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THE EXTENSION AND APPLICATION OF THE EQUIVALENCE OF NORMS ON A FINITE DIMENSIONAL SPACE

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Abstract. We introduce the concept of the equivalence of β-norms on a linear space which can be given different β-norms completely. Then for every n-dimensional β-normed space X, by using the norm of X: $\|x\| = (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} (\forall x = \sum_{i=1}^{n} x_i e_i \in X)$, we get a new β-normed space $(X, \|x\|^\beta)$, and get a conclusion that any β-norm on a finite dimensional β-normed space is equivalent to $\|x\|^\beta$. Further more, we prove that all of the β-norms on a finite dimensional linear space are equivalent. At last, we give an application of norm equivalence: Suppose X, Y are two n-dimensional real spaces, then the Banach-Mazur distance $d(X, \mathbb{R}^n) = c_1^{-1} c_2$, where c_1, c_2 are two constants concerned with the norm of X. We also give an estimation of $d(X, Y)$.

1. Preliminaries

Suppose X is an n-dimensional space($n \in \mathbb{N}$), $e_1, e_2, \cdots, e_n$ is a basis of X, $\|\cdot\|_1, \|\cdot\|_2$ are two different norms on X, it is well known of the following results

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in [1] : there exist two constants $c_1, c_2 > 0$, such that for $\forall x = \sum_{i=1}^{n} x_i e_i \in X$, we have

$$c_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} \leq \|x\|_i \leq c_2(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} \quad (i = 1, 2).$$

There also exist two constants $a, b > 0$, such that for $\forall x = \sum_{i=1}^{n} x_i e_i \in X$, we have

$$a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2.$$

Motivated by above results, we will discuss if there are similar results in β-normed spaces and discuss the equivalence of the two β-norms on any finite dimensional β-normed space. At last, we will discuss the applications of norm equivalence. Firstly, let us give some definitions.

Definition 1.1 ([2])**.** *Suppose β is a fixed number $0 < \beta \leq 1$, X is a linear space on $\mathbb{K}$, where $\mathbb{K}$ is real or complex, and $\|\cdot\|_\beta : X \mapsto \mathbb{R}^+ \cup 0$ is a functional. Then $(X, \|\cdot\|_\beta)$ is called a β-normed space if*

(1) $\|x\|_\beta \geq 0, \|x\|_\beta = 0 \Leftrightarrow x = \theta$;
(2) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$;
(3) $\|\alpha x\|_\beta = |\alpha|^\beta \|x\|_\beta$,

for all $\alpha \in \mathbb{K}$ and $x, y, z \in X$, where θ is a zero element in X, The functional $\|x\|_\beta$ is called a β-norm on X.

Definition 1.2 ([1])**.** *Suppose X is a real space. $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X. Then $\|\cdot\|_2$ is not weaker than $\|\cdot\|_1$ is that: for the arbitrary $\{x_n\} \subset X, x_0 \in X$, we have*

$$\|x_n - x_0\|_2 \to 0 \Rightarrow \|x_n - x_0\|_1 \to 0 \qquad (n \to \infty).$$

We say norm $\|\cdot\|_1$ and norm $\|\cdot\|_2$ are equivalent if norm $\|\cdot\|_1$ is not weaker than norm $\|\cdot\|_2$, and norm $\|\cdot\|_2$ is not weaker than norm $\|\cdot\|_1$.

There is also an equivalent conditions of the equivalence of two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X.

Theorem 1.3 ([1])**.** *Suppose X is a real space. $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if there exist two constants $a, b > 0$, such that*

$$a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2 \qquad (\forall x \in X).$$

The equivalence of β-norms has not a clear definition, and we do not know if there are two similar equivalent definitions of the equivalence of two β-norms on a β-normed space just like the equivalence of norms on a normed space. Also we do not know if all the β-norms on a finite dimensional β-normed space are equivalent.

