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MOMENT STABILITY FOR UNCERTAIN STOCHASTIC NEURAL NETWORKS WITH TIME-DELAYS

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Abstract. In this paper we study the moment stability for uncertain stochastic neural networks with time-delays

$$dx = [-(A + \triangle A(t))x(t) + f(t, x(t), x(t-\tau(t)))]dt + \sum_{j=1}^{m} g_j(t, x(t), x(t-\tau(t)))dw_j(t).$$

Using the variation-of-constants formula and comparison principle, we obtain new criteria for moment stability of the considered uncertain stochastic neural networks with time-delays.

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1. Introduction

Neural networks arise in important applications in combinatorial optimization, signal processing, pattern recognition and in solving nonlinear algebraic equations [1, 7, 8, 14]. There are only a few papers in the literature which discuss the stability of stochastic and delay equations; see [4, 5, 6, 12, 13, 15] and the references therein.

For notational purposes with $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ and $B = [b_{ij}(t)]_{n\times n}$, let

$$|x(t)|_1 = \sum_{i=1}^n |x_i(t)|$$

and

$$\|B(t)\|_3 = \sum_{i,j=1}^n |b_{ij}(t)|.$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; R)$ be the family of all bounded, $\mathcal{F}_0$-measurable functions. We denote by $C([-\tau, 0]; R)$ the family of all continuous functions $\varphi : [-\tau, 0] \to R$ with

$$\|\varphi\|_2 = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|_1,$$

where τ is a positive constant.

In this paper, using the variation-of-constants formula and comparison principle, we discuss the moment stability of the stochastic cellular neural network

$$\begin{aligned} dx \quad &= [-(A + \triangle A(t))x(t) + f(t, x(t), x(t-\tau(t)))]dt \\ &\quad + \sum_{j=1}^m g_j(t, x(t), x(t-\tau(t)))dw_j(t), \quad t \geq t_0, \end{aligned} \tag{1.1}$$

with the initial condition

$$x_{t_0}(s) = \varphi(s) \in C([-\overline{\tau}, 0]; R^n), \quad -\overline{\tau} \leq s \leq 0, \tag{1.2}$$

where $x_{t_0}(s) = x(t_0 + s)$, $\overline{\tau} = \max\{\tau(t)\}$, $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is the state vector, $A = diag(a_1, a_2, \cdots, a_n) > 0$, $\triangle A(t)$ represents the time-varying parameter uncertainties and is bounded. Here $w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^T \in R^m$ is a m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and $f : R^+ \times R^n \times C([-\overline{\tau}, 0]; R^n) \to R^n$ is the neuron activation function and we assume $f(t, 0, 0) = 0$. The stochastic disturbance terms, $g_j : R^+ \times R^n \times C([-\overline{\tau}, 0]; R^n) \to R^n$, can be viewed as stochastic perturbations on the neuron states and delayed neuron states.

In [10], Lei obtained necessary and sufficient conditions for the second moment to be bounded by using Laplace transform techniques. In [2], via the $\mathcal{H}-$representation technique and comparison principles, Zhao and Deng obtained the second order moment equations of nonlinear stochastic systems with-delays. This paper is largely motivated by [2, 10]. Only a few papers in the literature use the variation-of-constants formula to discuss the second moment stability of stochastic neural networks and the method in this paper is different from usual methods (see [3, 11]).

2. Preliminaries

Now we give the definitions of the pth moment stability and the pth moment boundedness. Let

$$x(t) = x(t_0, \varphi)(t), \quad t \geq t_0$$

denote solutions of Eq.(1.1) with the initial date $x_{t_0} = \varphi$. We denote by E the mathematical expectation.

Definition 2.1. For $p \geq 2$, the trivial solution of Eq.(1.1) is said to be stable in the pth moment if for each $\varepsilon > 0$, $t_0 \in R^+$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $E(\|\varphi\|_2^2) < \delta$ implies

$$E(\|x(t_0, \varphi)(t) - E(x(t_0, \varphi)(t)\|_2^p) < \varepsilon, \quad t \geq t_0 - \overline{\tau}. \tag{2.1}$$

Definition 2.2. For $p \geq 2$, the trivial solution of Eq.(1.1) is said to be uniformly stable in the pth moment if (2.1) holds with δ independent of t_0.

Definition 2.3. For $p \geq 2$, the trivial solution of Eq.(1.1) is said to be asymptotically stable in the pth moment if it is stable in pth moment and if for any $\varepsilon > 0$, $t_0 \in R^+$, there exists $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that $E(\|\varphi\|_2^2) < \delta$ implies

$$E(\|x(t_0, \varphi)(t) - E(x(t_0, \varphi)(t)\|_2^p) < \varepsilon, \quad t \geq t_0 + T. \tag{2.2}$$

Definition 2.4. For $p \geq 2$, the trivial solution of Eq.(1.1) is said to be uniformly asymptotically stable in the pth moment if (2.1) and (2.2) hold with δ, δ_0 and T independent of t_0.

Definition 2.5. The trivial solution of Eq.(1.1) is said to be the first moment exponentially stable if there exist two positive constants μ and β such that

$$\|Ex(t_0, \varphi)(t)\|_2 \leq \mu \|\varphi\|_2 e^{-\beta(t-t_0)}, \quad t \geq t_0,$$

for each $t_0 \in R^+$. When $p \geq 2$, Eq.(1.1) is said to be the pth moment exponentially stable if there exist two positive constants μ and β such that

$$E(\|x(t_0, \varphi)(t) - E(x(t_0, \varphi)(t))\|_2^p) \leq \mu \|\varphi\|_2^p e^{-\beta(t-t_0)}, \quad t \geq t_0.$$

Definition 2.6. For $p \geq 2$, Eq.(1.1) is said to be the pth moment bounded if there exists a positive constant $\widetilde{\mu} = \widetilde{\mu}(\|\varphi\|^p)$ such that

$$E(\|x(t_0, \varphi)(t) - E(x(t_0, \varphi)(t))\|_2^p) \leq \widetilde{\mu}, \quad t \geq t_0. \tag{2.3}$$

Otherwise, the pth moment is said to be unbounded.

