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ON THE STABILITY OF THE ORTHOGONAL PEXIDERIZED QUARTIC FUNCTIONAL EQUATIONS

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Abstract. In this paper, we consider orthogonal stability of 2-dimensional mixed type additive and quartic functional equation of the form

$$2f(2x+y)+2f(2x-y)=2g(x+y)+2g(x-y)+12g(x)-3g(y),$$

with $x \perp y$, where $\perp$ is orthogonality in the sense of Ratz.

1. Introduction

In 1940, Ulam [8] proposed the general Ulam stability problem: *Let G_1 be a group, G_2 a metric group with the metric d. Given $\varepsilon > 0$, does there exists $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality*

$$d\Big(h(xy)-h(x)h(y)\Big) < \delta, \quad (x, y \in G_1),$$

then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon, \quad (x \in G_1)?$$

Hyers [5] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, a generalized version of Hyers' theorem for approximate additive mappings was given by Aoki [1]. In 1978, Rassias [6]

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extended the theorem of Hyers by considering the unbounded cauchy difference inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p), \quad (\varepsilon \geq 0,\, p \in [0,1)). \tag{1.1}$$

Let us recall the orthogonality in the sense of Ratz.

Definition 1.1. ([7]) Suppose that X be a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(a) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(b) independence: if $x, y \in X - \{0\}$, then x, y are linearly independent;
(c) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in X$;
(d) the Thalesian property: Let P be a 2-dimensional subspace of X. If $x \in P$ and $\lambda \in \mathbb{R}^+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space equipped with a norm. Some examples of special interest are

(i) the trivial orthogonality on a vector space X defined by (a), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent,
(ii) the ordinary orthogonality on an inner product space $(X, (.,.))$ given by $x \perp y$ if and only if $(x, y) = 0$,
(iii) the Birkhoff- James orthogonality on a normed space $(X, \|.\|)$ defined by $x \perp y$ if and only if $\|x + y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly conditions (i) and (ii) are symmetric but (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \quad (x, y \in A, x \perp y), \tag{1.2}$$

in which $\perp$ is an abstract orthogonally was first investigated by Gudder and Strawther [4]. Ger and Sikkorska discussed the orthogonal stability of the equation (1.2) in [3]. Arunkumar and Hema Latha investigated the problem of the orthogonal stabilitx, y of a generalized quartic functional equation

$$\begin{aligned} &7[f(2x+y)+f(2x-y)] \\ &= 28[f(x+y)+f(x-y)] - 3[f(2y)-2f(y)] + 14[f(2x)-4f(x)], \end{aligned}$$

in Banach spaces [2].

In this paper, we deal with the next functional equation Pexider functions:

$$2f(2x+y)+2f(2x-y)=2g(x+y)+2g(x-y)+12g(x)-3g(y), \quad (1.3)$$

for all $x, y \in A$, with $x \perp y$. We will use the following notation

$$\begin{aligned} D_{f,g}(x,y) &= 2f(2x+y)+2f(2x-y)-2g(x+y) \\ &\quad -2g(x-y)-12g(x)+3g(y), \end{aligned} \quad (1.4)$$

for all $x, y \in A$, with $x \perp y$.

2. Main Results

In the class of real functionals $f, g : (X, \perp) \to \mathbb{R}$ defined on an orthogonality space in the sense of Ratz, let us consider the conditional equation (1.3). we describe its solutions first assuming that f, g are odd functionals, then even functionals, finally, using the decomposition of f, g into their even and odd parts, we describe the general solutions.

