aligned} \tag{3.1}$$

with $gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gy^2 \preceq gz^2$, $gx^3 \succeq gy^3 \succeq gz^3$, $\dots$, $gx^n \preceq gy^n \preceq gz^n$, such that F has the mixed g-monotone property. Assume that $F(X^n) \subseteq g(X)$ and both the mappings $F$and g commutes and continuous. If there exist $x_0^1, x_0^2, x_0^3, \dots, x_0^n \in X$ such that

$$\begin{cases}
gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n), \\
F(x_0^2, x_0^3, \dots, x_0^n, \dots, x_0^1) \preceq gx_0^2, \\
\vdots \\
F(x_0^n, x_0^1, \dots, x_0^2, \dots, x_0^{n-1}) \preceq gx_0^n.
\end{cases}$$

Then F and g have an n-tupled coincidence point in X.

Proof. Let $x_0^1, x_0^2, x_0^3, \dots, x_0^n \in X$ such that

$$\begin{cases}
gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n) \\
F(x_0^2, x_0^3, \dots, x_0^n, \dots, x_0^1) \preceq gx_0^2, \\
\vdots \\
F(x_0^n, x_0^1, \dots, x_0^2, \dots, x_0^{n-1}) \preceq gx_0^n.
\end{cases} \tag{3.2}$$

Since it is given that, $F(X^n) \subseteq g(X)$, we can choose $x_1^1, x_1^2, \ldots, x_1^n \in X$ such that

$$\begin{cases} gx_1^1 = F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n), \\ gx_1^2 = F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1), \\ \vdots \\ gx_1^n = F(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}). \end{cases} \tag{3.3}$$

Again, since $F(X^n) \subseteq g(X)$, we can choose $x_2^1, x_2^2, \ldots, x_2^n \in X$ such that

$$\begin{cases} gx_2^1 = F(x_1^1, x_1^2, x_1^3, \ldots, x_1^n), \\ gx_2^2 = F(x_1^2, x_1^3, \ldots, x_1^n, x_1^1), \\ \vdots \\ gx_2^n = F(x_1^n, x_1^1, x_1^2, \ldots, x_1^{n-1}). \end{cases}$$

Continuing this process, we can construct n sequences $\{x_m^1\}$, $\{x_m^2\}, \ldots,$ $\{x_m^n\}$ $(m \geq 0)$ in X such that

$$\begin{cases} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n), \\ gx_{m+1}^2 = F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1), \\ \vdots \\ gx_{m+1}^n = F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}). \end{cases} \tag{3.4}$$

Now, by the mathematical induction method, we shall show that for all $m \geq 0$,

$$gx_m^1 \preceq gx_{m+1}^1, \;\; gx_{m+1}^2 \preceq gx_m^2, \;\; gx_m^3 \preceq gx_{m+1}^3, \;\; \ldots \;\; , \;\; gx_{m+1}^n \preceq gx_m^n. \tag{3.5}$$

From (3.2) and (3.3), we obtain, $gx_0^1 \preceq gx_1^1, gx_1^2 \preceq gx_0^2, gx_0^3 \preceq gx_1^3, \ldots, gx_1^n \preceq gx_0^n$, that is (3.5) holds for $m = 0$. Suppose that, (3.4) holds for some $m > 0$. From mixed g-monotone property of F and (3.4), we have

$$\begin{aligned} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n) &\preceq F(x_{m+1}^1, x_m^2, x_m^3, \ldots, x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_m^3, \ldots, x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, \ldots, x_m^n) \\ &\vdots \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, \ldots, x_{m+1}^n) \\ &= gx_{m+2}^1, \end{aligned}$$

$$
\begin{aligned}
gx^2_{m+2} = F(x^2_{m+1}, x^3_{m+1}, \ldots, x^n_{m+1}, x^1_{m+1}) &\preceq F(x^2_{m+1}, x^3_{m+1}, \ldots, x^n_{m+1}, x^1_m) \\
&\preceq F(x^2_{m+1}, x^3_{m+1}, \ldots, x^n_m, x^1_m) \\
&\vdots \\
&\preceq F(x^2_{m+1}, x^3_m, \ldots, x^n_m, x^1_m) \\
&\preceq F(x^2_m, x^3_m, \ldots, x^n_m, x^1_m) \\
&= gx^1_{m+1}.
\end{aligned}
$$

Continuing the above process, yields

$$
\begin{aligned}
gx^n_{m+2} = F(x^n_{m+1}, x^1_{m+1}, x^2_{m+1}, \ldots, x^{n-1}_{m+1}) &\preceq F(x^n_m, x^1_m, x^2_m, \ldots, x^{n-1}_m) \\
&= gx^n_{m+1}.
\end{aligned}
$$

Therefore, from induction method one can easily show that, inequality (3.5) holds, for all $m \geq 0$. Hence

$$
\begin{cases}
gx^1_0 \preceq gx^1_1 \preceq gx^1_2 \preceq \ldots \preceq gx^1_m \preceq gx^1_{m+1} \preceq \ldots, \\
\ldots \preceq gx^2_{m+1} \preceq gx^2_m \preceq \ldots \preceq gx^2_2 \preceq gx^2_1 \preceq gx^2_0, \\
gx^3_0 \preceq gx^3_1 \preceq gx^3_2 \preceq \ldots \preceq gx^3_m \preceq gx^3_{m+1} \preceq \ldots, \\
\qquad \vdots \\
\ldots \preceq gx^n_{m+1} \preceq gx^n_m \preceq \ldots \preceq gx^n_2 \preceq gx^n_1 \preceq gx^n_0.
\end{cases} \tag{3.6}
$$

From (3.5), we obtain

$$
\begin{aligned}
&\phi\Bigg(\Big(G(gx^1_{m+1}, gx^1_m, gx^1_m) + G(gx^2_{m+1}, gx^2_m, gx^2_m) + \cdots \\
&\qquad + G(gx^n_{m+1}, gx^n_m, gx^n_m)\Big) \cdot n^{-1}\Bigg) \\
&= \phi\Bigg(\Big(G\big(F(x^1_m, x^2_m, \ldots, x^n_m), F(x^1_{m-1}, x^2_{m-1}, \cdots, x^n_{m-1}), \\
&\qquad F(x^1_{m-1}, x^2_{m-1}, \ldots, x^n_{m-1})\big) + G\big(F(x^2_m, x^3_m, \ldots, x^n_m, x^1_m), \\
&\qquad F(x^2_{m-1}, x^3_{m-1}, \ldots, x^n_{m-1}, x^1_{m-1}), F(x^2_{m-1}, x^3_{m-1}, \ldots, x^n_{m-1}, x^1_{m-1})\big) \\
&\qquad + \ldots + G\big(F(x^n_m, x^1_m, \ldots, x^{n-1}_m), F(x^n_{m-1}, x^1_{m-1}, \ldots, x^{n-1}_{m-1}), \\
&\qquad F(x^n_{m-1}, x^1_{m-1}, \ldots, x^{n-1}_{m-1})\big)\Big) \cdot n^{-1}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\le \phi\Big(\frac{1}{n}\Big(G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}) + G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}) \\
&\qquad + G(gx^3_m, gx^3_{m-1}, gx^3_{m-1}) + \cdots + G(gx^n_m, gx^n_{m-1}, gx^n_{m-1})\Big)\Big) \\
&\quad - \psi\Big(\frac{1}{n}\Big(G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}) + G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}) \\
&\qquad\quad + G(gx^3_m, gx^3_{m-1}, gx^3_{m-1}) + \ldots + G(gx^n_m, gx^n_{m-1}, gx^n_{m-1})\Big)\Big).
\end{aligned}
$$

Let

$$
\begin{aligned}
R_m = \frac{1}{n}\Big(&G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}) + G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}) \\
&+ G(gx^3_m, gx^3_{m-1}, gx^3_{m-1}) + \ldots + G(gx^n_m, gx^n_{m-1}, gx^n_{m-1})\Big).
\end{aligned} \tag{3.7}
$$