Let $A = (a_{i,j})$, $B = (b_{i,j})$ be $n \times m$ matrices, and denote $A \le B$ if $a_{i,j} \le b_{i,j}$ for all $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$. Denote by Θ the zero matrix, that is, all of the entries of Θ are 0.

Lemma 2.7. ([9], Comparison principle) *Assume that $P, Q \in C(R, R^{r\times r})$, $P(t) \ge \Theta$, $Q(t) \ge \Theta$ and $F(t) \in C(R, R^r)$. Let x, y be the solutions of the following systems*

$$\begin{cases} \frac{dx}{dt} \le P(t)x(t) + Q(t)x(t-\overline{\tau}) + F(t), \quad t \ge 0, \\ x(\theta) \le \phi(\theta), \quad -\overline{\tau} \le \theta \le 0 \end{cases}$$

and

$$\begin{cases} \frac{dy}{dt} = P(t)y(t) + Q(t)y(t-\overline{\tau}) + F(t), \quad t \ge 0, \\ y(\theta) \le \chi(\theta), \quad -\overline{\tau} \le \theta \le 0, \end{cases}$$

respectively, where $\phi, \chi \in C([-\overline{\tau}, 0]; R^r)$. Then $\phi(\theta) \le \chi(\theta)(-\overline{\tau} \le \theta \le 0)$ implies $x(t) \le y(t)$ for $t \ge 0$.

3. Moment boundedness of uncertain stochastic neural networks with time-delays

In this section we investigate the stability properties of nonlinear stochastic cellular neural network.

Throughout this paper, we always assume the following:

(H_1) there exists a positive constant α such that

$$\begin{aligned} &\|f(t, x(t), x(t-\tau(t))) - f(t, y(t), y(t-\tau(t)))\|_2 \\ &\le \alpha[\|x - y\|_1 + \|x(t-\tau(t)) - y(t-\tau(t))\|_2]; \end{aligned}$$

(H_2) there exist positive constants $\beta^{(j)}$, and positive definite constant matrices $D_j^{(0)}$, $\tilde{D}_j^{(0)}$, $D_j^{(1)}$, $\tilde{D}_j^{(1)}$, $D_j^{(2)}$ and $\tilde{D}_j^{(2)}$ such that

$$\begin{aligned} &\|g_j(t, x(t), x(t-\tau(t))) - g_j(t, y(t), y(t-\tau(t)))\|_2 \\ &\le \beta^{(j)}[\|x - y\|_1 + \|x(t-\tau(t)) - y(t-\tau(t))\|_2] \end{aligned}$$

and

$$\begin{aligned} &\|\tilde{D}_j^{(0)}\|_3 + x^T(t)\tilde{D}_j^{(1)}x(t) + x^T(t-\tau(t))\tilde{D}_j^{(2)}x(t-\tau(t)) \\ &\le g_j^T(t, x(t), x(t-\tau(t))) \times g_j(t, x(t), x(t-\tau(t))) \\ &\le \|D_j^{(0)}\|_3 + x^T(t)D_j^{(1)}x(t) + x^T(t-\tau(t))D_j^{(2)}x(t-\tau(t)), \end{aligned}$$

where T represents the transpose and $j = 1, 2, \cdots, m$.

Let $x(t) = x(t_0, \varphi)(t)$ be the solution of (1.1) and (1.2). Note from (1.1) and (1.2) that

$$\begin{aligned} x(t) &= \exp -A(t-t_0)\{\varphi(s) \\ &\quad + \int_{t_0}^{t} \exp(As)[-\triangle A(s)x(s) + f(s, x(s), x(s-\tau(s)))]ds \\ &\quad + \sum_{j=1}^{m} \int_{t_0}^{t} g_j(s, x(s), x(s-\tau(s))) \exp(As) dw(s)\}. \end{aligned} \tag{3.1}$$

From (H_1) and (H_2) we have that $f(\cdot,\cdot,\cdot)$ and $g_j(\cdot,\cdot,\cdot)$ satisfy a Lipschitz condition. Then there is a unique solution $x(t)$ of Equation (1.1) through (t, φ).

3.1. **The first moment stability.** Note from (3.1) we have

$$\begin{aligned} &\|Ex(t)\|_2 \\ &= \|\exp -A(t-t_0)\{\varphi(s) \\ &\quad + \int_{t_0}^{t} \exp(As)[-\triangle A(s)x(s) + f(s, x(s), x(s-\tau(s)))]ds\|_2. \end{aligned} \tag{3.2}$$

Now, we consider the following deterministic equation

$$\begin{cases} dx = [-(A + \triangle A(t))x(t) + f(t, x(t), x(t-\tau(t)))]dt, \quad t \geq t_0, \\ x_{t_0}(s) = \varphi(s) \in C([-\overline{\tau}, 0]; R^n), \quad -\overline{\tau} \leq s \leq 0. \end{cases} \tag{3.3}$$

Let $x_\varphi(t)$ be the solution of (3.3).

Theorem 3.1. *Suppose*

(H_3) *The solution of* (3.3) *is exponentially stable, i.e., there exist two positive constants* κ *and* λ *such that*

$$\|x_\varphi(t)\|_2 \leq \kappa \|\varphi\|_2 e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Then Eq.(1.1) *is first moment exponentially stable, i.e.,*

$$\|Ex(t)\|_2 = \|x_\varphi(t)\|_2 \leq \kappa \|\varphi\|_2 e^{-\lambda(t-t_0)}, \quad t \geq t_0. \tag{3.4}$$

Proof. The result follows from (H_3) and (3.2). □

In the following discussions, we always assume that Eq.(1.1) is first moment exponentially stable.