Lemma 2.1. *Let $f, g : (X, \perp) \to \mathbb{R}$ be odd real functionals satisfying* (1.3)*, then the solutions of* (1.3) *are* $f(x) = g(x) = 0$ *.*

Proof. Letting $x = 0$ in (1.3), oddness of f, g, we obtain $g(y) = 0$. Now by letting $y = 0$ in (1.3), we obtain $4f(2x) = 0$ for all $x \in X$. □

Theorem 2.2. *Let $f, g : (X, \perp) \to \mathbb{R}$ be real functionals satisfying* (1.3)*, then the solutions of* (1.3) *are given by*

$$\begin{aligned} f(x) &= Q(x) - f(0), \\ g(x) &= 4Q(x) - g(0), \end{aligned} \quad (2.1)$$

where $Q : (X, \perp) \to \mathbb{R}$ is orthogonality quartic functional.

Proof. According to Lemma (2.1), it is enough to assume $f, g : (X, \perp) \to \mathbb{R}$ be even real functionals satisfying (1.3). In equation (1.3), by letting $(x, y) = (0, 0)$, we obtain

$$4f(0) - 13g(0) = 0. \quad (2.2)$$

Replacing in (1.3), (x, y) by $(0, y)$, we obtain

$$4f(y) = g(y) + 12g(0). \quad (2.3)$$

By using (2.2) and (2.3), equation (1.3) may be rewritten as

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)-24f(0).$$

Moreovere, we have

$$\begin{aligned} & f(2x+y) - f(0) + f(2x-y) - f(0) \\ & = 4f(x+y) - 4f(0) + 4f(x-y) - 4f(0) \\ & \quad + 24f(x) - 24f(0) - 6f(y) + 6f(0). \end{aligned}$$

Now, setting $Q(x) = f(x) - f(0)$, we imply

$$Q(2x+y) + Q(2x-y) = 4Q(x+y) + 4Q(x-y) + 24Q(x) - 6Q(y).$$

Hence, Q is a quartic functional and we have

$$f(x) = Q(x) + f(0). \tag{2.4}$$

Also from (2.3) and (2.4), we have

$$g(x) = 4f(x) - 12g(0) = 4Q(x) + 4f(0) - 12g(0). \tag{2.5}$$

By adding and subtracting $g(0)$ in (2.5), and by using (2.2), we have

$$g(x) = 4Q(x) + g(0).$$

□

Through out this paper, let $(A, \perp)$ denote an orthogonality normed space with norm $\| \,.\, \|_A$ and $(B, \| \,.\, \|_B)$ is a Banach space.

In this section, we present the Hyers-Ulam-Aoki-Rassias stability of the orthogonal functional equation (1.3).

Theorem 2.3. *Let α and s, $(s < 1)$ be nonnegative real numbers, and $f_o, g_o : A \longrightarrow B$ are odd mappings satisfying*

$$\|D_{f_o,g_o}(x,y)\|_B \leq \alpha\{\|x\|_A^s + \|y\|_A^s\}, \tag{2.6}$$

for all $x, y \in A$, with $x \perp y$. Then there are unique orthogonally quartic mappings $\acute{Q}_{f_o} : A \longrightarrow B$ and $\acute{Q}_{g_o} : A \longrightarrow B$ such that

$$\|f_o(x) - \acute{Q}_{f_o}(x)\|_B \leq \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\frac{1}{1-2^{s-4}}\|x\|_A^s, \tag{2.7}$$

$$\| \, g_o(x) - \acute{Q}_{g_o}(x)\|_B \leq \frac{\alpha}{3}\frac{1}{1-2^{s-4}}\|x\|_A^s, \tag{2.8}$$

for all $x \in A$. The functions $\acute{Q}_{f_o}, \acute{Q}_{g_o}$ are defined by

$$\acute{Q}_{f_o} = \lim_{n\to\infty}\frac{f_o(2^n x)}{16^n}, \tag{2.9}$$

$$\acute{Q}_{g_o} = \lim_{n\to\infty}\frac{g_o(2^n x)}{16^n}, \tag{2.10}$$

for all $x \in A$.

