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MAKING NEW KKM SPACES FROM OLD

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Abstract. In our previous works [29, 30], we obtained three general KKM type theorems A, B, and C for abstract convex spaces. In a recent work [34], we showed that these three theorems are mutually equivalent. Actually, by adopting a method of making new abstract convex spaces from old, we gave a direct proof of Theorem C from Theorem B. In this paper, we apply our new method to ϕ_A-spaces and other spaces appeared in the KKM theory. Especially, we define some new abstract convex spaces called Ψ^l-spaces, Ψ^u-spaces, and Ψ-spaces generalizing ϕ_A-spaces.

1. Introduction

In 1929, Knaster, Kuratowski and Mazurkiewicz obtained a celebrated intersection theorem (the KKM theorem for short), which concerned with a particular type of multimaps called KKM maps later. The KKM theory is the study of applications of various equivalent formulations of the KKM theorem and their generalizations.

From 1961, Ky Fan showed that the KKM theorem provides the foundation for many of the modern essential results in diverse areas of mathematical sciences. Consequently, at the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961-1984. Then, the KKM theory was extended to convex spaces by Lassonde in 1983, and to c-spaces (or H-spaces) by Horvath

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in 1983-1993 and others. Since 1993 the theory is extended to generalized convex (G-convex) spaces in a sequence of articles of the present author and others; see [12].

While G-convex spaces were investigated by a large number of authors, the concept has been challenged by several authors who aimed to obtain more general concepts. In fact, a number of modifications or imitations of G-convex spaces followed. It is known in 2007-2009 [20-26] that all of such spaces belong to the class of ϕ_A-spaces. Furthermore, since 2006, all of the above mentioned classes of spaces are unified to that of abstract convex spaces [15, 22, 27], and the KKM theory tends to the research of such new spaces.

In our previous works [29, 30], we obtained three general KKM type theorems A, B, and C for abstract convex spaces. In a recent work [34], we showed that these three theorems are mutually equivalent. Actually, by adopting a method of making new abstract convex spaces from old, we gave a direct proof of Theorem C from Theorem B. In the present article, we apply our new method to ϕ_A-spaces and other spaces appeared in the KKM theory. Moreover, we obtain generalizations of ϕ_A-spaces. In fact, we define some new abstract convex spaces called Ψ^l-spaces, Ψ^u-spaces, and Ψ-spaces.

In Section 2, the basic concepts on our abstract convex spaces are given as a preliminary. Section 3 concerns with the standard forms of our general KKM type theorems and the method of making new KKM spaces from old. In Section 4, we recall that ϕ_A-spaces are KKM spaces. Section 5 concerns with generalizations of ϕ_A-spaces. In fact, we define new abstract convex spaces called Ψ^l-spaces, Ψ^u-spaces, and Ψ-spaces (which are KKM spaces). In Section 6, several variants of Ψ-spaces previously due to other authors are introduced. Finally, in Section 7, some historical remarks are added.

2. Abstract convex spaces and the KKM spaces

A multimap $F : X \multimap Y$ is a function $F : X \to 2^Y$ to the power set of Y and $F^- : Y \multimap X$ is defined by $F^-(y) := \{x \in X : y \in F(x)\}$ for $y \in Y$. Multimaps are also simply called maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

The following is the original Knaster-Kuratowski-Mazurkiewicz theorem and its open valued version:

Theorem (KKM)**.** *Let F_i $(0 \leq i \leq n)$ be $n+1$ closed [resp. open] subsets of an n-simplex $v_0v_1 \cdots v_n$. If the inclusion relation $v_{i_0}v_{i_1} \cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$*

holds for all faces $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, *then* $\bigcap_{i=0}^{n} F_i \neq \emptyset$.

Recall the following in [15, 22, 27] and others:

Definition 2.1. Let E be a topological space, D a nonempty set, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A : A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

If E is compact, then $(E, D; \Gamma)$ is called a *compact* abstract convex space.