3.2. **Second moment boundedness and stability.** Now we study the second moment.

Let

$$\begin{aligned} \widetilde{x}(t) &= x(t) - Ex(t) \\ &= \int_{t_0}^{t} [\sum_{j=1}^{m} \exp -A(t-s) g_j(s, x(s), x(s-\tau(s)))] dw_j \end{aligned} \tag{3.5}$$

and

$$\mathcal{V}(t) = E(\|\widetilde{x}(t)\|_2^2), \tag{3.6}$$

where $x(t) = x(t_0, \varphi)(t)$ is the solution of (1.1) and (1.2). Note that

$$\widetilde{x}(t_0) = x(t_0) - Ex(t_0) = \varphi(s) - E\varphi(s) = \varphi(s) - \varphi(s) = 0, \quad -\overline{\tau} \leq s \leq 0.$$

Then we have $\mathcal{V}(t_0) = 0$. Since $E(dw_j dw_k) = \delta_{jk}(j, k = 1, 2, \cdots, m)$, we have from (3.4) and (3.6) that

$$\begin{aligned}
&\mathcal{V}(t)\\
&= E(\|\widetilde{x}(t)\|_2^2) = E\| \textstyle\int_{t_0}^t \sum_{j=1}^m \exp A(s-t) g_j(s, x(s), x(s-\tau(s)))dw_j(s)\|_2^2\\
&\leq E \textstyle\int_{t_0}^t \|\exp A(s-t)\|_3^2 [\sum_{j=1}^m g_j^T(s, x(s), x(s-\tau(s)))\\
&\quad \times g_j(s, x(s), x(s-\tau(s)))]ds\\
&\leq E \textstyle\int_{t_0}^t n^2 e^{2\lambda_{\min}(A)(s-t)} [\sum_{j=1}^m (\|D_j^{(0)}\|_3\\
&\quad + x^T(s) D_j^{(1)} x(s) + x^T(s-\tau(s)) D_j^{(2)} x(s-\tau(s)))]ds\\
&= E \textstyle\int_{t_0}^t n^2 e^{2\lambda_{\min}(A)(s-t)} \{\sum_{j=1}^m [\|D_j^{(0)}\|_3 + (\widetilde{x}(s) + Ex(s))^T D_j^{(1)} (\widetilde{x}(s) + Ex(s))\\
&\quad + (\widetilde{x}(s-\tau(s)) + Ex(s-\tau(s)))^T D_j^{(2)} (\widetilde{x}(s-\tau(s)) + Ex(s-\tau(s)))]\}ds\\
&\leq \textstyle\int_{t_0}^t n^2 e^{2\lambda_{\min}(A)(s-t)} \{\sum_{j=1}^m [n\lambda_{\max}(D_j^{(0)}) + n\lambda_{\max}(D_j^{(1)})(E(\|\widetilde{x}(s)\|_2^2)\\
&\quad + (\|Ex(s)\|_2)^2) + n\lambda_{\max}(D_j^{(2)})(E(\|\widetilde{x}(s-\tau(s))\|_2^2)\\
&\quad + (E(\|x(s-\tau(s))\|_2))^2)]\}ds\\
&= \textstyle\int_{t_0}^t e^{2\lambda_{\min}(A)(s-t)} \{n^3 \sum_{j=1}^m [\lambda_{\max}(D_j^{(0)}) + \lambda_{\max}(D_j^{(1)})(\mathcal{V}(s) + (\|Ex(s)\|_2)^2)\\
&\quad + \lambda_{\max}(D_j^{(2)})(\mathcal{V}(s-\tau(s)) + (\|Ex(s-\tau(s))\|_2)^2)]\}ds\\
&\leq \textstyle\int_{t_0}^t e^{2\lambda_{\min}(A)(s-t)} \{n^3 \sum_{j=1}^m [\lambda_{\max}(D_j^{(0)}) + \lambda_{\max}(D_j^{(1)})(\mathcal{V}(s) + \kappa^2 \|\varphi\|_2^2 e^{-2\lambda s})\\
&\quad + \lambda_{\max}(D_j^{(2)})(\mathcal{V}(s-\tau(s)) + \kappa^2 e^{2\lambda\overline{\tau}} \|\varphi\|_2^2 e^{-2\lambda s})]\}ds\\
&\leq \textstyle\int_{t_0}^t e^{2\lambda_{\min}(A)(s-t)} \{n^3 \sum_{j=1}^m [\lambda_{\max}(D_j^{(0)}) + H_j + \lambda_{\max}(D_j^{(1)})\mathcal{V}(s)\\
&\quad + \lambda_{\max}(D_j^{(2)})\mathcal{V}(s-\tau(s))]\}ds\\
&= \textstyle\int_{t_0}^t e^{2\lambda_{\min}(A)(s-t)} [\overline{D}^{(0)} + \overline{D}^{(1)}\mathcal{V}(s) + \overline{D}^{(2)}\mathcal{V}(s-\tau(s))]ds,
\end{aligned} \tag{3.7}$$

where $\lambda_{\min}(A)$ represents the minimal eigenvalue of A, $\lambda_{\max}(D_j^{(0)})$, $\lambda_{\max}(D_j^{(1)})$ and $\lambda_{\max}(D_j^{(2)})$ represent the maximal eigenvalues of $D_j^{(0)}$, $D_j^{(1)}$ and $D_j^{(2)}(j = 1, 2, \cdots, m)$,

$$\overline{D}^{(0)} = n^3 \sum_{j=1}^m [\lambda_{\max}(D_j^{(0)}) + H_j], \quad \overline{D}^{(1)} = n^3 \sum_{j=1}^m \lambda_{\max}(D_j^{(1)}),$$