From the above inequality, we arrive at

$$
\phi(R_m) \le \phi(R_{m-1}) - \psi(R_{m-1}). \tag{3.8}
$$

Using the fact that, ψ is non negative function, this follows

$$
\phi(R_m) \le \phi(R_{m-1}) \quad \Rightarrow \quad R_m \le R_{m-1}.
$$

Hence $\{R_m\}$ is a positive non increasing sequence. Hence there exists $r \ge 0$ such that $R_m \to r$ as $m \to \infty$. Letting the limit as $m \to \infty$ in (3.8). Then by the properties of ϕ and ψ, we get

$$
\begin{aligned}
\phi(r) = \lim_{m\to\infty} \phi(R_m) &\le \lim_{m\to\infty} \phi(R_{m-1}) - \psi(R_{m-1}) \\
&= \phi(r) - \lim_{R_{m-1}\to r} \psi(R_{m-1}) < \phi(r).
\end{aligned}
$$

Leads to a contradiction, thus we find $r = 0$. Hence

$$
R_m \to 0 \;\text{ as }\; m \to \infty. \tag{3.9}
$$

Next, to show that $\{gx^1_m\}, \{gx^2_m\}, \{gx^3_m\}, \ldots, \{gx^n_m\}$ are G-cauchy sequences. Assume that, at least one of $\{gx^1_m\}, \{gx^2_m\}, \{gx^3_m\}, \ldots, \{gx^n_m\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ for which we can find sequences of positive integer $\{m(k)\}$ and $\{m(k)\}$ such that for all positive integer $k, l(k) > m(k) \ge k$. Let

$$
\begin{aligned}
A_k = \frac{1}{n}\Bigg(&G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{l(k)}\big) + G\big(gx^2_{m(k)}, gx^2_{m(k)}, gx^2_{l(k)}\big) + \cdots \\
&+ G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{l(k)}\big)\Bigg) \ge \varepsilon.
\end{aligned} \tag{3.10}
$$

Further, corresponding to $m(k)$ we can choose $l(k)$ in such a way that it is the smallest integer with $l(k) > m(k)$ and satisfying (3.10). Then

$$\frac{1}{n}\Bigg(G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{l(k)-1}\big) + G\big(gx^2_{m(k)}, gx^2_{m(k)}, gx^2_{l(k)-1}\big) + \cdots + G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{l(k)-1}\big)\Bigg) < \varepsilon. \tag{3.11}$$

Now, using the rectangular inequality of G-metric space and (3.11), we get

$$\varepsilon \le A_k = \frac{G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{l(k)}\big), G\big(gx^2_{m(k)}, gx^2_{m(k)}, gx^2_{l(k)}\big), \cdots, G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{l(k)}\big)}{n}$$

$$\le \frac{1}{n}\Bigg(G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{l(k)-1}\big) + G\big(gx^2_{m(k)}, gx^2_{m(k)}, gx^2_{l(k)-1}\big) + \cdots + G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{l(k)-1}\big)\Bigg) + \frac{1}{n}\Bigg(G\big(gx^1_{l(k)-1}, gx^1_{l(k)-1}, gx^1_{l(k)}\big) + G\big(gx^2_{l(k)-1}, gx^2_{l(k)-1}, gx^2_{l(k)}\big) + \cdots + G\big(gx^n_{l(k)-1}, gx^n_{l(k)-1}, gx^n_{l(k)}\big)\Bigg).$$

Therefore from (3.7) and (3.11), we obtain

$$\varepsilon \le A_k = \frac{1}{n}\Bigg(G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{l(k)}\big) + G\big(gx^2_{m(k)}, gx^2_{m(k)}, gx^2_{l(k)}\big) + \cdots + G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{l(k)}\big)\Bigg)$$
$$\le \varepsilon + R_{l(k)-1}.$$

Letting the limit as $k \to \infty$ in the above inequality and using (3.9), we have

$$\lim_{k\to\infty} A_k = \lim_{k\to\infty} \frac{1}{n}\Bigg(G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{l(k)}\big) + G\big(gx^2_{m(k)}, gx^2_{m(k)}, gx^2_{l(k)}\big) + \cdots + G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{l(k)}\big)\Bigg) = \varepsilon. \tag{3.12}$$

Again,

$$A_k = \left(\frac{G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{l(k)}\big) + G\big(gx^2_{m(k)}, gx^2_{m(k)}, gx^2_{l(k)}\big) + \ldots + G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{l(k)}\big)}{n}\right)$$

$$\leq \frac{1}{n}\begin{pmatrix} G\big(gx^1_{m(k)}, gx^1_{m(k)}, gx^1_{m(k)+1}\big)+\ldots+G\big(gx^n_{m(k)}, gx^n_{m(k)}, gx^n_{m(k)+1}\big) \\ +G\big(gx^1_{m(k)+1}, gx^1_{m(k)+1}, gx^1_{l(k)+1}\big)+\ldots+G\big(gx^n_{m(k)+1}, gx^n_{m(k)+1}, gx^n_{l(k)+1}\big) \\ +G\big(gx^1_{l(k)+1}, gx^1_{l(k)+1}, gx^1_{l(k)}\big)+\ldots+G\big(gx^n_{l(k)+1}, gx^n_{l(k)+1}, gx^n_{l(k)}\big) \end{pmatrix}$$

$$\leq R_{m(k)} + \left(\frac{G\big(gx^1_{l(k)+1}, gx^1_{l(k)+1}, gx^1_{l(k)}\big)+\ldots+G\big(gx^n_{l(k)+1}, gx^n_{l(k)+1}, gx^n_{l(k)}\big)}{n}\right)$$

$$+\left(\frac{G\big(gx^1_{m(k)+1}, gx^1_{m(k)+1}, gx^1_{l(k)+1}\big)+\ldots+G\big(gx^n_{m(k)+1}, gx^n_{m(k)+1}, gx^n_{l(k)+1}\big)}{n}\right).$$

Using that, $G(x,x,y) \leq 2G(x,y,y)$ for any $x, y \in X$, we obtain

$$\begin{aligned} A_k \leq R_{m(k)} + 2R_{l(k)} + \tfrac{1}{n}\Big(&G\big(gx^1_{m(k)+1}, gx^1_{m(k)+1}, gx^1_{l(k)+1}\big) \\ &+G\big(gx^2_{m(k)+1}, gx^2_{m(k)+1}, gx^2_{l(k)+1}\big) + \cdots \\ &+G\big(gx^n_{m(k)+1}, gx^n_{m(k)+1}, gx^n_{l(k)+1}\big)\Big). \end{aligned} \tag{3.13}$$

Next, we shall show that

$$\begin{aligned} \phi\Big(\frac{1}{n}\Big(&G\big(gx^1_{m(k)+1}, gx^1_{m(k)+1}, gx^1_{l(k)+1}\big)+G\big(gx^2_{m(k)+1}, gx^2_{m(k)+1}, gx^2_{l(k)+1}\big) \\ &+ \cdots + G\big(gx^n_{m(k)+1}, gx^n_{m(k)+1}, gx^n_{l(k)+1}\big)\Big)\Big) \leq \phi(A_k) - \psi(A_k). \end{aligned} \tag{3.14}$$

As $l(k) > m(k)$ and $gx^1_{m(k)} \preceq gx^1_{l(k)}$, $gx^2_{l(k)} \preceq gx^2_{m(k)}, \ldots,$ $gx^n_{l(k)} \preceq gx^n_{m(k)}$. Hence from inequality (3.1) and (3.4), we obtain

$$\begin{aligned} &\phi\left(\frac{1}{n}\Big(G\big(gx^1_{m(k)+1}, gx^1_{m(k)+1}, gx^1_{l(k)+1}\big) + G\big(gx^2_{m(k)+1}, gx^2_{m(k)+1}, gx^2_{l(k)+1}\big)\right. \\ &\qquad\left. + \cdots + G\big(gx^n_{m(k)+1}, gx^n_{m(k)+1}, gx^n_{l(k)+1}\big)\Big)\right) \\ &= \phi\left(\frac{1}{n}\left(G\big(F(x^1_m(k),\ldots,x^n_m(k)), F(x^1_l(k),\ldots,x^n_l(k)), F(x^1_l(k),\ldots,x^n_l(k))\big)\right.\right. \\ &\quad + G\big(F(x^2_m(k),\ldots,x^1_m(k)), F(x^2_l(k),\ldots,x^1_l(k)), F(x^2_l(k),\ldots,x^1_l(k))\big) \\ &\quad + \cdots \\ &\quad \left.\left.+G\big(F(x^n_m(k),\ldots,x^{n-1}_m(k)),F(x^n_l(k),\ldots,x^{n-1}_l(k)),F(x^n_l(k),\ldots,x^{n-1}_l(k))\big)\right)\right) \end{aligned}$$

$$\le \phi\Bigg(\frac{1}{n}\Big(G\big(gx^1_{m(k)}, gx^1_{l(k)}gx^1_{l(k)}\big) + G\big(gx^2_{m(k)}, gx^2_{l(k)}gx^2_{l(k)}\big) + \cdots + G\big(gx^n_{m(k)}, gx^n_{l(k)}gx^n_{l(k)}\big)\Big)\Bigg) - \psi\Bigg(\frac{1}{n}\Big(G\big(gx^1_{m(k)}, gx^1_{l(k)}gx^1_{l(k)}\big) + G\big(gx^2_{m(k)}, gx^2_{l(k)}gx^2_{l(k)}\big) + \cdots + G\big(gx^n_{m(k)}, gx^n_{l(k)}gx^n_{l(k)}\big)\Big)\Bigg).$$