Example 2.2. The following are typical examples of abstract convex spaces. Others can be seen in [27] and the references therein.
(1) A *convex space* $(X, D) = (X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$.
(2) A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where Δ_J is the face of Δ_n corresponding to $J \in \langle A\rangle$.

Definition 2.3. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition 2.4. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp. a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp. $F \in \mathfrak{KO}(E, D, Z)$].

We have plenty of examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps, see [16-18, 33].

Definition 2.5. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Example 2.6. (1) Known examples of KKM spaces are given in [27]. Recently, Kulpa and Szymanski [4] found some partial KKM spaces which are not KKM spaces.
(2) Let $E = \{0, 1\}$ with the discrete topology, $D = \mathbb{R}$ the set of real numbers, and $\Gamma : \langle D\rangle \multimap E$ is defined by $\Gamma(A) = \{0\}$ for $A \subset \mathbb{Q}$ the set of rational numbers, $\Gamma(A) = \{1\}$ for $A \subset \mathbb{R} \setminus \mathbb{Q}$, and $\Gamma(A) = E$ for other cases.

Let $G : D \multimap E$ be defined by $G(x) = \{0\}$ if $x \in \mathbb{Q}$ and $G(x) = \{1\}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then G is a KKM map, but $(E, D; \Gamma)$ is not a (partial) KKM space.

We had the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

3. New KKM spaces from old

In this section we introduce a method of making new KKM spaces from old.

In [29, 30], we gave standard forms of the general KKM type theorems as follows:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp. $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \multimap E$ a multimap satisfying*

(1) *G has closed [resp. open] values; and*
(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).*

Then $\{G(y)\}_{y \in D}$ has the finite intersection property.

Further, if $\bigcap_{y \in M} \overline{G(y)}$ is compact for some $M \in \langle D \rangle$, then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Recall that Theorem A is a simple consequence of the definitions of the partial KKM principle or the KKM principle.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.
(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).
(c) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).
(d) G is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type as follows:

Theorem B. *Let $(E, D; \Gamma)$ be a partial KKM space [that is, $1_E \in \mathfrak{KC}(E, D, E)$] and $G : D \multimap E$ a map such that*

(1) *$\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D \rangle$]; and*
(2) *there exists a nonempty compact subset K of E such that either*
  (i) *$\bigcap \{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D \rangle$; or*
  (ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$\overline{L_N} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have $K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset$. Furthermore,

(α) *if G is transfer closed-valued, then $K \cap \bigcap \{G(y) \mid y \in D\} \neq \emptyset$;*
(β) *if G is intersectionally closed-valued, then $\bigcap \{G(y) \mid y \in D\} \neq \emptyset$.*

Recall that conditions (i) and (ii) in Theorem B are usually called the *compactness conditions* or the *coercivity conditions*, and (ii) has numerous variations or particular forms appeared in a very large number of literature. Note that Theorem B can be easily deduced from the compact case of Theorem A; see [29, 30].

The following are recently given in [34].

Definition 3.1. Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \multimap Z$ a map. Let $\Lambda_A := F(\Gamma_A)$ for each $A \in \langle D \rangle$. Then $(Z, D; \Lambda)$ is called the *abstract convex space induced by F*.

Let $Y \subset Z$ and $D' \subset D$ such that $\Lambda_B \subset Y$ for each $B \in \langle D' \rangle$. Then Y is called a Λ-*convex subset* of Z relative to D', and $(Y, D'; \Lambda')$ a *subspace* of $(Z, D; \Lambda)$ whenever $\Lambda' = \Lambda|_{\langle D' \rangle}$.

Proposition 3.2. *A KKM map $G : D \multimap Z$ on an abstract convex space $(E, D; \Gamma)$ with respect to $F : E \multimap Z$ is simply a KKM map on the corresponding abstract convex space $(Z, D; \Lambda)$ induced by F.*

Proof. Simply note that $\Lambda_A := F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$. □

Proposition 3.3. *For an abstract convex space $(E, D; \Gamma)$, the corresponding abstract convex space $(Z, D; \Lambda)$ induced by $F : E \multimap Z$ is a partial KKM space if and only if $F \in \mathfrak{KC}(E, D, Z)$.*

The abstract convex space $(Z, D; \Lambda)$ induced by $F : E \multimap Z$ is a KKM space if and only if $F \in \mathfrak{KC}(E, D, Z) \cap \mathfrak{KO}(E, D, Z)$.