$$\overline{D}^{(2)} = n^3 \sum_{j=1}^{m} \lambda_{\max}(D_j^{(2)}),$$

$$H_j = \kappa^2 \|\varphi\|_2^2 \lambda_{\max}(D_j^{(1)}) + \kappa^2 e^{2\lambda\overline{\tau}} \|\varphi\|_2^2 \lambda_{\max}(D_j^{(2)}),$$

$$\exp A(s-t) = \begin{pmatrix} e^{a_1(s-t)} & 0 & 0 & \cdots & 0 \\ 0 & e^{a_2(s-t)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{a_n(s-t)} \end{pmatrix}$$

and

$$\|\exp A(s-t)\|_3 = \sum_{i=1}^{n} e^{a_i(s-t)};$$

here we used the relation

$$E(x(t) - Ex(t)) = E(x(t-\tau(t)) - Ex(t-\tau(t)) = 0.$$

Now we chose two functions $\phi(t)$, $\chi(t) \in C([-\overline{\tau}, 0]; R)$ with $\phi(0) = 0$ such that

$$\phi(t) \le \chi(t), \quad t \in [-\overline{\tau}, 0]. \tag{3.8}$$

Theorem 3.2. *Let* $(H_1) - (H_3)$ *be satisfied. Then*

$$\mathcal{V}(t) \le u(t), \quad t \ge t_0,$$

where $u(t) = u(t_0, \chi)(t)$ *is the solution of the comparison equation*

$$\begin{cases} \dot{u}(t) = (-2\lambda_{\min}(A) + \overline{D}^{(1)})u(t) + \overline{D}^{(2)} u(t-\tau(t)) + \overline{D}^{(0)}, \quad t \ge t_0, \\ u_{t_0}(s) = \chi(s), \quad s \in [-\overline{\tau}, 0]. \end{cases} \tag{3.9}$$

Proof. Let

$$\begin{cases} M(t) = \int_{t_0}^{t} e^{2\lambda_{\min}(A)(s-t)} [\overline{D}^{(0)} + \overline{D}^{(1)} \mathcal{V}(s) + \overline{D}^{(2)} \mathcal{V}(s-\tau(s))] ds, \quad t \ge t_0, \\ M_{t_0}(s) = \phi(s), \quad s \in [-\overline{\tau}, 0]. \end{cases} \tag{3.10}$$

We have from (3.7) and (3.10) that

$$\begin{aligned} &\dot{M}(t) \\ &= -2\lambda_{\min}(A) \int_{t_0}^{t} e^{2\lambda_{\min}(A)(s-t)} [\overline{D}^{(0)} + \overline{D}^{(1)} \mathcal{V}(s) + \overline{D}^{(2)} \mathcal{V}(s-\tau(s))] ds \\ &\quad + \overline{D}^{(0)} + \overline{D}^{(1)} \mathcal{V}(t) + \overline{D}^{(2)} \mathcal{V}(t-\tau(t)) \\ &\le -2\lambda_{\min}(A) M(t) + \overline{D}^{(0)} + \overline{D}^{(1)} M(t) + \overline{D}^{(2)} M(t-\tau(t)) \\ &= (-2\lambda_{\min}(A) + \overline{D}^{(1)}) M(t) + \overline{D}^{(2)} M(t-\tau(t)) + \overline{D}^{(0)}, \quad t \ge 0. \end{aligned}$$

Applying Lemma 2.7 (Comparison principle), from (3.8)-(3.10) we get $u(t) \geq M(t)$, $t \geq t_0$, thus

$$\mathcal{V}(t) \leq M(t) \leq u(t), \quad t \geq t_0. \tag{3.11}$$

The proof is complete. □

For convenience, we assume that the equilibrium of the system (3.9) is a trivial solution. From Theorem 3.2, we can obtain stability and boundedness criteria for (1.1) and (1.2).

Theorem 3.3. *If the assumptions of Theorem* 3.2 *are satisfied, and the trivial solution of the system* (3.9) *is stable, then the trivial solution of the system* (1.1) *and* (1.2) *is also stable in the 2th moment.*

Proof. Assume that the trivial solution $u(t) = u(t_0, \chi)(t)$ of (3.9) is stable. For $\varepsilon > 0$ there exists a positive number $\delta = \delta(t_0, \varepsilon)$ such that

$$\|u(t_0, \chi)(t)\|_2 \leq \varepsilon, \quad \|\chi\|_2 < \delta, \quad t \geq t_0 - \overline{\tau}. \tag{3.12}$$

We have from (3.12) that

$$0 = \mathcal{V}(t_0) \leq \|\chi\|_2 < \delta. \tag{3.13}$$

From (3.6), Theorem 3.2, (3.11)-(3.12) and the above, we get

$$E(\|\widetilde{x}(t)\|_2^2) = \mathcal{V}(t) \leq \|u(t_0, \chi)(t)\|_2 \leq \varepsilon, \quad t \geq t_0 - \overline{\tau}. \tag{3.14}$$

We see from (3.13) and (3.14) that the trivial solution of the system (1.1) and (1.2) is stable in the 2th moment. □

Theorem 3.4. *If the assumptions of Theorem* 3.2 *are satisfied, and the trivial solution of the system* (3.9) *is uniformly stable, then the trivial solution of the system* (1.1) *and* (1.2) *is also uniformly stable in the 2th moment.*

Proof. Follows as in Theorem 3.3. □

Theorem 3.5. *If the assumptions of Theorem* 3.2 *are satisfied, and the trivial solution of the system* (3.9) *is asymptotically stable, then the trivial solution of the system* (1.1) *and* (1.2) *is also asymptotically stable in the 2th moment.*