In this paper, we use the equivalence of two norms on a general normed space to give a clear definition of equivalence of two β-norms, by giving a counter example, we prove we can not use the inequation of β-norms like the inequation in Theorem 1.3 to define the equivalence of two β-norms. We also prove all the β-norms on a finite dimensional β-normed space are equivalent.

Definition 1.4. *Let X be a real or complex linear space , and $\|\cdot\|_{\beta_1}, \|\cdot\|_{\beta_2}$ are two β-norms on X. We say $\|\cdot\|_{\beta_2}$ is not weaker than $\|\cdot\|_{\beta_1}$ is that: for the arbitrary $\{x_n\} \subset X, x_0 \in X$, we have*

$$\|x_n - x_0\|_{\beta_2} \to 0 \Rightarrow \|x_n - x_0\|_{\beta_1} \to 0 \qquad (n \to \infty).$$

We say β-norm $\|\cdot\|_{\beta_1}$ and β-norm $\|\cdot\|_{\beta_2}$ are equivalent if β-norm $\|\cdot\|_{\beta_1}$ is not weaker than β-norm $\|\cdot\|_{\beta_2}$, and β-norm $\|\cdot\|_{\beta_2}$ is not weaker than β-norm $\|\cdot\|_{\beta_1}$.

Definition 1.5 ([3], [2])**.** *Suppose E and E_1 are two topology linear spaces. T is a mapping from E to E_1. M is an arbitrary bounded subset in E. We say T is bounded if TM is bounded in E_1. We say T is strongly bounded if there exists a $\rho > 0$, such that $\|Tx\| \leq \rho\|x\|, \forall x \in E$.*

For the sake of convenience, from now on, we use X to denote an arbitrary n-dimensional linear space which can be given different β-norms, $0 < \beta \leq 1, n \in \mathbb{N}$.

2. Any β-norm is equivalent to a new β-norm on X

In this part, we prove that any finite dimensional β-normed space X, can be given a new β-norm, any other β-norm on X is equivalent to the new β-norm.

Lemma 2.1. *Let $(X, \|x\|_\beta)$ be an n-dimensional β-normed space, then X can be given a new β-norm $\|x\|^\beta$ for $\forall x \in X$.*

Proof. Suppose $e_1, e_2, \cdots, e_n$ is a basis of X, let $\|x\|^\beta = (\sum\limits_{i=1}^{n} |\xi_i|^2)^{\frac{\beta}{2}}, \forall x = \sum\limits_{i=1}^{n} \xi_i e_i \in X$. It is easy to know that $\|x\|^\beta \geq 0$, and $\|x\|^\beta = 0$ if and only if $x = \theta, \|\alpha x\| = |\alpha|^\beta \|x\|^\beta$, to prove $\|x\|^\beta$ is a β-norm, we only need to show $\|x + y\|^\beta \leq \|x\|^\beta + \|x\|^\beta$.

In fact, $\|x\| = (\sum_{i=1}^{n} |\xi|^2)^{\frac{1}{2}}$ is a norm on X, so $\|x+y\| \le \|x\| + \|y\|$. Then

$$\|x+y\|^\beta \le (\|x\| + \|y\|)^\beta \le \|x\|^\beta + \|y\|^\beta.$$

So $\|x\|^\beta$ is a β-norm, $(X, \|x\|^\beta)$ is a new β-normed space. $\square$

Remark 2.2. *In fact, any linear space can be endowed with a norm (see* [4], *page* 35*). Here, we can say any finite dimensional linear space can be given a new β-norm.*

Theorem 2.3. *Suppose $\|\cdot\|_\beta$ is an arbitrary β-norm on X, then $\|\cdot\|_\beta$ is equivalent to the β-norm given in Lemma* 2.1.

Proof. We suppose X is a real linear space, when X is complex, the proof is similar.