Proof. $(Z, D; \Lambda)$ is a partial KKM space
$\Longleftrightarrow$ For every closed-valued KKM map $G : D \multimap Z$ (that is, $\Lambda_A = F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$), it has the finite intersection property of map-values.
$\Longleftrightarrow$ For every closed-valued KKM map $G : D \multimap Z$ with respect to F, it has the finite intersection property of map-values.
$\Longleftrightarrow$ $F \in \mathfrak{KC}(E, D, Z)$.

Similarly, for an open-valued KKM map $G : D \multimap Z$, it has the finite intersection property of map-values $\Longleftrightarrow$ $F \in \mathfrak{KO}(E, D, Z)$. □

4. On ϕ_A-spaces: Revisited

From 2007, the following became one of the main themes of the KKM theory [20-25]:

Definition 4.1. A space having a family $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-space

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle}) \text{ or simply } (X, D; \phi_A)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with the cardinality $|A| = n+1$. For a ϕ_A-space $(X, D; \phi_A)$, a subset C of X is said to be ϕ_A-convex relative to $D' \subset D$ if for each $A \in \langle D' \rangle$, we have $\phi_A(\Delta_{|A|-1}) \subset C$.

Every ϕ_A-space $(X; D; \phi_A)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with $|A| = n+1$ is a KKM space which is not G-convex in general.

If we put $\Gamma_A := \phi_A(\Delta_n)$, then any ϕ_A-space becomes an abstract convex space. Moreover, by different methods, ϕ_A-space can be made into a G-convex space; see [31].

Example 4.2. There are lots of examples of ϕ_A-spaces; see [20, 21, 23-27] and the references therein. So, the KKM theory was extended to the study of KKM maps on ϕ_A-spaces. Note that Khanh et al. studied ϕ_A-spaces using the name of GFC-spaces; see [35].

Definition 4.3. For a ϕ_A-space $(X, D; \phi_A)$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is called a *KKM map.*

This definition contains various previous particular ones.

Proposition 4.4. *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \phi_A)$ is a KKM map on the corresponding abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for all $A \in \langle D \rangle$ with $|A| = n + 1$.*

Proof. 1. From the definitions of a KKM map $T : D \multimap X$ on $(X, D; \phi_A)$ and $\Gamma_A := \phi_A(\Delta_n)$, we have

$$\Gamma_A = \phi_A(\Delta_{|A|-1}) \subset T(A);\ \Gamma_J = \phi_J(\Delta_{|J|-1}) \subset T(J)$$

for all $A, J \in \langle D \rangle$. This immediately implies T is a KKM on the abstract convex space $(X, D; \Gamma)$. □

Proof. 2. Define $\Gamma^T : \langle D \rangle \multimap X$ by $\Gamma^T_A := T(A)$ for each $A \in \langle D \rangle$. Then $(X, D; \Gamma^T)$ becomes an abstract convex space. Note that $\Gamma_A \subset T(A)$ for each $A \in \langle D \rangle$ and hence $T : D \multimap X$ is a KKM map on the abstract convex space $(X, D; \Gamma^T)$. □

Proposition 4.5. *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \phi_A)$ is a KKM map on a new abstract convex space $(X, D; \Gamma^T)$.*

Proof. Define $\Gamma^T : \langle D \rangle \multimap X$ by $\Gamma^T_A := T(A)$ for each $A \in \langle D \rangle$. Then $(X, D; \Gamma^T)$ becomes an abstract convex space. Note that $\Gamma^T_A \subset T(A)$ for each $A \in \langle D \rangle$ and hence $T : D \multimap X$ is a KKM map on the abstract convex space $(X, D; \Gamma^T)$. □

The following is a KKM theorem for ϕ_A-spaces. The proof is just a simple modification of the corresponding previous one.