Proof. By letting $(x, y) = (x, 0)$ in (2.6), we obtain

$$\|4f_o(2x) - 16g_o(x)\|_B \leq \alpha\|x\|_A^s, \tag{2.11}$$

for all $x \in A$. Setting (x, y) by $(0, y)$ in (2.6), we get

$$\|3g_o(y)\|_B \leq \alpha\|y\|_A^s, \qquad (y \in A) \tag{2.12}$$

or

$$\|g_o(y)\|_B \leq \frac{\alpha}{3}\|y\|_A^s, \qquad (y \in A). \tag{2.13}$$

Using (2.13) and (2.11), we have

$$\begin{aligned}\|4f_o(2x)\|_B &\leq \|4f_o(2x) - 16g_o(x)\|_B + \|16g_o(x)\|_B \\ &\leq \alpha\|x\|_A^s + \frac{16\alpha}{3}\|x\|_A^s = \frac{19\alpha}{3}\|x\|_A^s.\end{aligned} \tag{2.14}$$

Then

$$\| f_o(2x) \|_B \leq \frac{19\alpha}{12} \| x \|_A^s, \qquad (x \in A). \tag{2.15}$$

Replacing x by $\frac{x}{2}$ in (2.15), we get

$$\|f_o(x)\|_B \leq \frac{19\alpha}{12}\frac{1}{2^s}\|x\|_A^s, \tag{2.16}$$

for all $x \in A$. From (2.15) and (2.16), we have

$$\begin{aligned}\left\|\frac{1}{16}f_o(2x) - f_o(x)\right\|_B &\leq \left\|\frac{1}{16}f_o(2x)\right\|_B + \| - f_o(x)\|_B \\ &\leq \frac{1}{16}\frac{19\alpha}{12}\|x\|_A^s + \frac{19\alpha}{12}\frac{1}{2^s}\|x\|_A^s \\ &= \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\|x\|_A^s.\end{aligned} \tag{2.17}$$

Now replacing x by $2x$ and dividing by 16 in (2.17) and summing resulting inequality with (2.17), the following inequality is obtained

$$\left\|\frac{f_o(2^2x)}{16^2} - \frac{f_o(2x)}{16}\right\|_B \leq \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)2^{s-4}\|x\|_A^s, \tag{2.18}$$

for all $x \in A$. In general, using induction on a positive integer n, we obtain

$$\begin{aligned}\left\|\frac{f_o(2^nx)}{16^n} - f_o(x)\right\|_B &\leq \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\sum_{k=0}^{n-1} 2^{k(s-4)}\|x\|_A^s \\ &\leq \frac{19\alpha}{16}\left(\frac{1}{16} + \frac{1}{2^s}\right)\sum_{k=0}^{\infty} 2^{k(s-4)}\|x\|_A^s,\end{aligned} \tag{2.19}$$

for all $x \in A$. In order to prove the convergence of the sequence $\left\{\frac{f_o(2^n x)}{16^n}\right\}$, by replacing x by $2^n x$ and dividing by 2^m in (2.19), for any $m, n > 0$, we obtain

$$\begin{aligned}\left\|\frac{f_o(2^m 2^n x)}{16^m 16^n} - \frac{f_o(2^m x)}{16^m}\right\|_B &= \frac{1}{16^m}\left\|\frac{f_o(2^m 2^n x)}{16^n} - f_o(2^m x)\right\|_B \\ &\leq \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\sum_{k=0}^{n-1} 2^{m(s-4)} 2^{k(s-4)} \|x\|_A^s \quad (2.20) \\ &= \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\sum_{k=0}^{n-1} 2^{(s-4)(m+k)} \|x\|_A^s,\end{aligned}$$

for all $x \in A$. For $s < 4$, right hand side of (2.20) tends to zero as $m \to 0$ for all $x \in A$. Thus $\left\{\frac{f_o(2^n x)}{16^n}\right\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $\acute{Q}_{f_o} : A \longrightarrow B$ such that

$$\acute{Q}_{f_o}(x) = \lim_{n\to\infty} \frac{f_o(2^n x)}{16^n}, \qquad (x \in A).$$