This gives

$$\phi\Bigg(\tfrac{1}{n}\Big(G\big(gx^1_{m(k)+1}, gx^1_{m(k)+1}, gx^1_{l(k)+1}\big) + G\big(gx^2_{m(k)+1}, gx^2_{m(k)+1}, gx^2_{l(k)+1}\big) + \cdots + G\big(gx^n_{m(k)+1}, gx^n_{m(k)+1}, gx^n_{l(k)+1}\big)\Big)\Bigg) \le \phi(A_k) - \psi(A_k). \tag{3.15}$$

Hence, (3.14) holds, for each n and for all m. Therefore by the property of ϕ, (3.13) and (3.15) yields

$$\phi(A_k) \le \phi(R_{m(k)}) + 2\phi(R_{l(k)}) + \phi(A_k) - \psi(A_k).$$

Taking The limit as $k \to \infty$ in the above inequality, using (3.9), (3.12) and continuity of ϕ, we obtain

$$\phi(\varepsilon) = \lim_{k\to\infty} \phi(A_k) \le \phi(\varepsilon) - \lim_{k\to\infty} \psi(A_k) = \phi(\varepsilon) - \lim_{A_k\to\epsilon^+} \psi(A_k) < \phi(\varepsilon).$$

Which is a contradiction. Then we conclude that $\{gx^1_m\}$, $\{gx^2_m\}$, $\{gx^3_m\}, \dots,$ $\{gx^n_m\}$ are G-Cauchy sequences in the G-metric space (X, G) which is G-complete. Then there exist $x^1, x^2, \dots, x^n \in X$ such that

$$\begin{cases} \lim\limits_{m\to\infty} G(gx^1_m, gx^1_m, x^1) = \lim\limits_{m\to\infty} G(gx^1_m, x^1, x^1) = 0, \\ \lim\limits_{m\to\infty} G(gx^2_m, gx^2_m, x^2) = \lim\limits_{m\to\infty} G(gx^2_m, x^2, x^2) = 0, \\ \qquad\vdots \\ \lim\limits_{m\to\infty} G(gx^n_m, gx^n_m, x^n) = \lim\limits_{m\to\infty} G(gx^n_m, x^n, x^n) = 0. \end{cases} \tag{3.16}$$

Since, the mapping g is continuous, so from (3.16) we obtain

$$\begin{cases} \lim_{m\to\infty} G(g(gx_m^1), g(gx_m^1), gx^1) = \lim_{m\to\infty} G(g(gx_m^1), gx^1, gx^1) = 0, \\ \lim_{m\to\infty} G(g(gx_m^2), g(gx_m^2), gx^2) = \lim_{m\to\infty} G(g(gx_m^2), gx^2, gx^2) = 0, \\ \vdots \\ \lim_{m\to\infty} G(g(gx_m^n), g(gx_m^n), gx^n) = \lim_{m\to\infty} G(g(gx_m^n), gx^n, gx^n) = 0. \end{cases} \tag{3.17}$$

Hence, $g(gx_m^1)$ is convergent to $gx^1, g(gx_m^2)$ is convergent to $gx^2, \ldots, g(gx_m^n)$ is convergent to gx^n. Since,

$$\begin{cases} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n), \\ gx_{m+1}^2 = F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1), \\ gx_{m+1}^3 = F(x_m^3, \ldots, x_m^n, x_m^1, x_m^2), \\ \vdots \\ gx_{m+1}^n = F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}). \end{cases}$$

Thus, by the commutativity of F and g, we get

$$\begin{cases} g(gx_{m+1}^1) = g(F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n)) \\ \qquad = F(gx_m^1, gx_m^2, gx_m^3, \ldots, gx_m^n), \\ g(gx_{m+1}^2) = g(F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1)) \\ \qquad = F(gx_m^2, gx_m^3, \ldots, gx_m^n, gx_m^1), \\ \vdots \\ g(gx_{m+1}^n) = g(F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1})) \\ \qquad = F(gx_m^n, gx_m^1, gx_m^2, \ldots, gx_m^{n-1}). \end{cases} \tag{3.18}$$

Since the mapping F is continuous. Therefore taking the limit as $m \to \infty$ in (3.18) and from (3.16)-(3.17), we obtain

$$gx^1 = \lim_{m\to\infty} g(gx_{m+1}^1) = \lim_{m\to\infty} F(gx_m^1, gx_m^2, gx_m^3, \ldots, gx_m^n)$$
$$\Rightarrow gx^1 = F(x^1, x^2, x^3, \ldots, x^n).$$

Continuing in this way, we get

$$gx^2 = F(x^2, x^3, \ldots, x^n, x^1),$$
$$\vdots$$
$$gx^n = F(x^n, x^1, x^2, \ldots, x^{n-1}).$$

Hence the element $(x^1, x^2, x^3, \ldots, x^n) \in X^n$ is an n-tupled coincidence point of the mapping $F : X^n \to X$ and $g : X \to X$. $\square$

Next, the assumption the continuity of function F, along with the commutativity of mappings F and g are dropped and result is proved for $(X, G, \preceq)$ being regular.

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and there exists a G-metric space G on X. Let $F : X^n \to X$ and $g : X \to X$ be two symmetric (ϕ, ψ)-contractive mappings on X, that is*

$$\begin{aligned}
&\phi((G(F(x^1, x^2, \dots, x^n), F(y^1, y^2, \dots, y^n), F(z^1, z^2, \dots, z^n)) \\
&\qquad + G(F(x^2, \dots, x^n, x^1), F(y^2, \dots, y^n, y^1), F(z^2, \dots, z^n, z^1)) + \dots \\
&\qquad + G(F(x^n, x^1, \dots, x^{n-1}), F(y^n, y^1, \dots, y^{n-1}), F(z^n, z^1, \dots, z^{n-1})) \cdot n^{-1}) \\
&\ \le \phi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \dots + G(gx^n, gy^n, gz^n)}{n}\Big) \\
&\quad - \psi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \dots + G(gx^n, gy^n, gz^n)}{n}\Big),
\end{aligned}$$

with $gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gy^2 \preceq gz^2$, $gx^3 \succeq gy^3 \succeq gz^3, \dots,$ $gx^n \preceq gy^n \preceq gz^n$ such that F has the mixed g-monotone property. Suppose that $(X, G, \preceq)$ is regular. Assume that $F(X^n) \subseteq g(X)$ and $(g(X), G)$ is G-complete. If there exist $x_0^1, x_0^2, x_0^3, \dots, x_0^n \in X$ such that

$$\begin{cases}
gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n), \\
F(x_0^2, x_0^3, \dots, x_0^n, x_0^1) \preceq gx_0^2, \\
\vdots \\
F(x_0^n, x_0^1, x_0^2, \dots, x_0^{n-1}) \preceq gx_0^n.
\end{cases}$$

Then F and g have an n-tupled coincidence point in X.

