Nonlinear Functional Analysis and Applications
Vol. 20, No. 4 (2015), pp. 551-559

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A GENERALIZATION OF CARISTI KIRK'S THEOREM FOR COMMON FIXED POINTS ON G-METRIC SPACES

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Abstract. In this article, lower semi-continuous maps are used to generalize Cristi-Kirk's fixed point theorem on G-metric spaces. Some more general results are also obtained in G-metric spaces.

1. Introduction

Recently, Mustafa and Sims [8] introduced a new structure of generalized metric spaces, which are called G-metric spaces as generalization of metric space (X, d), to develop and introduce a new fixed point theory for various mappings in this new structure. Some authors [2, 9, 14] have proved some fixed point theorems in these spaces. Fixed point problems of contractive mappings in metric spaces endowed with a partially order have been studied in a number of works. Some recent references on this topic are the works noted in [1, 3, 6, 7]. Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. For example, Dhage [4] and Gähler [5] introduced

[0]Received March 18, 2015. Revised July 9, 2015.
[0]2010 Mathematics Subject Classification: 54H25, 47H10.
[0]Keywords: Fixed point, G-metric.

the concepts of 2-matric spaces and D-metric spaces, respectively, but some authors pointed out that these attempts are not valid (see [10, 11, 12, 13, 15]).

First, we present some known definitions and propositions in G-metric spaces.

Definition 1.1. ([8]) Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

- (G_1) $G(x, y, z) = 0$ if $x = y = z$,
- (G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
- (G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
- (G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables,
- (G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. ([8]) Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. A point $x \in X$ is said to be limit of $\{x_n\}$ iff

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0.$$

In this case, the sequence $\{x_n\}$ is said to be G-convergent to x.

Definition 1.3. ([8]) Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G-Cauchy iff

$$\lim_{n,\ m,\ l\to\infty} G(x_l, x_n, x_m) = 0.$$

(X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.4. ([8]) *In a G-metric space (X, G), the following are equivalent.*

1. *The sequence $\{x_n\}$ is G-Cauchy.*
2. *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Proposition 1.5. ([8]) *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 1.6. ([8]) *Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, it follows that*

(i) *if* $G(x,y,z)=0$ *then* $x=y=z$,
(ii) $G(x,y,z)\le G(x,x,y)+G(x,x,z)$,
(iii) $G(x,y,y)\le 2G(x,x,y)$,
(iv) $G(x,y,z)\le G(x,a,z)+G(a,y,z)$,
(v) $G(x,y,z)\le \frac{2}{3}[G(x,a,a)+G(y,a,a)+G(z,a,a)]$.

Proposition 1.7. ([8]) *Let* (X,G) *be a* G*-metric space. Then for a sequence* $\{x_n\}\subseteq X$ *and a point* $x\in X$*, the following are equivalent*
(i) $\{x_n\}$ *is* G*-convergent to* x,
(ii) $G(x_n,x_n,x)\to 0$ *as* $n\to\infty$,
(iii) $G(x_n,x,x)\to 0$ *as* $n\to\infty$,
(iv) $G(x_m,x_n,x)\to 0$ *as* $m,n\to\infty$.

We can find some examples and basic properties of G-metric spaces in Mustafa and Sims [8].

2. Main results

Lemma 2.1. *Let* (X,G) *be a* G*-metric space and* $\varphi: X\longrightarrow\mathbb{R}$*. Define the relation* $\preceq$ *on* X *as follows:*

$$x\preceq y \Longleftrightarrow G(x,y,y)\le\varphi(x)-\varphi(y).$$

Then $\preceq$ *is a (partial) order on* X *induced by* φ.

Proof. (i) It is easy to see that $x\preceq x$.
(ii) Let $x\preceq y$ then $G(x,y,y)\le\varphi(x)-\varphi(y)$. Also, if $y\preceq x$ then $G(y,x,x)\le\varphi(y)-\varphi(x)$. Therefore,

$$G(x,y,y)+G(y,x,x)\le 0,$$

thus $x=y$.
(iii) Let $x\preceq y$ then $G(x,y,y)\le\varphi(x)-\varphi(y)$. Also, if $y\preceq z$ then $G(y,z,z)\le\varphi(y)-\varphi(z)$. Therefore,

$$G(x,z,z)\le G(x,y,y)+G(y,z,z)\le\varphi(x)-\varphi(z),$$

thus $x\preceq z$. □

Definition 2.2. Let (X,G) be a G-metric space.
(i) Let $T: X\longrightarrow X$ be an arbitrary self-mapping on X such that

$$G(x,Tx,Tx)\le\varphi(x)-\varphi(Tx)$$

for all $x\in X$, then T is called a Caristi map on (X,G).
(ii) Let $S,T: X\longrightarrow X$ be two selfmappings on X such that

$$G(Sx,Tx,Tx)\le\varphi(Sx)-\varphi(Tx)$$

for all $x \in X$, then T is called a S-Caristi map on (X, G).