Theorem 4.6. *For a ϕ_A-space $(X, D; \phi_A)$, let $G : D \multimap X$ be a KKM map with closed values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property.* (*More precisely, for each $A \in \langle D\rangle$ with $|A| = n+1$, we have $\phi_A(\Delta_n) \cap \bigcap_{z\in A} G(z) \neq \emptyset$.*) *Further, if*

$$\bigcap_{z\in M} G(z) \text{ is compact for some } M \in \langle D\rangle, \qquad (*)$$

then we have $\bigcap_{z\in D} G(z) \neq \emptyset$.

Proof. Let $A = \{z_0, z_1, \ldots, z_n\} \in \langle D\rangle$. Since $G : D \multimap X$ is a KKM map, for each vertex e_i of Δ_n, we have $\phi_A(e_i) \subset G(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi^{-1}G(z_i)$ is a closed valued map since ϕ_A is continuous. Moreover, $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi^{-1}G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM theorem, $\Delta_n \supset \bigcap_{i=0}^{n} \phi^{-1}G(z_i) \neq \emptyset$ and hence $\phi_A(\Delta_n) \cap \bigcap_{z\in A} G(z) \neq \emptyset$. The second conclusion is clear. □

Theorem 4.7. *For a ϕ_A-space $(X, D; \phi_A)$, let $G : D \multimap X$ be a KKM map with open values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property.*

Proof. Let $A = \{z_0, z_1, \ldots, z_n\} \in \langle D\rangle$. Since $G : D \multimap X$ be a KKM map, for each vertex e_i of Δ_n, we have $\phi_A(e_i) \subset G(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi^{-1}G(z_i)$ is an open valued map since ϕ_A is continuous. Moreover, $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi^{-1}G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the KKM theorem for open valued KKM maps, $\Delta_n \supset \bigcap_{i=0}^{n} \phi^{-1}G(z_i) \neq \emptyset$ and hence $\phi_A(\Delta_n) \cap \bigcap_{z\in A} G(z) \neq \emptyset$. □

Corollary 4.8. *Every ϕ_A-space is a KKM space.*

The above proofs were given in [31]. However, we have another proofs by adopting our new method as follows:

Proofs of Theorems 4.6, 4.7 and Corollary 4.8. Let $(X, D; \phi_A)$ be a ϕ_A-space, $G : D \multimap X$ be a KKM map with closed (or open) values, and $A \in \langle D\rangle$. Then $(\Delta_{|A|-1}, A; \mathrm{co})$ is an abstract convex space where A can be regarded as the set of vertices of $\Delta_{|A|-1}$. Define a new abstract convex space $(X, A; \Lambda)$ induced by $\phi_A : \Delta_{|A|-1} \to X$ with $\Lambda(J) := \phi_A(\Delta_{|J|-1})$ for each $J \in \langle A\rangle$, where $\Delta_{|J|-1}$ is the face of $\Delta_{|A|-1}$ corresponding to $J \subset A$. Since ϕ_A is a continuous map, we know $\phi_A \in \mathfrak{KC}(\Delta_{|A|-1}, A, X) \cap \mathfrak{KO}(\Delta_{|A|-1}, A, X)$ (see [16-18]). Therefore, by Proposition 3.3, $(X, A; \Lambda)$ is a KKM space(Hence it satisfies Theorem A). Since $G|_A$ is a KKM map, we have $\bigcap_{y\in A} G(y) \neq \emptyset$. This means $(X, D; \phi_A)$ is a KKM space. □

5. Generalizations of ϕ_A-spaces

For topological spaces X and Y, a map $F : X \multimap Y$ with nonempty values, and a subset $B \subset Y$, we define

$$F^+(B) := \{x \in X : F(x) \subset B\}, \quad F^-(B) := \{x \in X : F(x) \cap B \neq \emptyset\}.$$

We say that F is

(i) *upper semicontinuous* (u.s.c.) iff $F^-(C)$ is closed in X for each closed set $C \subset Y$, or $F^+(O)$ is open in X for each open set $O \subset Y$;
(ii) *lower semicontinuous* (l.s.c.) iff $F^-(O)$ is open in X for each open set $O \subset Y$, or $F^+(C)$ is closed in X for each closed set $C \subset Y$; and
(iii) *continuous* (u.s.c.) iff it is u.s.c. and l.s.c.