Suppose $e_1, e_2, \cdots, e_n$ is a basis of X. For any $x = \sum_{i=1}^{n} \xi_i e_i \in X$, define a mapping $T : X \mapsto \mathbb{R}^n, T : x \mapsto (\xi_1, \xi_2, \cdots, \xi_n)$. Then T is a linear isomorphic mapping from X to $\mathbb{R}^n$.

$$\begin{aligned} \|x\|_\beta &= \|\sum_{i=1}^{n} \xi_i e_i\|_\beta \le \sum_{i=1}^{n} |\xi_i|^\beta \|e_i\|_\beta \\ &\le (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta}{2}} (\sum_{i=1}^{n} \|e_i\|_\beta^{\frac{2}{2-\beta}})^{\frac{2-\beta}{2}}. \end{aligned}$$

Let $c = (\sum_{i=1}^{n} \|e_i\|_\beta^{\frac{2}{2-\beta}})^{\frac{2-\beta}{2}}$. Then $\|x\|_\beta = \|\sum_{i=1}^{n} \xi_i e_i\|_\beta \le c(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta}{2}}$.

On the other hand, let A be the sphere of $\mathbb{R}^n$, that is

$$A = \{a = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n, (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} = 1\}.$$

For any $a = (\xi_1, \xi_2, \cdots, \xi_n) \in A$, there exists a $x = \sum_{i=1}^{n} \xi_i e_i \in X$, define a mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^+ \cup \{0\}$,

$$f(a) = \|\sum_{i=1}^{n} \xi_i e_i\|_\beta = \|x\|_\beta.$$

Then f is continuous. In fact, for any sequence $\{a_m\}, m \in \mathbb{N}, a_m \to a(m \to \infty)$, there exists $\{x_m\}$ and x in $X(m \in \mathbb{N})$, such that $a_m = Tx_m, a = Tx$, and

$$|f(a_m) - f(a)| = |\|x_m\|_\beta - \|x\|_\beta| \le \|x_m - x\|_\beta \le c(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i|^2)^{\frac{\beta}{2}}.$$

Where $a_m = (\xi_1^{(m)}, \xi_2^{(m)}, \cdots, \xi_n^{(m)}), a = (\xi_1, \xi_2, \cdots, \xi_n), n, m \in \mathbb{N}$.

Since in $\mathbb{R}^n, a_m \to a$ if and only if $\xi_i^{(m)} \to \xi_i, m \to \infty, i = 1, 2, \cdots, n$, we have $|f(a_m) - f(a)| \to 0$, then f is continuous. Again, $A \in \mathbb{R}^n$ is a bounded, closed set, then A is compact. Since continuous functional can get maximum and minimum, there exist $a_0 = Tx_0$ and $b_0 = Ty_0$ in A, such that

$$f(a_0) = \min\{f(a)|a \in A\}, f(b_0) = \max\{f(a)|a \in A\}.$$

Let $c_1 = f(a_0), c_2 = f(b_0)$, then $f(a_0) = \|x_0\|_\beta > 0, f(b_0) = \|y_0\|_\beta > 0$, for any $x = \sum\limits_{i=1}^{n} \xi_i e_i \in X$, and $x \neq \theta$, we have $\frac{Tx}{\|Tx\|} \in A$, where $\|Tx\| = (\sum\limits_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}$. Thus

$$c_1 = \min\{f(a)|a \in A\} \leq \|\frac{x}{\|Tx\|}\|_\beta = f(\frac{Tx}{\|Tx\|}) \leq \max\{f(a)|a \in A\} = c_2.$$

That is

$$c_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta}{2}} \leq \|x\|_\beta \leq c_2(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta}{2}} \qquad (x \neq \theta).$$

The above inequation is also right when $x = \theta$. Thus

$$c_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta}{2}} \leq \|x\|_\beta \leq c_2(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta}{2}} \qquad (x = \sum_{i=1}^{n} \xi_i e_i \in X).$$

For the arbitrary $\{x_m\} \subset X, x_0 \in X, m \in \mathbb{N}$, it is obvious to see that

$$\|x_m - x_0\|^\beta \to 0 \Leftrightarrow \|x_m - x_0\|_\beta \to 0 \qquad (n \to \infty).$$

By Definition 1.4 and the arbitrariness of the β-norm $\|\cdot\|_\beta$, we can get any β-norm $\|\cdot\|_\beta$ on X is equivalent to the β-norm $\|\cdot\|^\beta$ given in Lemma 2.1. $\square$

3. Any two β-norms on X are equivalent

In this part, we give one of our main theorems.