Proof. By Theorem 3.3, we just need to show the last part of Definition 2.3 is true. For given $\varepsilon > 0$, there exist a $\delta_1 > 0$ (δ_1 is independent of ε) and a $T = T(t_0, \varepsilon) > 0$ such that

$$\|u(t_0, \chi)(t)\|_2 \leq \varepsilon, \quad \|\chi\|_2 < \delta_1, \quad t \geq t_0 + T. \tag{3.15}$$

We have from (3.6) and (3.15) that

$$0 = \mathcal{V}(t_0) \leq \|\chi\|_2 < \delta_1. \tag{3.16}$$

From Theorem 3.2, (3.11) and (3.15)-(3.16), we get

$$E(\|\widetilde{x}(t)\|_2^2) = \mathcal{V}(t) \leq \|u(t_0, \chi)(t)\|_2 \leq \varepsilon, \quad t \geq t_0 + T. \tag{3.17}$$

Hence the trivial solution of (1.1) and (1.2) is asymptotically stable in the 2th moment. □

Theorem 3.6. *If the assumptions of Theorem* 3.2 *are satisfied, and the trivial solution of the system* (3.9) *is uniformly asymptotically stable, then the trivial solution of the system* (1.1) *and* (1.2) *is also uniformly asymptotically stable in the 2th moment.*

Proof. Follows as in Theorem 3.5. □

Theorem 3.7. *If the assumptions of Theorem* 3.2 *are satisfied, and the trivial solution of the system* (3.9) *is exponentially stable, then the trivial solution of the system* (1.1) *and* (1.2) *is also exponentially stable in the 2th moment.*

Proof. The argument is similar to that in Theorem 3.5. □

Theorem 3.8. *If the assumptions of Theorem* 3.2 *are satisfied, and the trivial solution of the system* (3.9) *is bounded, then the trivial solution of the system* (1.1) *and* (1.2) *is also bounded in the 2th moment.*

Proof. Assume that the trivial solution of (3.9) is bounded. For arbitrary $\delta > 0$ there exists a positive constant $\widetilde{\mu}$ such that

$$\|u(t_0, \chi)(t)\|_2 \leq \widetilde{\mu}, \quad \|\chi\|_2 < \delta, \quad t \geq t_0 - \overline{\tau}. \tag{3.18}$$

We have from (3.18) that

$$0 = \mathcal{V}(t_0) \leq \|\chi\|_2 < \delta. \tag{3.19}$$

From (3.6), Theorem 3.2, (3.18)-(3.19) and the above, we get

$$E(\|\widetilde{x}(t)\|_2^2) = \mathcal{V}(t) \leq \|u(t_0, \chi)(t)\|_2 \leq \widetilde{\mu}, \quad t \geq t_0 - \overline{\tau}.$$

Hence the trivial solution of the system (1.1) and (1.2) is bounded in the 2th moment. □

3.3. **Second moment unboundedness and instability.** Similar reasoning as in (3.7) we have from (3.5) and (3.6) that

$$
\begin{aligned}
&\mathcal{V}(t)\\
&= E(\|\widetilde{x}(t)\|_2^2) = E\|\textstyle\int_{t_0}^t \sum_{j=1}^m \exp A(s-t) g_j(s, x(s), x(s-\tau(s)))dw_j(s)\|_2^2\\
&= E\|\textstyle\int_{t_0}^t \exp 2A(s-t)[\sum_{j=1}^m g_j^T(s, x(s), x(s-\tau(s)))\\
&\quad \times g_j(s, x(s), x(s-\tau(s)))]ds\|_2\\
&\geq E\textstyle\int_{t_0}^t n^2 e^{2\lambda_{\max}(A)(s-t)}[\sum_{j=1}^m (\|\tilde{D}_j^{(0)}\|_3 + x^T(s)\tilde{D}_j^{(1)}x(s)\\
&\quad + x^T(s-\tau(s))\tilde{D}_j^{(2)}x(s-\tau(s)))]ds\\
&= E\textstyle\int_{t_0}^t n^2 e^{2\lambda_{\max}(A)(s-t)}\{\sum_{j=1}^m [\|\tilde{D}_j^{(0)}\|_3 + (\widetilde{x}(s)+Ex(s))^T\tilde{D}_j^{(1)}(\widetilde{x}(s)+Ex(s))\\
&\quad + (\widetilde{x}(s-\tau(s)) + Ex(s-\tau(s)))^T\tilde{D}_j^{(2)}(\widetilde{x}(s-\tau(s)) + Ex(s-\tau(s)))]\}ds\\
&\geq \textstyle\int_{t_0}^t n^2 e^{2\lambda_{\max}(A)(s-t)}\{\sum_{j=1}^m [n\lambda_{\min}(\tilde{D}_j^{(0)})\\
&\quad + n\lambda_{\min}(\tilde{D}_j^{(1)})(E(\|\widetilde{x}(s)\|_2^2) + (\|Ex(s)\|_2)^2)\\
&\quad + n\lambda_{\min}(\tilde{D}_j^{(2)})(E(\|\widetilde{x}(s-\tau(s))\|_2^2) + (E(\|x(s-\tau(s))\|_2))^2)]\}ds\\
&\geq \textstyle\int_{t_0}^t e^{2\lambda_{\max}(A)(s-t)}\{n^3\sum_{j=1}^m [\lambda_{\min}(\tilde{D}_j^{(0)}) + \lambda_{\min}(\tilde{D}_j^{(1)})\mathcal{V}(s)\\
&\quad + \lambda_{\min}(\tilde{D}_j^{(2)})\mathcal{V}(s-\tau(s))]\}ds\\
&= \textstyle\int_{t_0}^t e^{2\lambda_{\max}(A)(s-t)}[\hat{D}^{(0)} + \hat{D}^{(1)}\mathcal{V}(s) + \hat{D}^{(2)}\mathcal{V}(s-\tau(s))]ds,
\end{aligned}
\tag{3.20}
$$