Letting $n \to \infty$ in (2.13), implies

$$\begin{aligned}\left\|\lim_{n\to\infty} \frac{f_o(2^n x)}{16^n} - f_o(x)\right\|_B &\leq \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\sum_{k=0}^{\infty} 2^{k(s-4)} \|x\|_A^s \\ &= \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\frac{1}{1 - 2^{s-4}}\|x\|_A^s.\end{aligned}$$

Then formula (2.9), is satisfied. In order to prove that $\acute{Q}_{f_o}$ satisfies (1.3), we replace (x, y) by $(2^n x, 2^n y)$ in (2.6) and divide by 2^n, and so we deduce that

$$\begin{aligned}&\frac{1}{16^n}\|f_o(2^n(2x + y)) + f_o(2^n(2x - y)) - 2f_o(2^n(x + y)) \\ &\quad - 2f_o(2^n(x - y)) - 12f_o(2^n(x)) + 3f_o(2^n(y))\| \\ &\leq 2 \times 2^{n(s-4)}\{\|x\|_A^s + \|y\|_A^s\}.\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}&\acute{Q}_{f_o}(2x + y) + \acute{Q}_{f_o}(2x - y) \\ &= 2Q'_{f_o}(x + y) + 2\acute{Q}_{f_o}(x - y) - 12\acute{Q}_{f_o}(x) + 10\acute{Q}_{f_o}(x),\end{aligned}$$

for all $x, y \in A$ with $x \perp y$. Therefore $\acute{Q}_{f_o} : A \longrightarrow B$ is an orthogonally quartic mapping that satisfies (1.3). To prove the uniqueness of $\acute{Q}_{f_o}$, let $\grave{Q}_{f_o}$ be another orthogonally quartic mapping satisfying (1.3) and inquality (2.8).

Then

$$\begin{aligned}\|\acute{Q}_{f_o}(x)-\grave{Q}_{f_o}(x)\| &= \frac{1}{16^n}\|\acute{Q}_{f_o}(2^nx)-\grave{Q}_{f_o}(2^nx)\| \\ &\leq \frac{1}{16^n}(\|\acute{Q}_{f_o}(2^nx)-f_o(2^nx)\|+\|f_o(2^nx)-\grave{Q}_{f_o}(2^nx)\|) \\ &\leq \frac{2}{2^{n(4-s)}}\left(\frac{19\alpha}{12}\left(\frac{1}{16}+\frac{1}{2^s}\right)\frac{1}{1-2^{s-4}}\|x\|_A^s\right),\end{aligned}$$

which left hand side tends to zero as $n \longrightarrow \infty$ for all $x \in A$. Therefor $\acute{Q}_{f_o}$ is unique. From (2.13), we have

$$\|g_o(2x)\|_B \leq \frac{2^s\alpha}{3}\|x\|_A^s, \qquad (x \in A). \tag{2.21}$$

Now from (2.21) and (2.13), we have

$$\begin{aligned}\left\|\frac{1}{16}g_o(2x)-g_o(x)\right\|_B &\leq \left\|\frac{1}{16}g_o(2x)\right\|_B+\|-g_o(x)\|_B \\ &\leq \frac{\alpha}{3}(1+2^{s-4})\|x\|_A^s.\end{aligned} \tag{2.22}$$

By replacing x by $2x$ and dividing by 16 in (2.22) and summing resulting inequality with (2.22), we have

$$\left\|\frac{g_o(2^2x)}{16^2}-\frac{g_o(2x)}{16}\right\|_B \leq \frac{\alpha}{3}(1+2^{s-4})\frac{2^s}{16}\parallel x \parallel_A^s, \tag{2.23}$$

for all $x \in A$. In general, using induction on a positive integer n, we obtain

$$\begin{aligned}\left\|\frac{g_o(2^nx)}{16^n}-g_o(x)\right\|_B &\leq \frac{\alpha}{3}(1+2^{s-4})\sum_{k=0}^{n-1}2^{k(s-4)}\|x\|_A^s \\ &\leq \frac{\alpha}{3}(1+2^{s-4})\sum_{k=0}^{\infty}2^{k(s-4)}\|x\|_A^s,\end{aligned} \tag{2.24}$$