Theorem 3.1. *Let $\|x\|_{\beta_1}, \|x\|_{\beta_2}$ be two different β-norms on X, where $0 < \beta_1, \beta_2 \leq 1$ and $\beta_1 \neq \beta_2$. Then $\|x\|_{\beta_1}$ and $\|x\|\beta_2$ are equivalent.*

Proof. Let $e_1, e_2, \cdots, e_n$ be a basis of X. By Theorem 2.3, for $\|\cdot\|_{\beta_1}, \|\cdot\|_{\beta_2}$, there exist four constants $l_1, l_2, d_1, d_2 > 0$, such that

$$l_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_1}{2}} \leq \|x\|_{\beta_1} \leq l_2(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_1}{2}}, \tag{3.1}$$

and

$$d_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_2}{2}} \leq \|x\|_{\beta_2} \leq d_2(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_2}{2}}, \tag{3.2}$$

for $\forall x = \sum_{i=1}^{n} \xi_i e_i \in X$.

For the arbitrary $\{x_m\} \subset X, x_0 \in X, m \in \mathbb{N}$, define a mapping $T : X \mapsto \mathbb{R}^n, T : x \mapsto (\xi_1, \xi_2, \cdots, \xi_n)$. Suppose $Tx_m = (\xi_1^{(m)}, \xi_2^{(m)}, \cdots, \xi_n^{(m)}), Tx_0 = (\xi_1^{(0)}, \xi_2^{(0)}, \cdots, \xi_n^{(0)})$. If $\|x_m - x_0\|_{\beta_1} \to 0 (m \to \infty)$, then for any $\varepsilon > 0$, there exists a $M \in \mathbb{N}$, such that when $m > M, m \in \mathbb{N}$, we have

$$\|x_m - x_0\|_{\beta_1} < \varepsilon. \tag{3.3}$$

By (3.1), we can get

$$l_1\Big(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(0)}|\Big)^{\frac{\beta_1}{2}} \le \|x_m - x_0\|_{\beta_1} \le l_2\Big(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(0)}|\Big)^{\frac{\beta_1}{2}}. \tag{3.4}$$

By (3.3), (3.4), we can get for any $m > M, l_1(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(0)}|)^{\frac{\beta_1}{2}} < \varepsilon$, then

$$\xi_i^{(m)} \to \xi_i^{(0)} \qquad (1 \le i \le n, m \to \infty). \tag{3.5}$$

By the right side of (3.2) and (3.5), we can get

$$\|x_m - x_0\|_{\beta_2} \to 0 \qquad (m \to \infty).$$

By Definition 1.4, β-norm $\|\cdot\|_{\beta_1}$ is not weaker than β-norm $\|\cdot\|_{\beta_2}$. Similarly, we can get: β-norm $\|\cdot\|_{\beta_2}$ is not weaker than β-norm $\|\cdot\|_{\beta_1}$. That is $\|\cdot\|_{\beta_1}$ and $\|\cdot\|_{\beta_2}$ are equivalent. Hence our proof is complete. □

Theorem 3.2. *If there exist two constants $a, b > 0$, such that*

$$a\|x\|_{\beta_1} \le \|x\|_{\beta_2} \le b\|x\|_{\beta_1} \qquad (\forall x \in X), \tag{$*$}$$

then $\|\cdot\|_{\beta_1}$ and $\|\cdot\|_{\beta_2}$ are equivalent. But if $\|\cdot\|_{\beta_1}$ and $\|\cdot\|_{\beta_2}$ are equivalent, there is no need of the existence of a, b satisfying $(*)$.