Proof. Proceeding exactly as in Theorem 3.1 we have that $\{gx_m^1\}$, $\{gx_m^2\}, \dots,$ $\{gx_m^n\}$ are Cauchy sequences in the complete G-metric space. Then there exist $x^1, x^2, \dots, x^n \in X$ such that (gx_m^1) is convergent to gx^1, (gx_m^2) is convergent to $gx^2, \dots, (gx_m^n)$ is convergent to gx^n. That is,

$$\begin{cases}
\lim_{m\to\infty} G(gx_m^1, gx_m^1, gx^1) = \lim_{m\to\infty} G(gx_m^1, gx^1, gx^1) = 0, \\
\lim_{m\to\infty} G(gx_m^2, gx_m^2, gx^2) = \lim_{m\to\infty} G(gx_m^2, gx^2, gx^2) = 0, \\
\vdots \\
\lim_{m\to\infty} G(gx_m^n, gx_m^n, gx^n) = \lim_{m\to\infty} G(gx_m^n, gx^n, gx^n) = 0.
\end{cases} \tag{3.19}$$

Since, $\{gx_m^1\}, \{gx_m^3\}, \dots, \{gx_m^{n-1}\}$ are non decreasing and $\{gx_m^2\}$, $\{gx_m^4\}, \dots,$ $\{gx_m^n\}$ are non increasing. Using the regularity of $(X, G, \preceq)$, we obtain that

$$\begin{cases} gx_m^1 \preceq gx^1, \quad gx_m^2 \succeq gx^2, \\ gx_m^3 \preceq gx^3, \quad gx_m^4 \succeq gx^4, \\ \vdots \\ gx_m^{n-1} \preceq gx^{n-1}, \quad gx_m^n \succeq gx^n. \end{cases} \tag{3.20}$$

Now, using the rectangular inequality of G-metric space and (3.1), we get

$$\begin{aligned}
&\phi\Big(\Big(G(F(x^1,x^2,x^3,\dots,x^n),gx^1_{m+1},gx^1_{m+1})\\
&\quad + G(F(x^2,x^3,\dots,x^n,x^1),gx^2_{m+1},gx^2_{m+1}) + \dots\\
&\quad + G(F(x^n,x^1,x^2,\dots,x^{n-1}),gx^n_{m+1},gx^n_{m+1})\Big)\cdot n^{-1}\Big)\\
&= \phi\Big(\Big(G(F(x^1,x^2,x^3,\dots,x^n),F(x^1_m,x^2_m,x^3_m,\dots,x^n_m),F(x^1_m,x^2_m,x^3_m,\dots,x^n_m))\\
&\quad + G(F(x^2,x^3,\dots,x^n,x^1),F(x^2_m,x^3_m,\dots,x^n_m,x^1_m),F(x^2_m,x^3_m,\dots,x^n_m,x^1_m))\\
&\quad + \dots + G(F(x^n,x^1,x^2,\dots,x^{n-1}),F(x^n_m,x^1_m,x^2_m,\dots,x^{n-1}_m),\\
&\qquad F(x^n_m,x^1_m,x^2_m,\dots,x^{n-1}_m))\Big)\cdot n^{-1}\Big)\\
&\le \phi\Big(\frac{1}{n}\Big(G(gx^1,gx^1_m,gx^1_m)+G(gx^2,gx^2_m,gx^2_m)+G(gx^3,gx^3_m,gx^3_m)\\
&\quad + \dots + G(gx^n,gx^n_m,gx^n_m)\Big)\Big)\\
&\quad - \psi\Big(\frac{1}{n}\Big(G(gx^1,gx^1_m,gx^1_m)+G(gx^2,gx^2_m,gx^2_m)+G(gx^3,gx^3_m,gx^3_m)\\
&\quad + \dots + G(gx^n,gx^n_m,gx^n_m)\Big)\Big).
\end{aligned}$$

Taking the limit as $m \to \infty$, in above inequality and by the definition of ϕ and ψ and from (3.19) we obtain

$$\begin{aligned}
&\phi\Big(\lim_{m\to\infty}\Big((G(F(x^1,x^2,x^3,\dots,x^n),gx^1_{m+1},gx^1_{m+1})\\
&\qquad + G(F(x^2,x^3,\dots,x^n,x^1),gx^2_{m+1},gx^2_{m+1})\\
&\qquad + \dots + G(F(x^n,x^1,x^2,\dots,x^{n-1}),gx^n_{m+1},gx^n_{m+1})\cdot n^{-1}\Big)\Big)\\
&\le \phi\Big(\lim_{m\to\infty}\Big(\frac{G(gx^1,gx^1_m,gx^1_m)+G(gx^2,gx^2_m,gx^2_m)+\dots+G(gx^n,gx^n_m,gx^n_m)}{n}\Big)\Big)\\
&\quad - \psi\Big(\lim_{m\to\infty}\Big(\frac{G(gx^1,gx^1_m,gx^1_m)+G(gx^2,gx^2_m,gx^2_m)+\dots+G(gx^n,gx^n_m,gx^n_m)}{n}\Big)\Big).
\end{aligned}$$

This gives,

$$\lim_{m\to\infty}\Big(\Big(G(F(x^1,x^2,x^3,\ldots,x^n),gx^1_{m+1},gx^1_{m+1}) + G(F(x^2,x^3,\ldots,x^n,x^1),gx^2_{m+1},gx^2_{m+1}) + \cdots + G(F(x^n,x^1,x^2,\ldots,x^{n-1}),gx^n_{m+1},gx^n_{m+1})\Big)\cdot n^{-1}\Big) = 0.$$

Hence, one can easily acquire

$$\begin{cases} \lim\limits_{m\to\infty} G(F(x^1,x^2,x^3,\ldots,x^n),gx^1_{m+1},gx^1_{m+1}) = 0,\\ \lim\limits_{m\to\infty} G(F(x^2,x^3,\ldots,x^n,x^1),gx^2_{m+1},gx^2_{m+1}) = 0,\\ \vdots \\ \lim\limits_{m\to\infty} G(F(x^n,x^1,x^2,\ldots,x^{n-1}),gx^n_{m+1},gx^n_{m+1}) = 0. \end{cases} \tag{3.21}$$

On the other hand, from triangle inequality, we get

$$\begin{aligned} &G(F(x^1,x^2,\ldots,x^n),gx^1,gx^1) + G(F(x^2,x^3,\ldots,x^1),gx^2,gx^2) + \cdots \\ &+ G(F(x^n,x^1,\ldots,x^{n-1}),gx^n,gx^n) \\ &\le G(F(x^1,x^2,\ldots,x^n),gx^1_{m+1},gx^1_{m+1}) + G(F(x^2,x^3,\ldots,x^1),gx^2_{m+1},gx^2_{m+1}) \\ &\quad + \ldots + G(F(x^n,x^1,\ldots,x^{n-1}),gx^n_{m+1},gx^n_{m+1}) + G(gx^1_{m+1},gx^1,gx^1) \\ &\quad + G(gx^2_{m+1},gx^2,gx^2) + \ldots + G(gx^n_{m+1},gx^n,gx^n). \end{aligned}$$

Taking the limit as $m\to\infty$ and on using (3.18)-(3.20), we arrive at

$$\begin{aligned} &G(F(x^1,x^2,x^3,\ldots,x^n),gx^1,gx^1) + G(F(x^2,x^3,\ldots,x^n,x^1),gx^2,gx^2) \\ &+ \ldots + G(F(x^1,x^2,x^3,\ldots,x^{n-1}),gx^n,gx^n) = 0. \end{aligned}$$

That is,

$$\begin{cases} G(F(x^1,x^2,x^3,\ldots,x^n),gx^1,gx^1) = 0,\\ G(F(x^2,x^3,\ldots,x^n,x^1),gx^2,gx^2) = 0,\\ \cdots \\ G(F(x^1,x^2,x^3,\ldots,x^{n-1}),gx^n,gx^n) = 0. \end{cases}$$

That gives,

$$\begin{gathered} F(x^1,x^2,x^3,\ldots,x^n) = gx^1,\\ F(x^2,x^3,\ldots,x^n,x^1) = gx^2,\\ F(x^3,\ldots,x^n,x^1,x^2) = gx^3.\\ \vdots \\ F(x^1,x^2,x^3,\ldots,x^{n-1}) = gx^n. \end{gathered}$$

Hence the element$(x^1,x^2,x^3,\ldots,x^n)\in X^n$ is an n-tupled coincidence point of the mappings $F: X^n\to X$ and $g: X\to X$ and this makes end to the proof. □

Remark 3.3. Restricting 'n' to 2, Theorem 3.1 and Theorem 3.2 reduce to Theorem 18 and Theorem 20 of Jain and Tas [7]. Thus our theorem is a proper generalization of results [7]. Again in light of Examples 21 and 23 given in [7], we assert that for $n = 2$, Theorem 3.1 is a generalizes the main results of Choudhary and Maity [5], Mohiuddin et al. [15] and Nassine [18] as our contractive condition is more general than that of [5], [15] and [18].