Theorem 2.3. *Let (X, G) be a complete G-metric space and $\varphi : X \longrightarrow \mathbb{R}$ be a lower semi-continuous function which is bounded below and $\preceq$ the order introduced by φ. Let $S, T : X \longrightarrow X$ be two self-mappings such that T is a S-Caristi map on (X, G). If $S(X)$ be a closed subspace of X then there exists $z \in X$ such that $Sz = Tz$.*

Proof. For each $x \in X$, define

$$\begin{aligned} H(x) &= \{z \in X : Sx \preceq z\}, \\ \alpha(x) &= \inf\{\varphi(z) : z \in H(x)\}. \end{aligned} \tag{2.1}$$

Since $Sx \in H(x)$, then $H(x) \neq \emptyset$. From (2.1), we have $\alpha(x) \leq \varphi(Sx)$.

Take $x \in X$ and say $x = x_0$. We construct a sequence $\{x_n\}$ in the following way:

$$\begin{aligned} &x_1 := Sx, \\ &Sx_{n+1} \in H(x_n) \text{ such that } \varphi(Sx_{n+1}) \leq \alpha(x_n) + \frac{1}{n}, \ \forall\, n \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Thus, one can easily observe that

$$\begin{aligned} &G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq \varphi(Sx_n) - \varphi(Sx_{n+1}), \\ &\alpha(x_n) \leq \varphi(Sx_{n+1}) \leq \alpha(x_n) + \frac{1}{n}, \ \forall\, n \in \mathbb{N}. \end{aligned} \tag{2.3}$$

Note that (2.3) implies that $\{\varphi(Sx_n)\}$ is a decreasing sequence of real numbers and it is bounded. Therefore, the sequence $\{\varphi(Sx_n)\}$ is convergent to some positive real number, say L. Thus, regarding (2.3), we have

$$L = \lim_{n\to\infty} \varphi(Sx_n) = \lim_{n\to\infty} \alpha(x_n). \tag{2.4}$$

From (2.3) and (2.4), for each $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$\varphi(Sx_n) \leq L + \frac{1}{k}, \ \forall\, n \geq N_k. \tag{2.5}$$

Regarding the monotonicity of $\{\varphi(Sx_n)\}$, for $m \geq n \geq N_k$, we have

$$L \leq \varphi(Sx_m) \leq \varphi(Sx_n) \leq L + \frac{1}{k}. \tag{2.6}$$

Thus, we obtain

$$\varphi(Sx_n) - \varphi(Sx_m) < \frac{1}{k}, \ \forall\, m \geq n \geq N_k. \tag{2.7}$$

On the other hand, taking (2.3) into account, together with the triangle inequality, we observe that

$$\begin{aligned} G(Sx_n, Sx_{n+2}, Sx_{n+2}) &\le G(Sx_n, Sx_{n+1}, Sx_{n+1}) + G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) \\ &\le \varphi(Sx_{n+1}) - \varphi(Sx_{n+2}) + \varphi(Sx_n) - \varphi(Sx_{n+1}). \end{aligned}$$

Thus

$$G(Sx_n, Sx_{n+2}, Sx_{n+2}) \le \varphi(Sx_n) - \varphi(Sx_{n+2}). \tag{2.8}$$

Analogously,

$$\begin{aligned} G(Sx_n, Sx_{n+3}, Sx_{n+3}) &\le G(Sx_n, Sx_{n+2}, Sx_{n+2}) + G(Sx_{n+2}, Sx_{n+3}, Sx_{n+3}) \\ &\le \varphi(Sx_n) - \varphi(Sx_{n+2}) + \varphi(Sx_{n+2}) - \varphi(Sx_{n+3}). \end{aligned}$$