Proof. If there exist a, b satisfying $(*)$, then the Cauchy sequences in $(X, \|\cdot\|_{\beta_1})$ and $(X, \|\cdot\|_{\beta_2})$ are same. It is easy to see $\|\cdot\|_{\beta_1}$ and $\|\cdot\|_{\beta_2}$ are equivalent. The rest proof of this theorem will be finished by a counter example.

In fact, suppose X is a one-dimensional real linear space. Especially, suppose $X = \mathbb{R}$. For $\forall \xi \in \mathbb{R}$, define

$$\|x\|_{\beta_1} = |\xi|^{\beta_1}, \|x\|_{\beta_2} = |\xi|^{\beta_2}.$$

Then $\|\cdot\|_{\beta_1}$ and $\|\cdot\|_{\beta_2}$ are two different β-norms on X, $\|\cdot\|_{\beta_1}$ is equivalent to $\|\cdot\|_{\beta_2}$. But there is obviously no existence of two constants a, b, such that

$$a\|x\|_{\beta_1} \le \|x\|_{\beta_2} \le b\|x\|_{\beta_1} \qquad (\forall x \in X).$$

□

4. Some corollaries

In this part, we show that some properties which a finite dimensional normed space shares are also shared by a finite dimensional β-normed space.

Corollary 4.1. *Any finite dimensional β-normed space X is complete, therefor any finite dimensional subspace of a β-normed space is closed.*

Proof. Let $e_1, e_2, \cdots, e_n$ be a basis of X. If $\{x_m\}$ is a cauchy sequence in X, by the theorem 2.3 and follow the signals of Theorem 2.3, we have: $T\{x_m\}$ is a cauchy sequence in $\mathbb{R}^n$. Since $(\mathbb{R}^n, \|x\|)$ is complete, where $\|x\| = (\sum\limits_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}$ for $\forall x = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$, then there exists an $a = Tx$ in $\mathbb{R}^n$, such that $T(\{x_m\}) \to a = Tx$, then $x_m \to x \in X(m \to \infty)$, so X is complete.

Since X is complete, it must be closed. So any finite dimensional subspace of a β-normed space is closed. □

Corollary 4.2. *Linear operators on a finite dimensional β-normed space are bounded.*

Proof. Suppose $(X, \|\cdot\|_{\beta_1}), (Y, \|\cdot\|_{\beta_2})$ are two β-normed spaces, $\dim(X) = n, e_1, e_2, \cdots, e_n$ is a basis of X, $T : X \to Y$ is a linear operator. For any $x = \sum\limits_{i=1}^{n} \xi_i e_i \in X$, then

$$\begin{aligned}
\|Tx\|_{\beta_2} &= \|\sum_{i=1}^{n} \xi_i Te_i\|_{\beta_2} \le \sum_{i=1}^{n} |\xi_i|^{\beta_2} \|Te_i\|_{\beta_2}, \\
&\le (\sum_{i=1}^{n} \|Te_i\|_{\beta_2}^{\frac{2}{2-\beta_2}})^{\frac{2-\beta_2}{2}} (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_2}{2}}, \\
&\le (\sum_{i=1}^{n} \|Te_i\|_{\beta_2}^{\frac{2}{2-\beta_2}})^{\frac{2-\beta_2}{2}} \frac{(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_2}{2}}}{(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_1}{2}}} (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_1}{2}} \quad (x \ne \theta).
\end{aligned}$$

By Theorem 2.3, there exists a constant $c_1 > 0$, such that

$$c_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_1}{2}} \le \|x\|_{\beta_1}.$$

Let

$$r = \frac{(\sum_{i=1}^{n} \|Te_i\|_{\beta_2}^{\frac{2}{2-\beta_2}})^{\frac{2-\beta_2}{2}} (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_2}{2}}}{c_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_1}{2}}}.$$

Then

$$\|Tx\|_{\beta_2} \le rc_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{\beta_2}{2}} \le r\|x\|_{\beta_1} \qquad (x \ne \theta). \tag{4.1}$$