where $\lambda_{\max}(A)$ represents the maximal eigenvalue of A, $\lambda_{\min}(\tilde{D}_j^{(0)})$, $\lambda_{\min}(\tilde{D}_j^{(1)})$ and $\lambda_{\min}(\tilde{D}_j^{(2)})$ represent the minimal eigenvalues of $\tilde{D}_j^{(0)}$, $\tilde{D}_j^{(1)}$ and $\tilde{D}_j^{(2)}(j = 1, 2, \cdots, m)$,

$$
\begin{aligned}
&\hat{D}^{(0)} = n^3\textstyle\sum_{j=1}^m \lambda_{\min}(\tilde{D}_j^{(0)}), \quad \hat{D}^{(1)} = n^3\sum_{j=1}^m \lambda_{\min}(\tilde{D}_j^{(1)}),\\
&\hat{D}^{(2)} = n^3\textstyle\sum_{j=1}^m \lambda_{\min}(\tilde{D}_j^{(2)}).
\end{aligned}
$$

Now we chose two functions $\overline{\phi}(t), \overline{\chi}(t) \in C([-\overline{\tau}, 0]; R)$ with $\overline{\chi}(0) = 0$ such that

$$\overline{\phi}(t) \leq \overline{\chi}(t), \quad t \in [-\overline{\tau}, 0]. \tag{3.21}$$

Theorem 3.9. *Let $(H_1) - (H_3)$ be satisfied. Then*

$$\mathcal{V}(t) \geq u(t), \quad t \geq t_0,$$

where $u(t) = u(t_0, \overline{\phi})(t)$ *is the solution of the comparison equation*

$$\begin{cases} \dot{u}(t) = (-2\lambda_{\max}(A) + \hat{D}^{(1)})u(t) + \hat{D}^{(2)}u(t-\tau(t)) + \hat{D}^{(0)}, \quad t \geq t_0, \\ u_{t_0}(s) = \overline{\phi}(s), \quad s \in [-\overline{\tau}, 0]. \end{cases} \tag{3.22}$$

Proof. Let

$$\begin{cases} M(t) = \int_{t_0}^{t} e^{2\lambda_{\max}(A)(s-t)}[\hat{D}^{(0)} + \hat{D}^{(1)}\mathcal{V}(s) + \hat{D}^{(2)}\mathcal{V}(s-\tau(s))]ds, \quad t \geq t_0, \\ M_{t_0}(s) = \overline{\chi}(s), \quad s \in [-\overline{\tau}, 0]. \end{cases} \tag{3.23}$$

We have from (3.20) and (3.23) that

$$\begin{aligned} &\dot{M}(t) \\ &= -2\lambda_{\max}(A) \int_{t_0}^{t} e^{2\lambda_{\max}(A)(s-t)}[\hat{D}^{(0)} + \hat{D}^{(1)}\mathcal{V}(s) + \hat{D}^{(2)}\mathcal{V}(s-\tau(s))]ds \\ &\quad + \hat{D}^{(0)} + \hat{D}^{(1)}\mathcal{V}(t) + \hat{D}^{(2)}\mathcal{V}(t-\tau(t)) \\ &\geq -2\lambda_{\max}(A)M(t) + \hat{D}^{(0)} + \hat{D}^{(1)}M(t) + \hat{D}^{(2)}M(t-\tau(t)) \\ &= (-2\lambda_{\max}(A) + \hat{D}^{(1)})M(t) + \hat{D}^{(2)}M(t-\tau(t)) + \hat{D}^{(0)}. \end{aligned}$$

Applying Lemma 2.7 (Comparison principle), from (3.21)-(3.23) we get $M(t) \geq u(t)$, $t \geq t_0$, thus

$$\mathcal{V}(t) \geq M(t) \geq u(t), \quad t \geq t_0.$$

The proof is complete. □

For convenience, we assume that the equilibrium of the system (3.22) is a trivial solution. From Theorem 3.9, we can obtain unboundedness and instability criteria for (1.1) and (1.2).

Theorem 3.10. *If the assumptions of Theorem* 3.9 *are satisfied, and the trivial solution of the system* (3.22) *is unbounded, then the trivial solution of the system* (1.1) *and* (1.2) *is also unbounded in the 2th moment.*

Proof. Assume that the trivial solution of (3.22) is unbounded. For an arbitrary positive constant $\widetilde{\nu}$, there exist a positive constant $\widetilde{\delta}$ and a $\widetilde{t} \geq t_0$ such that

$$\|u(t_0, \overline{\phi})(\widetilde{t})\|_2 > \widetilde{\nu}, \quad \|\overline{\phi}\|_2 < \widetilde{\delta}. \tag{3.24}$$

We have from (3.24) that

$$0 = \mathcal{V}(t_0) \leq \|\overline{\phi}\|_2 < \delta. \tag{3.25}$$

From (3.6), Theorem 3.9, (3.18)-(3.19) and the above, we get

$$E(\|\widetilde{x}(\widetilde{t})\|_2^2) = \mathcal{V}(\widetilde{t}) \geq \|u(t_0, \overline{\phi})(\widetilde{t})\|_2 > \widetilde{\nu}.$$

Hence the trivial solution of the system (1.1) and (1.2) is unbounded in the 2th moment. □

Theorem 3.11. *If the assumptions of Theorem* 3.9 *are satisfied, and the trivial solution of the system* (3.22) *is instable, then the trivial solution of the system* (1.1) *and* (1.2) *is also instable in the 2th moment.*