In this section, we give another examples of (partial) KKM spaces extending ϕ_A-spaces:

Definition 5.1. An abstract convex space $(X, D; \Gamma)$ is called a Ψ^l-space [resp. Ψ^u-space] if, for each $N \in \langle D \rangle$, there exists a l.s.c. [resp. an u.s.c.] map $F : \Delta_n \multimap X$ such that $F(\Delta_J) \subset \Gamma(J)$ for all $J \subset N = \{y_0, y_1, \ldots, y_n\}$, where Δ_J denotes the face of Δ_n corresponding to $J \subset N$.

An abstract convex space $(X, D; \Gamma)$ is called a Ψ-space if, for each $N \in \langle D \rangle$, there exists a continuous multimap $F : \Delta_n \multimap X$ such that $F(\Delta_J) \subset \Gamma(J)$ for all $J \subset N$.

A Ψ^l-space [resp. Ψ^u-space] can be denoted by $(X, D; \Psi^l)$ [resp. $(X, D; \Psi^u)$], and a Ψ-space by $(X, D; \Psi)$.

Lemma 5.2. *Any $(X, D; \Psi)$ is a Ψ^l-space [resp. Ψ^u-space].*

Example 5.3. (1) Any ϕ_A-space is a Ψ-space. Hence G-convex spaces are Ψ-spaces.
(2) A similar concept adopting u.s.c. maps with nonempty compact values instead of mere u.s.c. maps is called a *pseudo H-space* in [6]; see Section 6.
(3) In [3], the concept of Ψ^l-spaces are implicitly given without any practical examples.

Lemma 5.4. *Let $V = \{e^0, e^1, \ldots, e^n\}$, X be a topological space, and $(\Delta_n, V; \mathrm{co})$ be the abstract convex space as in the original KKM theorem. Let $G : V \multimap X$ be a closed [resp. an open] valued multimap. If G is a KKM map with respect to a l.s.c. [resp. an u.s.c.] map $F : \Delta_n \multimap X$, then $\bigcap_{i=0}^n G(e^i) \neq \emptyset$.*

Proof. Since $F(\Delta_J) \subset G(J)$ for each $J \subset V$, we have $\Delta_J \subset F^+G(J)$ for each $J \subset D$. Then $F^+G : V \multimap \Delta_n$ is a KKM map. Moreover, it is closed-valued [resp. open-valued] since F is l.s.c. [resp. u.s.c.]. Therefore, by the original

KKM theorem, we have $\bigcap_{i=0}^n F^+G(e^i) = F^+(\bigcap_{i=0}^n G(e^i)) \neq \emptyset$. Consequently, $\bigcap_{i=0}^n G(e^i)) \neq \emptyset$. □

From this Lemma, we have the following:

Theorem 5.5. *Let $(X, D; \Gamma)$ be an abstract convex space and $G : D \multimap X$ be a KKM map with closed [resp. open] values. If $(X, D; \Gamma)$ is a Ψ^l-space [resp. Ψ^u-space], then $\{G(y) : y \in D\}$ has the finite intersection property. Further if $\overline{G(y)}$ is compact for some $y \in D$, then $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$.*

Proof. Given $N = \{y_0, y_1, \ldots, y_n\} \subset D$, we prove $\bigcap_{i=0}^n G(y_i) \neq \emptyset$. Since G is a KKM map, we have

$$\Gamma(J) \subset \bigcup_{y\in J} G(y), \quad \forall\, J \subset N.$$

Then there exists a l.s.c. [resp. an u.s.c.] map $F : \Delta_n \multimap X$ such that

$$F(\Delta_J) \subset \Gamma(J) \subset \bigcup_{y\in J} G(y), \quad \forall\, J \subset N.$$