(4.1) also holds when $x=\theta$. So

$$\|Tx\|_{\beta_2} \le r\|x\|_{\beta_1} \qquad (\forall x = \sum_{i=1}^{n} \xi_i e_i \in X)$$

If $\varepsilon > 0$, then there is a positive $\delta = \frac{\varepsilon}{r} > 0$, such that when $\|x-\theta\|_{\beta_1} < \delta$, then

$$\|Tx - T\theta\|_{\beta_2} \le r\|x\|_{\beta_1} < \varepsilon.$$

So T is continuous at θ.

T is continuous at θ, then there is an open ball V centered at θ, such that $\|Tx\|_{\beta_2} < 1$, whenever $x \in V$. For each bounded subset B of X, there is a positive t_B, such that $B \subset t_B V$, and so $\|Tx\|_{\beta_2} < t_B^{\beta_2}$ if $x \in B$. Thus T is bounded. Hence, our proof is complete. □

Remark 4.3. *Linear operators on a finite dimensional β-normed space may not be strongly bounded.*

Example 4.4. *Suppose $(l_{(n)}^{\beta_1}), (l_{(n)}^{\beta_2}), 0 < \beta_1 < \beta_2 < 1$ are two n-dimensional β-normed space, then $(l_{(n)}^{\beta_1}) \subset (l_{(n)}^{\beta_2})$. Let a linear operator I:*

$$(l_{(n)}^{\beta_1}) \mapsto (l_{(n)}^{\beta_2}) : Ix = x \qquad (\forall x \in (l_{(n)}^{\beta_1}).$$

We claim T is bounded, but not strongly bounded.

In fact, suppose K is a bounded set in $(l_{(n)}^{\beta_1})$, for each $x = (\xi_1, \xi_2, \cdots, \xi_n) \in K$, $\|x\|_{\beta_1} \le M_1$, where $M_1 > 0$ is a constant, then $\|\frac{x}{(M_1)^{\frac{1}{\beta_1}}}\|_{\beta_1} \le 1$. So $|\frac{\xi_i}{(M_1)^{\frac{1}{\beta_1}}}| \le 1, 1 \le i \le n$. Since $\beta_1 < \beta_2$, then $\|\frac{x}{(M_1)^{\frac{1}{\beta_1}}}\|_{\beta_2} \le 1$. That is $\|x\|_{\beta_2} \le (M_1)^{\frac{\beta_2}{\beta_1}}$. So I is bounded.

But I is not strongly bounded. In fact, if there exists a $\rho > 0$, such that

$$\|x\|_{\beta_2} = \|Ix\|_{\beta_2} \le \rho\|x\|_{\beta_1} \quad (\forall x = (\xi_1, \xi_2, \cdots, \xi_n) \in (l_{(n)}^{\beta_1})).$$

That is $\sum\limits_{k=1}^{n} |\xi_k|^{\beta_2} \le \rho \sum\limits_{k=1}^{n} |\xi_k|^{\beta_1}$. But if we let $x_m = (m, 0, 0, \cdots, 0) \in (l_{(n)}^{\beta_1}), m \in \mathbb{N}$, we can get

$$m^{\beta_2 - \beta_1} = \frac{m^{\beta_2}}{m^{\beta_1}} \le \rho \qquad (\forall m \in \mathbb{N}).$$

Notice that, $\beta_2 - \beta_1 > 0$, so the above inequation is obviously not exist. So I is not strongly bounded.

5. An application of the equivalence of norms

Banach-Mazur distance is one of the most basic conceptions in Banach local theories. It can denote how closed the two isomorphic Banach spaces are. Suppose X is an n-dimensional real Banach space, an estimation about $d(X, l^1_{(n)})$ is given in [5], [7] : $d(X, l^1_{(n)}) \leq n$. Another classic estimation given by T.John [6] in 1948 is: $d(X, l^2_{(n)}) \leq \sqrt{n}$. Here, we will use the equivalence of norms on a finite dimensional normed space to give an exact method to calculate $d(X, \mathbb{R}^n)$, not an estimation about $d(X, \mathbb{R}^n)$, where $\mathbb{R}^n$ is an Euclid space with a norm $\|x\| = (\sum\limits_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}$ for $\forall x = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$. At last, we give an estimation about $d(X, Y)$, where X, Y are two n-dimensional real normed spaces.