for all $x \in A$. Since $\left\{\frac{g_o(2^nx)}{16^n}\right\}$ is Cauchy sequence (The proof is similar to the first part) and B is complete, there exists a mapping $\acute{Q}_{g_o} : A \longrightarrow B$ such that

$$\acute{Q}_{g_o}(x) = \lim_{n\to\infty}\frac{g_o(2^nx)}{16^n}, \qquad (x \in A).$$

Letting $n \to \infty$ in (2.24), we arrive the formula (2.8) for all $x \in A$. To prove $\acute{Q}_{g_o}$ satisfies (1.3) and is unique, the proof is similar to the first part. □

Theorem 2.4. *Let α and $s(s<4)$ be nonnegative real number and $f_e, g_e : A \longrightarrow B$ are even mappings satisfying*

$$\|D_{f_e,g_e}(x,y)\|_B \leq \alpha\{\|x\|_A^s+\|y\|_A^s\}, \tag{2.25}$$

for all $x, y \in A$, with $x \perp y$. Then there are unique orthogonally quartic mappings $\acute{Q}_{f_e} : A \longrightarrow B$ and $\acute{Q}_{g_e} : A \longrightarrow B$ such that

$$\|f_e(x) - \acute{Q}_{f_e}(x)\|_B \leq \frac{17\alpha}{64}\left(\frac{1}{1-2^{s-4}}\right)\|x\|_A^s, \tag{2.26}$$

$$\|g_e(x) - \acute{Q}_{g_e}(x)\|_B \leq \frac{\alpha}{16}(1-2^s)\left(\frac{1}{1-2^{s-4}}\right)\|x\|_A^s, \tag{2.27}$$

for all $y \in A$. The functions $\acute{Q}_{f_e}, \acute{Q}_{g_e}$ are defined by

$$\acute{Q}_{f_e} = \lim_{n\to\infty} \frac{f_e(2^n x)}{16^n}, \tag{2.28}$$

$$\acute{Q}_{g_e} = \lim_{n\to\infty} \frac{g_e(2^n x)}{16^n}, \tag{2.29}$$

for all $x \in A$.

Proof. By letting $y = 0$ in (2.25), we obtain

$$\|4f_e(2x) - 16g_e(x)\|_B \leq \alpha\|x\|_A^s, \tag{2.30}$$

for all $x \in A$. Setting x by zero in (2.25), we have

$$\|4f_e(y) - g_e(y)\|_B \leq \alpha\|y\|_A^s, \tag{2.31}$$

for all $x \in A$. From (2.30) and (2.31), we get

$$\begin{aligned}
&\left\|\frac{1}{16}f_e(2x) - f_e(x)\right\|_B \\
&\leq \left\|\frac{1}{16}f_e(2x) - \frac{1}{4}g_e(x)\right\|_B + \left\|-f_e(y) + \frac{1}{4}g_e(y)\right\|_B \\
&\leq \frac{17\alpha}{64}\|x\|_A^s.
\end{aligned} \tag{2.32}$$

Now replacing x by $2x$ and dividing by 16 in (2.32) and summing resulting inequality with (2.32), also using induction on a positive integer n , we obtain

$$\begin{aligned}
\left\|\frac{f_e(2^n x)}{16^n} - f_e(x)\right\|_B &\leq \frac{17\alpha}{64}\sum_{k=0}^{n-1} 2^{k(s-4)}\|x\|_A^s \\
&\leq \frac{17\alpha}{64}\sum_{k=0}^{\infty} 2^{k(s-4)}\|x\|_A^s,
\end{aligned} \tag{2.33}$$

for all $x \in A$. Since $\left\{\frac{f_o(2^n x)}{16^n}\right\}$ is Cauchy sequence (The proof is similar to Theorem (2.3)) and B is complete, there exists a mapping $\acute{Q}_{f_e} : A \longrightarrow B$ such that

$$\acute{Q}_{f_e}(x) = \lim_{n\to\infty} \frac{f_e(2^n x)}{16^n}, \qquad (x \in A).$$