Remark 3.4. Again restricting 'n' to 2, and choosing $\phi, \psi : [0, \infty) \to [0, \infty)$, such that $\phi = \frac{t}{2}$ and $\psi(t) = \frac{(1-k)t}{2}$, $0 \leq k \leq 1$, (3.1) reduces to Corollary 2.5 of Karpinar et al. [8].

Next, in view of concept given in Examples 2.1 and 2.3 of [8] and Examples 21, 23 of [7], we present an example which illustrates the weakness of Theorem 2.1 of Mustafa [16] and shows that Theorem 3.1 is more general than Theorem 2.1 [16] since the contractive condition (3.1) is more general than the Condition (2.1) of Mustafa [16].

Example 3.5. Let $X = \Re$ with usual ordering. Define $G : X^3 \to X$ by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Let $g : X \to X$ and $F : X^4 \to X$ be defined by, $gx = \frac{x}{2}$ for all $x \in X$ and for all $(x^1, x^2, \ldots, x^n) \in X$

$$F(x^1, x^2, x^3, x^4) = \frac{x^1 - 2x^2 + x^3 - 2x^4}{16}.$$

Then

(a) (X, G) is complete ordered G-metric space,
(b) F and g have the mixed g-monotone property,
(c) (F, g) is commutative,
(d) $F(X^4) \subseteq g(X)$.

Now, all the conditions of Theorem 2.1 of [16] are satisfied except the contractive condition (2.1). Suppose, to the contrary that there exist functions ϕ and ψ(define as in [16]) such that contractive condition of Theorem 2.1 of [16] holds. Then, we must have

$$\begin{aligned}
&\phi\Big(G\Big(\frac{x^1 - 2x^2 + x^3 - 2x^4}{16}, \frac{y^1 - 2y^2 + y^3 - 2y^4}{16}, \frac{z^1 - 2z^2 + z^3 - 2z^4}{16}\Big)\Big) \\
&\leq \frac{1}{4}\phi\Big(G\Big(\frac{x^1}{2}, \frac{y^1}{2}, \frac{z^1}{2}\Big) + G\Big(\frac{x^2}{2}, \frac{y^2}{2}, \frac{z^2}{2}\Big) + G\Big(\frac{x^3}{2}, \frac{y^3}{2}, \frac{z^3}{2}\Big) + G\Big(\frac{x^4}{2}, \frac{y^4}{2}, \frac{z^4}{2}\Big)\Big) \\
&\quad - \psi\left(\frac{G\Big(\frac{x^1}{2}, \frac{y^1}{2}, \frac{z^1}{2}\Big) + G\Big(\frac{x^2}{2}, \frac{y^2}{2}, \frac{z^2}{2}\Big) + G\Big(\frac{x^3}{2}, \frac{y^3}{2}, \frac{z^3}{2}\Big) + G\Big(\frac{x^4}{2}, \frac{y^4}{2}, \frac{z^4}{2}\Big)}{4}\right).
\end{aligned}$$

Now, by the definition of $G(x, y, z)$, we get

$$\begin{aligned}
&\phi\Big(\Big|\frac{(x^1-2x^2+x^3-2x^4)-(y^1-2y^2+y^3-2y^4)}{16}\Big| \\
&\quad+\Big|\frac{(y^1-2y^2+y^3-2y^4)-(z^1-2z^2+z^3-2z^4)}{16}\Big| \\
&\quad+\Big|\frac{(z^1-2z^2+z^3-2z^4)-(x^1-2x^2+x^3-2x^4)}{16}\Big|\Big) \\
&\le \frac{1}{4}\phi\Big(\Big|\frac{x^1}{2}-\frac{y^1}{2}\Big|+\Big|\frac{y^1}{2}-\frac{z^1}{2}\Big|+\Big|\frac{z^1}{2}-\frac{x^1}{2}\Big|+\Big|\frac{x^2}{2}-\frac{y^2}{2}\Big| \\
&\qquad+\Big|\frac{y^2}{2}-\frac{z^2}{2}\Big|+\Big|\frac{z^2}{2}-\frac{x^2}{2}\Big|+\Big|\frac{x^3}{2}-\frac{y^3}{2}\Big|+\Big|\frac{y^3}{2}-\frac{z^3}{2}\Big| \\
&\qquad+\Big|\frac{z^3}{2}-\frac{x^3}{2}\Big|+\Big|\frac{x^4}{2}-\frac{y^4}{2}\Big|+\Big|\frac{y^4}{2}-\frac{z^4}{2}\Big|+\Big|\frac{z^4}{2}-\frac{x^4}{2}\Big|\Big) \\
&\quad-\psi\Bigg(\Big(\Big|\frac{x^1}{2}-\frac{y^1}{2}\Big|+\Big|\frac{y^1}{2}-\frac{z^1}{2}\Big|+\Big|\frac{z^1}{2}-\frac{x^1}{2}\Big|+\Big|\frac{x^2}{2}-\frac{y^2}{2}\Big| \\
&\qquad+\Big|\frac{y^2}{2}-\frac{z^2}{2}\Big|+\Big|\frac{z^2}{2}-\frac{x^2}{2}\Big|+\Big|\frac{x^3}{2}-\frac{y^3}{2}\Big|+\Big|\frac{y^3}{2}-\frac{z^3}{2}\Big| \\
&\qquad+\Big|\frac{z^3}{2}-\frac{x^3}{2}\Big|+\Big|\frac{x^4}{2}-\frac{y^4}{2}\Big|+\Big|\frac{y^4}{2}-\frac{z^4}{2}\Big|+\Big|\frac{z^4}{2}-\frac{x^4}{2}\Big|\Big)\cdot 4^{-1}\Bigg).
\end{aligned}$$

Taking $x^1 = y^1 = z^1, x^3 = y^3 = z^3$ and $x^4 = y^4 = z^4$, we get

$$\begin{aligned}
&\phi\Big(\Big|\frac{x^2-y^2}{8}\Big|+\Big|\frac{y^2-z^2}{8}\Big|+\Big|\frac{z^2-x^2}{8}\Big|\Big) \\
&\le \frac{1}{4}\phi\Big(\Big|\frac{x^2}{2}-\frac{y^2}{2}\Big|+\Big|\frac{y^2}{2}-\frac{z^2}{2}\Big|+\Big|\frac{z^2}{2}-\frac{x^2}{2}\Big|\Big) \\
&\quad-\psi\Big(\Big(\Big|\frac{x^2}{2}-\frac{y^2}{2}\Big|+\Big|\frac{y^2}{2}-\frac{z^2}{2}\Big|+\Big|\frac{z^2}{2}-\frac{x^2}{2}\Big|\Big)\cdot 4^{-1}\Big).
\end{aligned}$$

Taking $\frac{\left|\frac{x^2}{2}-\frac{y^2}{2}\right|+\left|\frac{y^2}{2}-\frac{z^2}{2}\right|+\left|\frac{z^2}{2}-\frac{x^2}{2}\right|}{4} = k$, we acquire $\phi(k) \le \frac{1}{4}\phi(4k) - \psi(k)$. Since, function ϕ satisfies the sub additive property than we have $\frac{1}{4}\phi(4k) \le \phi(k)$. Thus, we obtain $\psi(k) \le 0$, for all $k > 0$; i.e., $\psi(k) = 0$, which contradicts the definition of ψ. This provides that, function F does not satisfy the contractive condition (2.1) of Mustafa [16].

Let us reconsider the same example, we show that the contractive condition (3.1) of Theorem 3.1 is satisfied for the above functions i.e., the following segment demonstrates that Theorem 3.1 is an extension of Theorem 2.1 [16].

Taking first four terms i.e., set $n = 4$ in the contractive condition (3.1) of Theorem 3.1.