Definition 5.1 ([5])**.** *Given two Banach spaces E, F and the isomorphic mapping T from E to F. The Banach-Mazur distance between them is defined by $d(X, Y) = \inf\{\|T\|\|T^{-1}\|, T : E \mapsto F\}$.*

Theorem 5.2. *Suppose $(X, \|\cdot\|_X)$ is an n-dimensional normed space with a basis $e_1, e_2, \cdots, e_n$. $\mathbb{R}^n$ is an n-dimensional real Euclid space with a norm $\|x\| = (\sum\limits_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}$ for $\forall x = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$. Then for any isomorphic mapping T from $\mathbb{R}^n$ to X, we have*

$$d(X, \mathbb{R}^n) = \inf\{\|T\|\|T^{-1}\|, T : \mathbb{R}^n \mapsto X\} = c_1^{-1} c_2.$$

Where $d(X, \mathbb{R}^n)$ denotes the Banach-Mazur distance between $\mathbb{R}^n$ and X. And $c_1 = \min\{\|\sum\limits_{i=1}^{n} \xi_i e_i\|_X\}, c_2 = \max\{\|\sum\limits_{i=1}^{n} \xi_i e_i\|_X\}$ for $(\sum\limits_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} = 1$.

Proof. Since two finite linear spaces are isomorphic, we can suppose T is an arbitrary isomorphic mapping from $\mathbb{R}^n$ to X, where $Tx = \sum\limits_{i=1}^{n} \eta_i e_i$ for $\forall x = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$. Among these isomorphic mappings, we denote the isomorphic T_0 as $T_0 x = \sum\limits_{i=1}^{n} \xi_i e_i$. Since X is a special β-normed space, by Theorem 2.3, there exist two constants c_1, c_2, such that

$$c_1 (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} \leq \|\sum_{i=1}^{n} \xi_i e_i\|_X \leq c_2 (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}.$$

Meantime, by

$$\|T_0\| = \sup\{\|T_0 x\|_X, \|x\| = 1\},$$

and

$$c_2 = \max\{\|\sum_{i=1}^{n} \xi_i e_i\|_X | (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} = 1\},$$

we can see $\|T_0\| = c_2$. On the other hand, for $\forall y = T_0x$,

$$\begin{aligned}\|T_0^{-1}\| &= \inf\{\rho, \|T_0^{-1}y\| \le \rho\|y\|_X\}\\ &= \inf\{\rho, \|y\|_X \ge \tfrac{1}{\rho}\|T_0^{-1}y\|\}\\ &= \inf\{\tfrac{1}{k}, k\|T_0^{-1}y\| \le \|y\|_X\}.\end{aligned} \tag{5.1}$$

By Theorem 2.3, for $\forall x \in \mathbb{R}^n$, there is a $c_1 = \min\{\|\sum_{i=1}^{n} \xi_i e_i\|_X\}$, where $(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} = 1$, such that $\|T_0x\|_X \ge c_1\|x\|$. Then we have

$$c_1 = \sup\{l : \|T_0x\|_X \ge l\|x\|\}. \tag{5.2}$$

Otherwise, if $c_1 \ne \sup\{l : \|T_0x\|_X \ge l\|x\|\}$, there exists a $c_0 > c_1$, such that $\|T_0x\|_X \ge c_0\|x\|$, then $\|T_0x\|_X \ge c_0$ for $\forall x \in \mathbb{R}^n, \|x\| = 1$, which is a contradiction with $c_1 = \min\{\|T_0x\|_X \text{ for } \forall x \in \mathbb{R}^n, \|x\| = 1\}, c_1 < c_0$. By (5.1) and (5.2), we can get $c_1^{-1} = \|T_0^{-1}\|$, and

$$d(X, \mathbb{R}^n) = \inf\{\|T\|\|T^{-1}\|, T : \mathbb{R}^n \mapsto X\} \le \|T_0\|\|T_0^{-1}\| = c_1^{-1}c_2. \tag{5.3}$$