Letting $n \to \infty$ in (2.33), the formula (2.26) is satisfied for all $x \in A$. To prove $\acute{Q}_{f_e}$ satisfies (1.3) and is unique, the proof is similar to the proof of Theorem (2.3). From (2.14), we have

$$\begin{aligned} &\left\| \frac{1}{16} g_e(2x) - g_e(x) \right\|_B \\ &\le \left\| -\frac{1}{4} f_e(2x) + \frac{1}{16} g_e(2x) \right\|_B + \left\| -g_e(x) + \frac{1}{4} f_e(2x) \right\|_B \\ &\le \frac{\alpha}{16}(1+2^s)\|x\|_A^s, \qquad (x \in A). \end{aligned} \tag{2.34}$$

Now replacing x by $2x$ and dividing by 16 in (2.34) and summing resulting inequality with (2.34), and, using induction on a positive integer n, we obtain

$$\begin{aligned} \left\| \frac{g_e(2^n x)}{16^n} - g_e(x) \right\|_B &\le \frac{\alpha}{16}(1+2^{s-2}) \sum_{k=0}^{n-1} 2^{k(s)} \|x\|_A^s \\ &\le \frac{\alpha}{16}(1+2^s) \sum_{k=0}^{\infty} 2^{k(s-4)} \|x\|_A^s, \end{aligned} \tag{2.35}$$

for all $x \in A$. Since $\left\{ \frac{g_e(2^n x)}{16^n} \right\}$ is Cauchy sequence (The proof is similar to Theorem (3.1)) and B is complete, there exists a mapping $\acute{Q}_{g_e} : A \longrightarrow B$ such that

$$\acute{Q}_{g_e}(x) = \lim_{n\to\infty} \frac{g_e(2^n x)}{16^n}, \qquad (x \in A).$$

Letting $n \to \infty$ in (2.35), the formula (2.27) is correct for all $x \in A$. To prove $\acute{Q}_{g_e}$ satisfies (1.3) and is unique, the proof is similar to the proof of Theorem (3.1). □

Theorem 2.5. *Let α and $s(s < 4)$ be nonnegative real number and $f, g : A \longrightarrow B$ are mappings satisfying*

$$\|D_{f,g}(x,y)\|_B \le \alpha\{\|x\|_A^s + \|y\|_A^s\}, \tag{2.36}$$

for all $x, y \in A$, with $x \perp y$. Then there are unique orthogonally quartic mappings $Q_f : A \longrightarrow B$ and $Q_g : A \longrightarrow B$ such that

$$\begin{aligned} &\|f(x) - Q_f(x)\|_B \\ &\le \left\{ \frac{17\alpha}{64}\left(\frac{1}{1-2^{s-4}}\right) + \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\left(\frac{1}{1-2^{s-4}}\right) \right\} \|x\|_A^s, \end{aligned} \tag{2.37}$$

$$\|g(x) - Q_g(x)\|_B \le \left\{ \frac{\alpha(1+2^s)}{16}\left(\frac{1}{1-2^{s-4}}\right) + \frac{\alpha}{3}\frac{1}{1-2^{s-4}} \right\} \|x\|_A^s, \tag{2.38}$$

for all $x \in A$. The functions Q_f, Q_g are defined by

$$Q_f(x) = \lim_{n\to\infty} \frac{f(2^n x)}{16^n}, \tag{2.39}$$

$$Q_g(x) = \lim_{n\to\infty} \frac{g(2^n x)}{16^n}, \tag{2.40}$$

for all $x \in A$.