From Lemma 5.4, it follows that $\bigcap_{i=0}^n G(y_i) \neq \emptyset$. The second part is clear. □

From the definition of a KKM map of an abstract convex space, we can define the following:

Definition 5.6. For a Ψ^l-space $(X, D; \Psi^l)$, a map $G : D \multimap X$ is called a KKM map if, for each $N \in \langle D\rangle$, there exists a l.s.c. map $F : \Delta_n \multimap X$ such that $F(\Delta_J) \subset \Gamma(J)$ for all $J \subset N$. For a Ψ^u-space, we can define a KKM map similarly.

Remark 5.7. A KKM map for a Ψ^l-space is called a *generalized L-KKM map* in [3], where it is said to contain the so-called generalized R-KKM maps. The author of [2] claimed as follows: "The class of generalized *R-KKM* mappings includes those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, *G-H*-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses."

Proposition 5.8. *A KKM map $G : D \multimap X$ on a Ψ^l-space $(X, D; \Psi^l)$ is a KKM map on a new abstract convex space $(X, D; \Gamma)$.*

Proof. Define $\Gamma : \langle D\rangle \multimap X$ by $\Gamma(A) := G(A)$ for each $A \in \langle D\rangle$. Then $(X, D; \Gamma)$ becomes an abstract convex space. Note that $\Gamma(A) \subset G(A)$ for each

$A \in \langle D \rangle$ and hence $G : D \multimap X$ is a KKM map on the abstract convex space $(X, D; \Gamma)$. □

Proposition 5.8 also holds for Ψ^u-spaces.

Now we have the following diagram:

$$\begin{array}{ccccc} \phi_A\text{-space} & \Longrightarrow & \Psi\text{-space} & \Longrightarrow & \Psi^l\text{-space} \\ & & \Downarrow & & \Downarrow \\ & & \text{KKM space} & \Longrightarrow & \text{Partial KKM space} \end{array}$$

In fact, we have the following:

Corollary 5.9. *Every Ψ-space is a KKM space. Every Φ^l_A-space is a partial KKM space.*

6. Some variants of abstract convex spaces

In this section, we collect some possible variants of abstract convex spaces or Ψ-spaces. Some of them may or may not have concrete examples at present.

6.1. **Youness type E-convex spaces.** In [37], a class of sets and a class of functions called E-convex sets and E-convex functions are introduced by relaxing the definitions of convex sets and convex functions. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions. The optimality results for E-convex programming problems are established in [37].

Definition 6.1. ([37]) A set $M \subset \mathbb{R}^n$ is said to be *E-convex* iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that $(1 - \lambda)E(x) + \lambda E(y) \in M$, for each $x, y \in M$ and $0 < \lambda < 1$.

There is an example of an E-convex set, which is not convex [37].

Example 6.2. An E-convex set is an example of an abstract convex space and hence can be applied the KKM theory.

In fact, let D be a subset of M. Then $(M, D; \Gamma)$ is an abstract convex space, where $\Gamma : \langle D \rangle \to M$ is defined by

$$\begin{aligned} \Gamma\{x_0, x_1, \ldots, x_n\} &= \operatorname{co} E\{x_0, x_1, \ldots, x_n\} \\ &= \left\{ \sum_{i=0}^{n} \lambda_i E(x_i) \ : \ 0 \le \lambda_i \le 1, \ \sum_{i=0}^{n} \lambda_i = 1 \right\} \end{aligned}$$

for each $\{x_0, x_1, \ldots, x_n\} \in \langle D\rangle$. Hence, every E-convex set $M \subset \mathbb{R}^n$ is an abstract convex space $(M, M; \Gamma)$. Moreover, it is an example of ϕ_A-spaces or GFC-spaces and hence partially KKM spaces. Therefore, it satisfies so many results in the KKM theory as shown in [27] and many other articles.