Thus

$$G(Sx_n, Sx_{n+3}, Sx_{n+3}) \le \varphi(Sx_n) - \varphi(Sx_{n+3}). \tag{2.9}$$

By induction, we obtain that

$$G(Sx_n, Sx_m, Sx_m) \le \varphi(Sx_n) - \varphi(Sx_m), \ \ \forall\, m \ge n \tag{2.10}$$

and taking (2.7) into account, (2.10) turns into

$$G(Sx_n, Sx_m, Sx_m) \le \varphi(Sx_n) - \varphi(Sx_m) < \frac{1}{k}, \ \ \forall\, m \ge n \ge N_k. \tag{2.11}$$

Since the sequence $\{\varphi(Sx_n)\}$ is convergent which implies that the right-hand side of (2.11) tends to zero. That is $\{Sx_n\}$ is a Cauchy sequence in the G-metric space (X, G). Since (X, G) is complete then the sequence $\{Sx_n\}$ converges in the G-metric space (X, G), say $\lim\limits_{n\to\infty} G(Sx_n, Sx_n, x^*) = 0$. Since $S(X)$ is a closed subspace of X, there exists $z \in X$ such that $\lim_{n\to\infty} Sx_n = x^* = Sz$.

On the other hand, with the triangle inequality, we observe that

$$\begin{aligned} G(Sx_n, Tz, Tz) &\le G(Sx_n, Sz, Sz) + G(Sz, Tz, Tz) \\ &\le \varphi(Sx_n) - \varphi(Sz) + \varphi(Sz) - \varphi(Tz). \end{aligned}$$

That is

$$G(Sx_n, Tz, Tz) \le \varphi(Sx_n) - \varphi(Tz).$$

Hence, $Tz \in H(x_n)$ for all $n \in \mathbb{N}$ which yields that $\alpha(x_n) \le \varphi(Tz)$ for all $n \in \mathbb{N}$. From (2.4), the inequality $L \le \varphi(Tz)$ is obtained. Moreover, by lower semi-continuous of φ, we have

$$\varphi(Sz) \le \liminf_{n\to\infty} \varphi(Sx_n) = L \le \varphi(Tz).$$

Since T is S-Caristi for each $x \in X$, then we have $\varphi(Tz) \le \varphi(Sz)$. Hence $\varphi(Tz) = \varphi(Sz)$. Therefore,

$$G(Sz, Tz, Tz) \le \varphi(Sz) - \varphi(Tz) = 0.$$

Regarding definition, $Tz = Sz$. □

Corollary 2.4. *Let (X, G) be a complete G-metric and $\varphi : X \longrightarrow \mathbb{R}$ be a lower semi-continuous function which is bounded below and $\preceq$ the order introduced by φ. Let $T : X \longrightarrow X$ be a self-mapping such that T be a Caristi map on (X, G). Then there exists $z \in X$ such that $Tz = z$.*

Theorem 2.5. *Let (X, G) be a complete G-metric and let $T : X \longrightarrow X$ be a selfmap, satisfying for all $x, y, z \in X$ and $0 < k < \frac{1}{3}$ the condition*

$$\frac{1}{2}G(x, x, Tx) \leq G(x, y, z) \Longrightarrow G(Tx, Ty, Tz) \leq kG(x, y, z). \tag{2.12}$$

Then T has a unique fixed point in X.

Proof. Putting $y = x$ and $z = Tx$ in (2.12). Hence from

$$\frac{1}{2}G(x, x, Tx) \leq G(x, x, Tx),$$

it follows

$$G(Tx, Tx, T^2x) \leq kG(x, x, Tx), \tag{2.13}$$

for every $x \in X$. Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ by $x_1 = Tx_0$ and $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. By (2.13), we have

$$\begin{aligned} G(x_{n+1}, x_{n+1}, x_{n+2}) &= G(Tx_n, Tx_n, T^2x_n) \\ &\leq kG(x_n, x_n, Tx_n) = kG(x_n, x_n, x_{n+1}) \\ &\vdots \\ &\leq k^nG(x_0, x_0, x_1). \end{aligned}$$

Hence, $G(x_{n+1}, x_{n+2}, x_{n+2}) \leq 2G(x_{n+1}, x_{n+1}, x_{n+2}) \leq 2k^nG(x_0, x_0, x_1)$. Also, by Axioms G_5 of Definition of G-metric spaces, we have

$$\begin{aligned} &G(x_n, x_m, x_m) \\ &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq 2k^nG(x_0, x_0, x_1) + 2k^{n+1}G(x_0, x_0, x_1) + \cdots + 2k^{m-1}G(x_0, x_0, x_1) \\ &= 2\frac{k^n - k^m}{1 - k}G(x_0, x_0, x_1) \\ &\leq 2\frac{k^n}{1 - k}G(x_0, x_0, x_1) \longrightarrow 0. \end{aligned}$$