Proof. Assume that the trivial solution of (3.22) is instable. For an arbitrary positive constant $\widetilde{\varepsilon}$, there exist a positive constant $\widehat{\delta}$ and a $\widehat{t} \geq t_0$ such that

$$\|u(t_0, \overline{\phi})(\widehat{t})\|_2 > \widetilde{\varepsilon}, \quad \|\overline{\phi}\|_2 < \widehat{\delta}. \tag{3.26}$$

We have from (3.26) that

$$0 = \mathcal{V}(t_0) \leq \|\overline{\phi}\|_2 < \widehat{\delta}. \tag{3.27}$$

From (3.6), Theorem 3.9, (3.26)-(3.27) and the above, we get

$$E(\|\widetilde{x}(\widehat{t})\|_2^2) = \mathcal{V}(\widehat{t}) \geq \|u(t_0, \overline{\phi})(\widehat{t})\|_2 > \widetilde{\varepsilon}.$$

Hence the trivial solution of the system (1.1) and (1.2) is instable in the 2th moment. □

Remark 3.12. The system (1.1) can be generalized to the general form

$$\begin{aligned} & dx \\ & = [-(A+\triangle A(t))x(t)+(B+\triangle B(t))f(t, x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t))) \\ & \quad + \textstyle\sum_{p=1}^{k}(W_p + \triangle W_p(t)) \int_{t-r_p(t)}^{t} g_p(x(s))ds]dt \\ & \quad + \textstyle\sum_{j=1}^{l} h_j(t, x(t), x(t-\sigma_j(t)))dw(t). \end{aligned}$$

Example 3.13. Consider the following stochastic neutral cellular network

$$\begin{aligned} & dx(t) \\ & = [-(a+\triangle a(t))x(t) + f(t, x(t), x(t-\tau))]dt \\ & \quad +[\sigma_0(t) + \sigma_1(t)x(t) + \sigma_2(t)x(t-\tau)]dw(t), \end{aligned} \tag{3.28}$$

with the initial condition $x_0 = \varphi$, where

$$\varphi = \{\varphi(\theta), -\tau \leq \theta \leq 0\} \in C^b_{\mathcal{F}_0}([-\tau, 0]; R), \tag{3.29}$$

τ, a are two positive constants, $|\triangle a(t)|$ is bounded, $\sigma_j(t) \in C(R)$ $(j = 0, 1, 2)$ with $|\sigma_j(t)|$ bounded and $-2a + \sigma_1^2 + 2|\sigma_1\sigma_2| + \sigma_2^2 \neq 0$.

Now we chose two functions $\widetilde{\phi}(t)$, $\widetilde{\chi}(t) \in C([-\overline{\tau}, 0]; R)$ with $\widetilde{\phi}(0) = 0$ such that

$$\widetilde{\phi}(t) \leq \widetilde{\chi}(t), \quad t \in [-\overline{\tau}, 0].$$

Conclusion 3.14. Suppose

(1) the assumptions (H_1) and (H_3) are satisfied;
(2) there exists a positive constant γ such that

$$2(2a-\sigma_1^2-|\sigma_1\sigma_2|-\gamma)>\sigma_2^2+|\sigma_1\sigma_2|$$

and

$$2\gamma>\sigma_2^2+|\sigma_1\sigma_2|.$$

Then

$$\mathcal{V}(t)\leq u(t),\quad t\geq 0,$$

where $u(t)=u(0,\widetilde{\chi})(t)$ is the solution of the comparison equation

$$\begin{cases} \dot{u}(t)=(-2a+\sigma_1^2+|\sigma_1\sigma_2|)u(t)+(\sigma_2^2+|\sigma_1\sigma_2|)u(t-\tau) \\ \qquad +(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2,\quad t\geq 0, \\ u_0(s)=\widetilde{\chi}(s),\quad s\in[-\tau,0]. \end{cases}\tag{3.30}$$

Proof. Obviously condition (H_2) holds since $\sigma_j(t)\in C(R)(j=0,1,2)$ with $|\sigma_j(t)|$ bounded. Then there is a unique solution of Equation (3.28) and (3.29) through (t,φ) from (H_1) and (H_3). By the definition of $\mathcal{V}(t_0,\varphi)(t)$, we have from (3.6) that

$$\begin{aligned} &\mathcal{V}(t) \\ &=\int_0^t\exp\{-2a(t-s)\}\|[\sigma_1^2\mathcal{V}(s)+\sigma_2^2\mathcal{V}(s-\tau) \\ &\quad +2\sigma_1\sigma_2E(\widetilde{x}(s)\widetilde{x}(s-\tau))]ds+F(t)\|_2 \\ &\leq\int_0^t\exp\{-2a(t-s)\}\|[\sigma_1^2\mathcal{V}(s)+\sigma_2^2\mathcal{V}(s-\tau) \\ &\quad +2\sigma_1\sigma_2E(\widetilde{x}(s)\widetilde{x}(s-\tau))]\|_2ds+\|F(t)\|_2 \\ &\leq\int_0^t\exp\{-2a(t-s)\}[\sigma_1^2\mathcal{V}(s)+\sigma_2^2\mathcal{V}(s-\tau) \\ &\quad +|\sigma_1\sigma_2|(\mathcal{V}(s)+\mathcal{V}(s-\tau))]ds+\|F(t)\|_2, \end{aligned}\tag{3.31}$$

where $t_0=0$ and

$$F(t)=\int_0^t\exp\{-2a(t-s)\}[\sigma_0(s)+\sigma_1(s)Ex(s)+\sigma_2(s)Ex(s-\tau)]^2ds;\tag{3.32}$$

here we used the relation

$$\begin{aligned} \|E(\widetilde{x}(t)\widetilde{x}(t-\tau))\|_2 &\leq(E(|\widetilde{x}(t)|_1^2))^{\frac{1}{2}}(E(\|\widetilde{x}(t-\tau)\|_2^2))^{\frac{1}{2}} \\ &=\mathcal{V}^{\frac{1}{2}}(t)\mathcal{V}^{\frac{1}{2}}(t-\tau)\leq\frac{\mathcal{V}(t)+\mathcal{V}(t-\tau)}{2}. \end{aligned}$$