More generally, we have the following way of making new abstract convex spaces from old:

Definition 6.3. Let $(E, D; \Gamma)$ be an absolute convex space and let $F : D \multimap D$ be a map. Then $(E, D; \Gamma^F)$ is called an abstract F-convex space whenever

$$\Gamma^F(A) := \mathrm{co}_\Gamma(F(A)) \quad \text{for each } A \in \langle D\rangle.$$

Note that this concept reduces to that of an abstract convex space whenever $F = 1_D$, the identity map on D.

6.2. **ψ-space.** We can consider a ψ_A-space $(X, D; \{\psi_A\}_{A\in\langle D\rangle})$, similar to a ϕ_A-space, where $\psi_A : [0,1]^n \to X$ is continuous for each $A \in \langle D\rangle$ with $|A| = n+1$. Such types of spaces are given by Michael [9], Llinares [7], and Cain and Gonzáles [1]. For each $n \geq 0$, considering continuous functions $g_n : \Delta_n \to [0,1]^n$ given by

$$g_n : u = \sum_{i=0}^{n} \lambda_i(u)e_i \mapsto (\lambda_0(u), \cdots, \lambda_{n-1}(u))$$

for $u \in \Delta_n$ and by putting $\phi_A := \psi_A g_n$, a ψ_A-space becomes a ϕ_A-space.

6.3. **Lin-Yao's pseudo H-spaces.** In 2003 [6], its authors introduced the following:

Definition 6.4. ([6]) Let X be a topological space, D be a nonempty set. The triple (X, D, q) is said to be a pseudo H-space if for each nonempty finite subset A of D, the restricted mapping $q : \Delta_{|A|-1} \to 2^X$ is upper semicontinuous with nonempty compact values, where $\Delta_{|A|-1}$ is an $(|A|-1)$-simplex with vertices $\{e_1, e_2, \ldots, e_{|A|}\}$. If $D = X$, the triple (X, D, q) is written by (X, q).

Its authors incorrectly observed that a G-convex space $(X, D; \Gamma)$ with $|D| < \infty$ is an example of pseudo H-space and gave no other proper example. Therefore, they might obtain some statements on their spaces, but it seems to be not practical.

Now, by defining $\Gamma_A := q(\Delta_{|A|-1})$ for each nonempty finite subset A of D, then (X, D, q) can be an abstract convex space $(X, D; \Gamma)$. Therefore the basic theorems in the recently developed abstract convex space theory can be applied.

Moreover, the following is given:

Definition 6.5. ([5]) Let (X, D, q) be a pseudo H-space. A mapping $F : D \to 2^X$ is a q-map if for each nonempty finite subset A of D, $q(\Delta_{|A|-1}) \subset \bigcup_{x\in A} F(x)$ and $q(\Delta_{|J|-1}) \subset \bigcup_{x\in J} F(x)$ for all nonempty finite subset J of A, where $\Delta_{|J|-1}$ is the convex hull of $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{k+1}}\}$ if $A = \{a_1, a_2, \ldots, a_{n+1}\}$, $J = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{k+1}}\}$.

The authors of [5] then showed that, under a strong restriction, a q-map can be a KKM map and that $1_X \in \mathfrak{KC}(X, X)$. From this KKM theorem, they deduced routine intersection result, Fan-Browder type fixed point theorems, a selection theorem, a Ky Fan type minimax inequality, and an application to abstract economies. However, they fail to give any proper example of their space which is not a G-convex space.

Later, the authors of [6] incorrectly stated that if the map q is single-valued and we set $\Gamma(A) = q(\Delta_{|A|-1})$ for each nonempty finite subset A of X, then (X, D, Γ) forms a G-convex space. Note also that their Example 1 can not be an example of their space.

6.4. **Ψ^l-space of Fang et al.** Our Ψ^l-spaces are motivated by Fang and Huang [3] as follows:

Definition 6.6. ([3], Definition 2.1.) Let X be a nonempty set and Y be a topological space. A set-valued mapping $G : X \to 2^Y$ is called a generalized L-KKM mapping if, for any $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ (where some elements in N may be same), there exists a lower semicontinuous mapping $\varphi_N : \Delta_n \to 2^Y$ such that for each $\{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\}$,

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} G(x_{i_j}),$$

where $\Delta_k = co(\{e_{i_0}, \ldots, e_{i_k}\})$.