For (x^1, x^2, x^3, x^4), (y^1, y^2, y^3, y^4), $(z^1, z^2, z^3, z^4) \in X$ with $gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gy^2 \preceq gz^2$, $gx^3 \succeq gy^3 \succeq gz^3$, $gx^4 \preceq gy^4 \preceq gz^4$, we get the following inequality

$$\begin{aligned}
&\phi((G(F(x^1, x^2, x^3, x^4), F(y^1, y^2, y^3, y^4)F(z^1, z^2, z^3, z^4)) \\
&\qquad + G(F(x^2, x^3, x^4, x^1), F(y^2, y^3, y^4, y^1)F(z^2, z^3, z^4, z^1)) \\
&\qquad + G(F(x^3, x^4, x^1, x^2), F(y^3, y^4, y^1, y^2)F(z^3, z^4, z^1, z^2)) \\
&\qquad + G(F(x^4, x^1, x^2, x^3), F(y^4, y^1, y^2, y^3)F(z^4, z^1, z^2, z^3))) \cdot 4^{-1}) \\
&\leq \phi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + G(gx^3, gy^3, gz^3)}{4} \\
&\qquad + \frac{G(gx^4, gy^4, gz^4)}{4}\Big) - \psi\Big(\frac{G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2)}{4} \\
&\qquad + \frac{G(gx^3, gy^3, gz^3) + G(gx^4, gy^4, gz^4)}{4}\Big).
\end{aligned} \tag{3.22}$$

Next, we shall show that (3.22) holds for the above example. Then, for $gx^1 \geq gy^1 \geq gz^1$, $gx^2 \leq gy^2 \leq gz^2$, $gx^3 \geq gy^3 \geq gz^3$, $gx^4 \leq gy^4 \leq gz^4$, we have

$$\begin{aligned}
&G(F(x^1, x^2, x^3, x^4), F(y^1, y^2, y^3, y^4)F(z^1, z^2, z^3, z^4)) \\
&\leq \frac{1}{16}(|x^1 - y^1| + |y^1 - z^1| + |z^1 - x^1| + |x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|) \\
&\quad \frac{1}{8}(|x^2 - y^2| + |y^2 - z^2| + |z^2 - x^2| + |x^4 - y^4| + |y^4 - z^4| + |z^4 - x^4|).
\end{aligned} \tag{3.23}$$

Similarly,

$$\begin{aligned}
&G(F(x^2, x^3, x^4, x^1), F(y^2, y^3, y^4, y^1)F(z^2, z^3, z^4, z^1)) \\
&\leq \frac{1}{8}(|x^1 - y^1| + |y^1 - z^1| + |z^1 - x^1| + |x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|) \\
&\quad \frac{1}{16}(|x^2 - y^2| + |y^2 - z^2| + |z^2 - x^2| + |x^4 - y^4| + |y^4 - z^4| + |z^4 - x^4|).
\end{aligned} \tag{3.24}$$

Inductively, one can easily show that

$$\begin{aligned}
&G(F(x^3, x^4, x^1, x^2), F(y^3, y^4, y^1, y^2)F(z^3, z^4, z^1, z^2)) \\
&\leq \frac{1}{16}(|x^1 - y^1| + |y^1 - z^1| + |z^1 - x^1| + |x^3 - y^3| + |y^3 - z^3| + |z^3 - x^3|) \\
&\quad \frac{1}{8}(|x^2 - y^2| + |y^2 - z^2| + |z^2 - x^2| + |x^4 - y^4| + |y^4 - z^4| + |z^4 - x^4|).
\end{aligned} \tag{3.25}$$

And

$$\begin{aligned} &G(F(x^4,x^1,x^2,x^3),F(y^4,y^1,y^2,y^3)F(z^4,z^1,z^2,z^3)) \\ &\leq \frac{1}{8}(|x^1-y^1|+|y^1-z^1|+|z^1-x^1|+|x^3-y^3|+|y^3-z^3|+|z^3-x^3|) \\ &\quad \frac{1}{16}(|x^2-y^2|+|y^2-z^2|+|z^2-x^2|+|x^4-y^4|+|y^4-z^4|+|z^4-x^4|). \end{aligned} \tag{3.26}$$

On using the inequalities (3.23)-(3.26), we get

$$\begin{aligned} &G(F(x^1,x^2,x^3,x^4),F(y^1,y^2,y^3,y^4)F(z^1,z^2,z^3,z^4)) \\ &\quad + G(F(x^2,x^3,x^4,x^1),F(y^2,y^3,y^4,y^1)F(z^2,z^3,z^4,z^1)) \\ &\quad + G(F(x^3,x^4,x^1,x^2),F(y^3,y^4,y^1,y^2)F(z^3,z^4,z^1,z^2)) \\ &\quad + G(F(x^4,x^1,x^2,x^3),F(y^4,y^1,y^2,y^3)F(z^4,z^1,z^2,z^3)) \\ &\leq \frac{3}{4}(G(gx^1,gy^1,gz^1)+G(gx^2,gy^2,gz^2)+G(gx^3,gy^3,gz^3)+G(gx^4,gy^4,gz^4)). \end{aligned}$$

Clearly, inequality (3.1) holds with $\phi(t)=\frac{1}{2}t$ and $\psi(t)=\frac{1}{8}t$. Hence, all the conditions of Theorem 3.1 are satisfied and $(0,0,0,\dots,0)$ is an n-tupled coincidence point of F and g.

Next result involves existence and uniqueness of n-tupled common fixed point.

Theorem 3.6. *In addition to the hypotheses of Theorem* 3.1*, suppose that for every* $(x^1,x^2,x^3,\dots,x^n)$, $(y^1,y^2,y^3,\dots,y^n)\in X^n$ *there exists* $(z^1,z^2,z^3,\dots,z^n)\in X^n$ *such that* $(F(z^1,z^2,\dots,z^n),\dots,F(z^n,z^1,\dots,z^{n-1}))\in X^n$ *is comparable with* $(F(x^1,x^2,\dots,x^n),\dots,F(x^n,x^1,\dots,x^{n-1}))$ *and* $(F(y^1,y^2,\dots,y^n),\dots,F(y^n,y^1,\dots,y^{n-1}))$. *Then F and g have a unique n-tupled common fixed point.*

Proof. From Theorem 3.1 the set of n-tupled coincidence points of F and g is non empty. Suppose that $(x^1,x^2,x^3,\dots,x^n),(y^1,y^2,y^3,\dots,y^n)$ are two n-tupled coincidence points, that is

$$\begin{cases} F(x^1,x^2,x^3,\dots,x^n)=gx^1; & F(y^1,y^2,y^3,\dots,y^n)=gy^1, \\ F(x^2,x^3,\dots,x^n,x^1)=gx^2; & F(y^2,y^3,\dots,y^n,y^1)=gy^2, \\ & \vdots \\ F(x^n,x^1,x^2,\dots,x^{n-1})=gx^n; & F(y^n,y^1,y^2,\dots,y^{n-1})=gy^n. \end{cases}$$

Now, we shall show that $gx^1=gy^1,gx^2=gy^2,\dots,gx^n=gy^n$. By assumption, there exists $(z^1,z^2,z^3,\dots,z^n)\in X^n$ such that $(F(z^1,z^2,z^3,\dots,z^n),F(z^2,z^3,\dots,z^n,z^1),\dots,F(z^n,z^1,z^2,\dots,z^{n-1}))$ is comparable with $(F(x^1,x^2,x^3,\dots,x^n),F(x^2,x^3,\dots,x^n,x^1),\dots,F(x^n,x^1,x^2,\dots,x^{n-1}))$ and $(F(y^1,y^2,y^3,$

$\ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(y^n, y^1, y^2, \ldots, y^{n-1}))$. Set, $z_0^1 = z^1, z_0^2 = z^2, \ldots, z_0^n = z^n$ and take $z^1, z^2, z^3, \ldots, z^n \in X$ such that

$$\begin{cases} gz_1^1 = F(z_0^1, z_0^2, z_0^3, \ldots, z_0^n), \\ gz_1^2 = F(z_0^2, z_0^3, \ldots, z_0^n, z_0^1), \\ \vdots \\ gz_1^n = F(z_0^n, z_0^1, z_0^2, \ldots, z_0^{n-1}). \end{cases}$$