Also, we have from (H_3), (3.31) and (3.32) that

$$\begin{aligned} &\mathcal{V}(t) \\ &\leq \int_0^t \exp\{-2a(t-s)\}[\sigma_1^2\mathcal{V}(s)+\sigma_2^2\mathcal{V}(s-\tau) \\ &\quad +|\sigma_1\sigma_2|(\mathcal{V}(s)+\mathcal{V}(s-\tau)) \\ &\quad +(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2]ds \\ &\leq \int_0^t \exp\{-2a(t-s)\}[\sigma_1^2\mathcal{V}(s)+\sigma_2^2\mathcal{V}(s-\tau) \\ &\quad +|\sigma_1\sigma_2|(\mathcal{V}(s)+\mathcal{V}(s-\tau)) \\ &\quad +(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2]ds. \end{aligned} \tag{3.33}$$

Let

$$\begin{cases} M(t) = \int_0^t \exp\{-2a(t-s)\}[\sigma_1^2\mathcal{V}(s)+\sigma_2^2\mathcal{V}(s-\tau) \\ \qquad\qquad +|\sigma_1\sigma_2|(\mathcal{V}(s)+\mathcal{V}(s-\tau)) \\ \qquad\qquad +(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2, \\ M_0(s) = \widetilde{\phi}(s), \quad s\in[-\tau,0], \end{cases} \tag{3.34}$$

where $\widetilde{\phi}(0)=0$. We have from (3.34) that

$$\begin{aligned} &\dot{M}(t) \\ &= -2a\int_0^t \exp\{-2a(t-s)\}[\sigma_1^2\mathcal{V}(s)+\sigma_2^2\mathcal{V}(s-\tau) \\ &\quad +|\sigma_1\sigma_2|(\mathcal{V}(s)+\mathcal{V}(s-\tau)) \\ &\quad +(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2]ds \\ &\quad +[(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2 \\ &\quad +\sigma_1^2\mathcal{V}(t)+\sigma_2^2\mathcal{V}(t-\tau)+|\sigma_1\sigma_2|(\mathcal{V}(t)+\mathcal{V}(t-\tau))] \\ &\leq -2aM(t)+[\sigma_1^2M(t)+\sigma_2^2M(t-\tau)+|\sigma_1\sigma_2|(M(t)+M(t-\tau))] \\ &\quad +(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2 \\ &= (-2a+\sigma_1^2+|\sigma_1\sigma_2|)M(t)+(\sigma_2^2+|\sigma_1\sigma_2|)M(t-\tau) \\ &\quad +(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2e^{\lambda\tau})^2. \end{aligned} \tag{3.35}$$

Applying Lemma 2.7 (Comparison principle), from (3.33)-(3.35), we get $u(t)\geq M(t)$, $t\geq 0$, and so

$$\mathcal{V}(t)\leq M(t)\leq u(t), \quad t\geq 0. \tag{3.36}$$

Let

$$y(t) = u(t) + \frac{(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2 e^{\lambda\tau})^2}{-2a+\sigma_1^2+2|\sigma_1\sigma_2|+\sigma_2^2}. \tag{3.37}$$

Then we have from (3.30) and (3.37) that

$$\begin{cases} \dot{y}(t) = (-2a+\sigma_1^2+|\sigma_1\sigma_2|)y(t)+(\sigma_2^2+|\sigma_1\sigma_2|)y(t-\tau), & t \geq 0, \\ y_0(s) = u_0(s) + \frac{(|\sigma_0|+|\sigma_1|\kappa\|\varphi\|_2+|\sigma_2|\kappa\|\varphi\|_2 e^{\lambda\tau})^2}{-2a+\sigma_1^2+2|\sigma_1\sigma_2|+\sigma_2^2}, & s \in [-\tau, 0]. \end{cases} \tag{3.38}$$

Also, let

$$V(t,y) = \tfrac{1}{2}y^2(t) + \gamma \int_{-\tau}^{0} y^2(\theta)d\theta, \quad \gamma > 0, \tag{3.39}$$

where $y(t)$ is the solution of (3.38). Then we have from condition (2), (3.38) and (3.39) that

$$\begin{aligned} &\dot{V}(t,y) \\ &= (-2a+\sigma_1^2+|\sigma_1\sigma_2|+\gamma)y^2(t) + (\sigma_2^2+|\sigma_1\sigma_2|)y(t)y(t-\tau) - \gamma y^2(t-\tau) \\ &\leq (-2a+\sigma_1^2+|\sigma_1\sigma_2|+\gamma)y^2(t) + \tfrac{(\sigma_2^2+|\sigma_1\sigma_2|)}{2}[y^2(t)+y^2(t-\tau)] - \gamma y^2(t-\tau) \\ &= [-2a+\sigma_1^2+|\sigma_1\sigma_2|+\gamma+\tfrac{(\sigma_2^2+|\sigma_1\sigma_2|)}{2}]y^2(t) + [-\gamma+\tfrac{(\sigma_2^2+|\sigma_1\sigma_2|)}{2}]y^2(t-\tau) \\ &\leq 0. \end{aligned}$$

This implies that

$$\tfrac{1}{2}y^2(t) \leq V(t,y) \leq [\tfrac{1}{2}+\gamma\tau] \max_{-\tau\leq s\leq 0} |y(s)|_1^2, \quad t \geq 0.$$

Hence Equation (3.38) is asymptotically stable. Furthermore, Equation (3.30) is asymptotically stable from (3.37). Applying Lemma 2.7 (Comparison principle) and (3.36), we see that the stochastic system (3.28) and (3.29) is 2th moment asymptotically stable. □

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