Proof. Define

$$f(x) = f_e(x) + f_o(x),$$
$$g(x) = g_e(x) + g_o(x),$$

for all $x \in A$. In (2.36), we have

$$\begin{aligned}&\|f_e(2x+y) + f_o(2x+y) + 2f_e(2x-y) + 2f_o(2x-y)\\ &\quad - 12g_e(x) - 2g_e(x+y) - 2g_o(x+y) - 2g_e(x-y)\\ &\quad - 2g_o(x-y) - 12g_o(x) + 3g_e(y) + 3g_o(y)\|\\ &\le \alpha\{\|x\|_A^s + \|y\|_A^s\}.\end{aligned} \tag{2.41}$$

Replacing (x, y) by $(-x, -y)$ in (2.41), and since $f_e(-x) = f_e(x)$, $f_o(-x) = -f_o(x)$, we have

$$\begin{aligned}&\|f_e(2x+y) - f_o(2x+y) + 2f_e(2x-y) - 2f_o(2x-y)\\ &\quad - 12g_e(x) - 2g_e(x+y) + 2g_o(x+y) - 2g_e(x-y)\\ &\quad + 2g_o(x-y) + 12g_o(x) + 3g_e(y) - 3g_o(y)\|\\ &\le \alpha\{\|x\|_A^s + \|y\|_A^s\}.\end{aligned} \tag{2.42}$$

Then

$$\begin{aligned}&\| - f_e(2x+y) + f_o(2x+y) - 2f_e(2x-y) + 2f_o(2x-y)\\ &\quad - 12g_o(x) + 12g_e(x) + 2g_e(x+y) - 2g_o(x+y)\\ &\quad + 2g_e(x-y) - 2g_o(x-y) - 3g_e(y) + 3g_o(y)\|\\ &\le \alpha\{\|x\|_A^s + \|y\|_A^s\}.\end{aligned} \tag{2.43}$$

By summing (2.41) and (2.42), we get

$$\|D_{f_e,g_e}(x,y)\|_B \le \alpha\{\|x\|_A^s + \|y\|_A^s\}.$$

By summing (2.41) and (2.43), we obtain

$$\|D_{f_o,g_o}(x,y)\|_B \le \alpha\{\|x\|_A^s + \|y\|_A^s\}.$$

By Theorem (2.3) and (2.4), we have

$$\|f_e(x) - \acute{Q}_{f_e}(x)\|_B \le \frac{17\alpha}{64}\left(\frac{1}{1-2^{s-4}}\right)\|x\|_A^s, \tag{2.44}$$

$$\|f_o(x) - \acute{Q}_{f_o}(x)\|_B \leq \frac{19\alpha}{12}\left(\frac{1}{16} + \frac{1}{2^s}\right)\left(\frac{1}{1-2^{s-4}}\right)\|x\|_A^s \tag{2.45}$$

and

$$\|g_e(x) - \acute{Q}_{g_e}(x)\|_B \leq \frac{\alpha(1+2^s)}{16}\frac{1}{1-2^{s-4}}\|x\|_A^s, \tag{2.46}$$

$$\|g_o(x) - \acute{Q}_{g_o}(x)\|_B \leq \frac{\alpha}{3}\frac{1}{1-2^{s-4}}\|x\|_A^s. \tag{2.47}$$

From (2.44), (2.45), (2.46) and (2.47), we have

$$\|f(x)-Q_f(x)\|_B \leq \left\{\frac{17\alpha}{64}\left(\frac{1}{1-2^{s-4}}\right) + \frac{19\alpha}{12}\left(\frac{1}{16}+\frac{1}{2^s}\right)\left(\frac{1}{1-2^{s-4}}\right)\right\}\|x\|_A^s,$$

$$\|g(x) - Q_g(x)\|_B \leq \left\{\frac{\alpha}{16}(1-2^{s-2})\left(\frac{1}{1-2^s}\right) + \frac{\alpha}{3}\frac{1}{1-2^{s-4}}\right\}\|x\|_A^s,$$

for all $x \in A$. Hence the proof completes. □

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