Then similarly, as in the proof of Theorem 3.1 one can inductively define sequences $\{gz_m^1\}, \{gz_m^2\}, \{gz_m^3\}, \ldots, \{gz_m^n\}$ in X such that

$$\begin{cases} gz_{m+1}^1 = F(z_m^1, z_m^2, z_m^3, \ldots, z_m^n), \\ gz_{m+1}^2 = F(z_m^2, z_m^3, \ldots, z_m^n, z_m^1), \\ \vdots \\ gz_{m+1}^n = F(z_m^n, z_m^1, z_m^2, \ldots, z_m^{n-1}). \end{cases}$$

Further, set $x_0^1 = x^1, x_0^2 = x^2, \ldots, x_0^n = x^n$ and $y_0^1 = y^1, y_0^2 = y^2, \ldots, y_0^n = y^n$ and on the same way define the sequences $\{gx_m^1\}, \{gx_m^2\}, \{gx_m^3\}, \ldots, \{gx_m^n\}$ and $\{gy_m^1\}, \{gy_m^2\}, \{gy_m^3\}, \ldots, \{gy_m^n\}$. Then we can easily show that

$$\begin{cases} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n); \quad gy_{m+1}^1 = F(y_m^1, y_m^2, y_m^3, \ldots, y_m^n), \\ gx_{m+1}^2 = F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1); \quad gy_{m+1}^2 = F(y_m^2, y_m^3, \ldots, y_m^n, y_m^1), \\ \vdots \\ gx_{m+1}^n = F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}); \quad gy_{m+1}^n = F(y_m^n, y_m^1, y_m^2, \ldots, y_m^{n-1}). \end{cases}$$

Since, it is given that

$$\begin{aligned} &(F(x^1, x^2, x^3, \ldots, x^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})) \\ &= (gx_1^1, gx_1^2, \ldots, gx_1^n) = (gx^1, gx^2, \ldots, gx^n) \end{aligned}$$

and

$$\begin{aligned} &(F(z^1, z^2, z^3, \ldots, z^n), F(z^2, z^3, \ldots, z^n, z^1), \ldots, F(z^n, z^1, z^2, \ldots, z^{n-1})) \\ &= (gz_1^1, gz_1^2, \ldots, gz_1^n) \end{aligned}$$

are comparable, then $gx^1 \preceq gz_1^1$, $gz_1^2 \preceq gx^2$, $gx^3 \preceq gz_1^3, \ldots,$ $gz_1^n \preceq gx^n$.

In a similar way, we can show that for all $m \geq 1$,

$$gx^1 \preceq gz_m^1, gz_m^2 \preceq gx^2, gx^3 \preceq gz_m^3, \ldots, gz_m^n \preceq gx^n.$$

Therefore, the inequality (3.1) yields

$$
\begin{aligned}
&\phi\Big(\tfrac{G(gx^1,gx^1,gz^1_{m+1})+G(gx^2,gx^2,gz^2_{m+1})+G(gx^3,gx^3,gz^3_{m+1})+\ldots+G(gx^n,gx^n,gz^n_{m+1})}{n}\Big)\\
&=\phi\Bigg(\frac{1}{n}\Bigg(G(F(x^1,x^2,\ldots,x^n),F(x^1,x^2,\ldots,x^n),F(z^1_m,z^2_m,\ldots,z^n_m))\\
&\quad+G(F(x^2,x^3,\ldots,x^n,x^1),F(x^2,x^3,\ldots,x^n,x^1),F(z^2_m,z^3_m,\ldots,z^n_m,z^1_m))\\
&\quad+\ \vdots\\
&\quad+G(F(x^n,x^1,\ldots,x^{n-1}),F(x^n,x^1,\ldots,x^{n-1}),F(z^n_m,z^1_m,\ldots,z^{n-1}_m))\Bigg)\Bigg)\\
&\le\phi\left(\tfrac{G(gx^1,gx^1,gz^1_m)+G(gx^2,gx^2,gz^2_m)+G(gx^3,gx^3,gz^3_m)+\ldots+G(gx^n,gx^n,gz^n_m)}{n}\right)\\
&\quad-\psi\left(\tfrac{G(gx^1,gx^1,gz^1_m)+G(gx^2,gx^2,gz^2_m)+G(gx^3,gx^3,gz^3_m)+\ldots+G(gx^n,gx^n,gz^n_m)}{n}\right).
\end{aligned}
\tag{3.27}
$$

Set,

$$
\begin{aligned}
\alpha_m &= G(gx^1,gx^1,gz^1_{m+1})+G(gx^2,gx^2,gz^2_{m+1})\\
&\quad+G(gx^3,gx^3,gz^3_{m+1})+\ldots+G(gx^n,gx^n,gz^n_{m+1}).
\end{aligned}
$$

Then, from (3.27), we get

$$\phi(\alpha_m)\le\phi(\alpha_{m-1})-\psi(\alpha_{m-1}). \tag{3.28}$$

Since, ψ is non negative, therefore we obtain $\phi(\alpha_m)\le\phi(\alpha_{m-1})$. From above inequalities and monotone property of ϕ, we have $\alpha_m\le\alpha_{m-1}$. Therefore $\{\alpha_m\}$ is a monotonically decreasing sequence of nonnegative real numbers. So, there exists $\alpha\ge 0$ such that $\alpha_m\to\alpha$ as $m\to\infty$. Letting the limit as $m\to\infty$ in (3.28), one can get

$$\phi(\alpha)\le\phi(\alpha)-\lim_{m\to\infty}\psi(\alpha_m)=\phi(\alpha)-\lim_{\alpha_m\to\alpha^+}\psi(\alpha_m)<\phi(\alpha).$$

This gives, $\alpha_m\to 0$ as $m\to\infty$. Then

$$
\begin{aligned}
&\lim_{m\to\infty}G(gx^1,gx^1,gz^1_{m+1})=0,\ \ \lim_{m\to\infty}G(gx^2,gx^2,gz^2_{m+1})=0,\\
&\cdots,\ \ \lim_{m\to\infty}G(gx^n,gx^n,gz^n_{m+1})=0.
\end{aligned}
$$

Similarly, one can show that

$$
\begin{aligned}
&\lim_{m\to\infty}G(gy^1,gy^1,gz^1_{m+1})=0,\ \ \lim_{m\to\infty}G(gy^2,gy^2,gz^2_{m+1})=0,\\
&\cdots,\ \ \lim_{m\to\infty}G(gy^n,gy^n,gz^n_{m+1})=0.
\end{aligned}
$$

Now, by the property(G5), we have

$$\begin{cases} G(gx^1, gx^1, gy^1) \leq G(gx^1, gx^1, gz^1_{m+1}) + G(gz^1_{m+1}, gz^1_{m+1}, gy^1) \to 0, \\ G(gx^2, gx^2, gy^2) \leq G(gx^2, gx^2, gz^2_{m+1}) + G(gz^2_{m+1}, gz^2_{m+1}, gy^2) \to 0, \\ \vdots \\ G(gx^n, gx^n, gy^n) \leq G(gx^n, gx^n, gz^n_{m+1}) + G(gz^n_{m+1}, gz^n_{m+1}, gy^n) \to 0, \end{cases}$$

as $m \to \infty$. From the above inequality, we get

$$gx^1 = gy^1, \quad gx^2 = gy^2, \quad \dots \quad , \quad gx^n = gy^n. \tag{3.29}$$

Since,

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = gx^1, \\ F(x^2, x^3, \dots, x^n, x^1) = gx^2, \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = gx^n. \end{cases}$$

And, the pair (F, g) is commutative then

$$\begin{cases} F(gx^1, gx^2, gx^3, \dots, gx^n) = ggx^1, \\ F(gx^2, gx^3, \dots, gx^n, gx^1) = ggx^2, \\ \vdots \\ F(gx^n, gx^1, gx^2, \dots, gx^{n-1}) = ggx^n. \end{cases}$$

Now, put $gx^1 = u^1, gx^2 = u^2, \dots, gx^n = u^n$, we arrive at

$$\begin{cases} F(u^1, u^2, u^3, \dots, u^n) = gu^1, \\ F(u^2, u^3, \dots, u^n, u^1) = gu^2, \\ \vdots \\ F(u^n, u^1, u^2, \dots, u^{n-1}) = gu^n. \end{cases} \tag{3.30}$$