Hence, $\{x_n\}$ is a G-Cauchy sequence. Since X is G-complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. That is,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z.$$

Let us prove now that

$$G(z, Tx, Tx) \leq kG(z, x, x),$$

holds for each $x \neq z$. Since $G(x_n, x_n, Tx_n) \to 0$ and $G(x_n, x, x) \to G(z, x, x) \neq 0$, it follows that there exists a $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2}G(x_n, x_n, Tx_n) \leq G(x_n, x, x),$$

holds for every $n \geq n_0$. Assumption (2.12) implies that for such n

$$G(Tx_n, Tx, Tx) \leq kG(x_n, x, x),$$

thus as $n \to \infty$ (and continuity of G), we get that

$$G(z, Tx, Tx) \leq kG(z, x, x). \tag{2.14}$$

On the other hand,

$$G(z, Tz, Tz) \leq G(z, Tx, Tx) + G(Tx, Tz, Tz).$$

Therefore for $x \neq z$ by using (2.14), we have

$$G(z, Tz, Tz) \leq kG(z, x, x) + G(Tx, Tz, Tz). \tag{2.15}$$

We prove that $Tz = z$. For, if $Tz \neq z$, putting $x = Tz$ in (2.15) inequality we get

$$G(z, Tz, Tz) \leq kG(z, Tz, Tz) + G(T^2z, Tz, Tz),$$

by using (2.13) we have

$$G(z, Tz, Tz) \leq kG(z, Tz, Tz) + kG(z, z, Tz),$$

thus

$$(1 - k)G(z, Tz, Tz) \leq kG(Tz, z, z) \leq 2kG(z, Tz, Tz).$$

Since $k < \frac{1}{3}$ it follows that $\frac{2k}{1-k} < 1$, hence

$$G(z, Tz, Tz) \leq \frac{2k}{1-k}G(z, Tz, Tz) < G(z, Tz, Tz),$$

which is contradiction. Thus, we have proved that z is a fixed point of T. The uniqueness of the fixed point follows easily from (2.12). Indeed, if y, z are two fixed points of T,

$$0 = \frac{1}{2}G(z, z, z) = \frac{1}{2}G(z, z, Tz) \leq G(z, y, y),$$

then (2.12) implies that

$$G(z, y, y) = G(Tz, Ty, Ty) \leq kG(z, y, y) < G(z, y, y),$$

whereform $y = z$. □

Example 2.6. Let $X = [0, \infty)$ and $G(x, y) = |x - y| + |y - z| + |x - z|$, then (X, G) is a G-metric space. Suppose $T : X \to X$ such that $Tx = \frac{3x}{8}$ and $Sx = \frac{x}{2}$ for all $x \in X$ and $\varphi : X \to [0, \infty)$ such that $\varphi(x) = 8x$. Then

$$G(Sx, Tx, Tx) = 2\left|\frac{x}{2} - \frac{3x}{8}\right| = \frac{x}{4},$$

and $\varphi(Sx) - \varphi(Tx) = x$. Other conditions of Theorem 2.3 are also satisfied. Therefore T and S have coincidence point; indeed $x = 0$ is the required point.

Example 2.7. Let $X = [0, \pi]$ and $G(x, y) = |x - y| + |y - z| + |x - z|$. If define the relation $\preceq$ on X as follows:

$$x \preceq y \Longleftrightarrow y \leq x.$$

Then $\preceq$ is a (partial) order on X induced by φ and (X, G) is a G-metric space. Suppose $T : X \to X$ such that $Tx = \sin(x)$ for all $x \in X$ and $\varphi : X \to [0, \infty)$ such that $\varphi(x) = 4x$. Then

$$G(x, Tx, Tx) = 2|x - \sin(x)| = 2x - 2\sin(x)$$

and $\varphi(x) - \varphi(Tx) = 4x - 4\sin(x)$. Other conditions of Corollary 2.4 are also satisfied. Then there exists $0 \in X$ such that $T0 = 0$, indeed $x = 0$ is the required point.

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