All arguments in [3] are based on the following:

Theorem 6.7. ([3], Theorem 3.1.) *Let X be a nonempty set, Y be a topological space, and $G : X \to 2^Y$ be a generalized L-KKM mapping such that for each $x \in X$ and $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ (where some elements in N may be same), $G(x)\cap\varphi_N(\Delta_n)$ is closed in $\varphi_N(\Delta_n)$, where $\varphi_N : \Delta_n \to 2^Y$ is the lower semicontinuous mapping in touch with N in Definition* 2.1. *Then*

$$\varphi_N(\Delta_n) \cap \bigcap_{i=0}^{n} G(x_i) \neq \emptyset.$$

This follows from Theorem 5.5 by putting $G(x) \cap \varphi_N(\Delta_n)$ instead of $G(x)$.

6.5. **FWC-spaces of Lu and Zhang.** The following was introduced:

Definition 6.8. ([8], Definition 2.3.) A triple $(Y, D; \varphi_N)$ is said to be a finite weakly convex space (shortly, an FWC-space) if Y, D are two nonempty sets and for each $N = \{u_0, \dots, u_n\} \in \langle D \rangle$ where some elements in N may be same, there exists a set-valued mapping $\varphi_N : \Delta_n \to 2^Y$ with nonempty values. When $D \subset Y$, the space is denoted by $(Y \supset D; \varphi_N)$. In case $Y = D$, let $(Y; \varphi_N) := (Y, Y; \varphi_N)$.

Its authors stated: "It is worthwhile noticing that Y and D in Definition 2.3 do not possess any linear, convex and topological structure and so the set-valued mapping φ_N has no continuity requirement. Even Y is a topological space, it is easy to see that convex subsets of topological vector spaces, Lassonde's convex spaces, H-spaces introduced by Horvath, G-convex spaces introduced by Park and Kim, L-convex spaces introduced by Ben-El-Mechaiekh et al., G-H-spaces introduced by Verma, pseudo H-spaces introduced by Lai et al., GFC-spaces due to Khanh et al., FC-spaces due to Ding, and many other topological spaces with abstract convex structure are all particular forms of FWC-spaces." For the references, see [8]. Here L-spaces are carelessly called L-convex spaces as many peoples do.

Recall that FC-spaces $(Y; \varphi_N)$ due to Ding is a particular form of FWC-spaces for topological spaces Y and continuous φ_N. According to the definitions of FC-spaces and FWC-spaces, for each $N = \{u_0, \dots, u_n\} \in \langle D \rangle$ where some elements in N may be same, there should be an infinite number of maps $\varphi_N : \Delta_n \to 2^Y$.

Note that all of the preceding examples of FWC-spaces are known to be KKM spaces (that is, abstract convex spaces satisfying abstract form of the KKM theorem and its open-valued version); see Section 2.

For further comments on FWC-spaces, see [36].

7. Historical remarks

In 2006-09, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [15, 22, 27] and the references therein. Some corrections of [27] were given in [32].

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A partial KKM space is an abstract convex space satisfying the partial KKM principle. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. Now the KKM theory becomes the study of spaces satisfying the partial KKM principle.

In our work [27], we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [27] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

For topologies of abstract convex spaces; see [13, 28].

Our study on abstract convex spaces is closely related to the fixed point theory of our better admissible class $\mathfrak{B}$ of multimaps. The origin of the class of multimaps is given in [10, 11] as a generalization of the admissible class $\mathfrak{A}_c^\kappa$ due to Park earlier. Later, for topological spaces X and Y, we defined the *"better" admissible class* $\mathfrak{B}$ of maps from X into Y [14, 19, 33]. A number of authors imitated our definition, sometimes incorrectly.

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