Hence, $(u^1, u^2, u^3, \dots, u^n)$ is an n-tupled coincidence point of F and g. Now, put $y^1 = u^1, y^2 = u^2, \dots, y^n = u^n$ in (3.29), we get

$$gx^1 = gu^1, \quad gx^2 = gu^2, \quad \dots \quad , \quad gx^n = gu^n.$$

This gives,

$$gu^1 = u^1, \quad gu^2 = u^2, \quad \dots \quad , \quad gu^n = u^n. \tag{3.31}$$

From (3.30) and (3.31), we get

$$\begin{cases} F(u^1, u^2, u^3, \dots, u^n) = gu^1 = u^1, \\ F(u^2, u^3, \dots, u^n, u^1) = gu^2 = u^2, \\ \vdots \\ F(u^n, u^1, u^2, \dots, u^{n-1}) = gu^n = u^n. \end{cases}$$

Thus, $(u^1, u^2, u^3, \ldots, u^n)$ is n-tupled common fixed point of F and g. To prove the uniqueness, suppose that $(v^1, v^2, v^3, \ldots, v^n)$ is an another n-tupled common fixed point of F and g. From (3.29), we obtain that

$$\begin{cases} gu^1 = u^1 = gv^1 = v^1, \\ gu^2 = u^2 = gv^2 = v^2, \\ \qquad \vdots \\ gu^n = u^n = gv^n = v^n. \end{cases}$$

And this makes end to the proof. □

Now following example illustrates the usability of Theorem 3.1.

Example 3.7. Let $X = \mathbb{R}$ with usual ordering. Define $G : X^3 \to X$ by $G(x, y, z) = \max\{|x - y| + |y - z| + |z - x|\}$. Let $g : X \to X$ and $F : X^n \to X$ be defined by, $gx = \frac{x}{4}$ for all $x \in X$ and for all $(x^1, x^2, \ldots, x^n) \in X$

$$F(x^1, x^2, \ldots, x^n) = \frac{x^1 - x^2 + x^3 - \ldots - x^n}{16n}.$$

Taking $\phi = \frac{3t}{4}$ for all $t \in [0, \infty)$ and $\psi = \frac{9t}{16}$ for all $t \in [0, \infty)$. Then

(a) (X, G) is complete ordered G-metric space.
(b) F and g have the mixed g-monotone property.
(c) (F, g) is commutative.
(d) $F(X^n) \subseteq g(X)$.
(e) For the verification of contractive condition let $(x^1, x^2, \ldots, x^n)$, $(y^1, y^2, \ldots, y^n)$ and $(z^1, z^2, \ldots, z^n)$ are in X with $gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gy^2 \preceq gz^2$, $gx^3 \succeq gy^3 \succeq gz^3, \ldots,$ $gx^n \preceq gy^n \preceq gz^n$, then

$$\begin{aligned} &G(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n), F(z^1, z^2, \ldots, z^n)) \\ &= G(\frac{x^1 - x^2 + \ldots - x^n}{16n}, \frac{y^1 - y^2 + \ldots - y^n}{16n}, \frac{z^1 - z^2 + \ldots - z^n}{16n}) \\ &\leq \frac{1}{16}((|x^1 - z^1| + |x^2 - z^2| + \ldots + |x^n - z^n|)) \\ &\leq \frac{1}{4n}\big(G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \ldots + G(gx^n, gy^n, gz^n)\big). \end{aligned} \tag{3.32}$$

Again,

$$\begin{aligned} &G\big(F(x^2, \ldots, x^n, x^1), F(y^2, \ldots, y^n, y^1), F(z^2, \ldots, z^n, z^1)\big) \\ &= G\Big(\frac{x^2 - \ldots + x^n - x^1}{16n}, \frac{y^2 - \ldots + y^n - y^1}{16n}, \frac{z^2 - \ldots + z^n - z^1}{16n}\Big) \\ &\leq \frac{1}{4n}\big(G(gx^2, gy^2, gz^2) + \ldots + G(gx^n, gy^n, gz^n) + G(gx^1, gy^1, gz^1)\big). \end{aligned} \tag{3.33}$$

In a similar way, one can easily show that

$$\begin{aligned}&G(F(x^n, x^1, \ldots, x^{n-1}), F(y^n, y^1, \ldots, y^{n-1}), F(z^n, z^1, \ldots, z^{n-1}))\\&\leq \frac{1}{4n}(G(gx^n, gy^n, gz^n)+G(gx^1, gy^1, gz^1)+\cdots\\&\quad+G(gx^{n-1}, gy^{n-1}, gz^{n-1})).\end{aligned} \tag{3.34}$$

Now, on using (3.32)-(3.34), we have

$$\begin{aligned}&(G(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n), F(z^1, z^2, \ldots, z^n))\\&+G(F(x^2, \ldots, x^n, x^1), F(y^2, \ldots, y^n, y^1), F(z^2, \ldots, z^n, z^1))\\&+\ldots+G(F(x^n, x^1, \ldots, x^{n-1}), F(y^n, y^1, \ldots, y^{n-1}), F(z^n, z^1, \ldots, z^{n-1})))\\&\leq \frac{n}{4n}(G(gx^1, gy^1, gz^1)+G(gx^2, gy^2, gz^2)+\ldots+G(gx^n, gy^n, gz^n))\end{aligned}$$

implies that

$$\begin{aligned}&(G(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n), F(z^1, z^2, \ldots, z^n))\\&+G(F(x^2, \ldots, x^n, x^1), F(y^2, \ldots, y^n, y^1), F(z^2, \ldots, z^n, z^1))+\cdots\\&+G(F(x^n, x^1, \ldots, x^{n-1}), F(y^n, y^1, \ldots, y^{n-1}), F(z^n, z^1, \ldots, z^{n-1})))\cdot n^{-1}\\&\leq \frac{1}{4n}(G(gx^1, gy^1, gz^1)+G(gx^2, gy^2, gz^2)+\ldots+G(gx^n, gy^n, gz^n)).\end{aligned}$$

Clearly, inequality (3.4) hold with $\phi(t) = \frac{3}{4}t$ and $\psi(t) = \frac{9}{16}t$. Hence, all the conditions of Theorem 3.1 are satisfied and $(0, 0, 0, \ldots, 0)$ is an n-tupled coincidence point of F and g.

Setting $g = I_x$ in Theorem 3.1 and Theorem 3.2, resulting the following corollary.

Corollary 3.8. *Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space such that (X, G) is G-complete. Let $F : X^n \to X$ a (ϕ, ψ)-contractive mappings on X, such that*

$$\begin{aligned}&\phi((G(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n), F(z^1, z^2, \ldots, z^n))\\&+G(F(x^2, \ldots, x^n, x^1), F(y^2, \ldots, y^n, y^1), F(z^2, \ldots, z^n, z^1))+\cdots\\&+G(F(x^n, x^1, \ldots, x^{n-1}), F(y^n, y^1, \ldots, y^{n-1}), F(z^n, z^1, \ldots, z^{n-1})))\cdot n^{-1})\\&\leq \phi\Big(\frac{G(x^1, y^1, z^1)+G(x^2, y^2, z^2)+\ldots+G(x^n, y^n, z^n)}{n}\Big)\\&\quad-\psi\Big(\frac{G(x^1, y^1, z^1)+G(x^2, y^2, z^2)+\ldots+G(x^n, y^n, z^n)}{n}\Big),\end{aligned} \tag{3.35}$$

with $x^1 \succeq y^1 \succeq z^1$, $x^2 \preceq y^2 \preceq z^2$, $x^3 \succeq y^3 \succeq z^3, \ldots, x^n \preceq y^n \preceq z^n$, *such that* F *has the mixed monotone property. Also assume that, either*

(a) F *is continuous, or*
(b) $(X, G, \preceq)$ *is regular.*

If there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ *such that*

$$\begin{cases} x^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n), \\ F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \preceq x_0^2, \\ x_0^3 \preceq F(x_0^3, \ldots, x_0^n, x_0^1, x_0^2), \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \ldots, n^{n-1}) \preceq x_0^n. \end{cases}$$

Then F *has an* n*-tupled fixed point in* X.

Remark 3.9. If we choose $n = 2$ in Corollary 3.8, we obtain Corollary 22 in [7].

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