Next, we will prove $d(X, \mathbb{R}^n) \ge c_1^{-1}c_2$. In fact, for the arbitrary isomorphic mapping T and $x = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n, Tx = \sum_{i=1}^{n} \eta_i e_i$, since

$$\|x\| = \|T^{-1}Tx\| \le \|T^{-1}\|\|Tx\|,$$

we have $\|T^{-1}\|^{-1}\|x\| \le \|Tx\|_X$, so

$$\|T^{-1}\|^{-1}\|x\| \le \|Tx\|_X \le \|T\|\|x\|. \tag{5.4}$$

By Theorem 2.3, there exist $c_1, c_2 > 0$, such that

$$c_1(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}} \le \|\sum_{i=1}^{n} \eta_i e_i\|_X \le c_2(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}}. \tag{5.5}$$

By (5.5), when $x \ne \theta$, we can get

$$\begin{aligned} c_1(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}} &= c_1 \frac{(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}} (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}\\ &\le \|\sum_{i=1}^{n} \eta_i e_i\|_X\\ &\le c_2(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}} = c_2 \frac{(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}} (\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}. \end{aligned} \tag{5.6}$$

By the proof of Theorem 2.3 and T is isomorphic, there exist x_1 and x_2 in $\mathbb{R}^n$, such that $\|Tx_1\|_X = c_1, \|Tx_2\|_X = c_2$, by (5.4) and (5.6), we can get

$$c_1 \frac{(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}} \geq \|T^{-1}\|^{-1}, \quad c_2 \frac{(\sum_{i=1}^{n} |\eta_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}}} \leq \|T\|. \tag{5.7}$$

By (5.7), we can get $\|T\|\|T^{-1}\| \geq c_1^{-1} c_2$. Then

$$d(X, \mathbb{R}^n) = \inf\{\|T\|\|T^{-1}\| |T : \mathbb{R}^n \mapsto X\} \geq c_1^{-1} c_2. \tag{5.8}$$

By (5.3) and (5.8), we can get $d(X, \mathbb{R}^n) = c_1^{-1} c_2$. □

Theorem 5.3. *Suppose* $(X_1, \|\cdot\|_1), (X_2, \|\cdot\|_2)$ *are two n-dimensional real normed spaces. Then the Banach-Mazur distance* $d(X_1, X_2) \leq d_1^{-1} d_2 l_1^{-1} l_2$. *Where* $d_1 = \min\{\|\sum_{i=1}^{n} \xi_i e_i\|_1\}, d_2 = \max\{\|\sum_{i=1}^{n} \xi_i e_i\|_1\}, l_1 = \min\{\|\sum_{i=1}^{n} \xi_i e_i\|_2\}, l_2 = \max\{\|\sum_{i=1}^{n} \xi_i e_i\|_2\}$ *for* $\forall x = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n, \|x\| = 1$.

Proof. By theorem 2.3, there exist four constants d_1, d_2, l_1, l_2, such that

$$d_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} \leq \|x\|_1 \leq d_2(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}},$$

and

$$l_1(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}} \leq \|x\|_2 \leq l_2(\sum_{i=1}^{n} |\xi_i|^2)^{\frac{1}{2}},$$

for $\forall x = \sum_{i=1}^{n} \xi_i e_i \in X$. By Theorem 5.2, $d(X_1, \mathbb{R}^n) = d_1^{-1} d_2, d(X_2, \mathbb{R}^n) = l_1^{-1} l_2$. Since for three isomorphic Banach spaces X, Y, Z, the Banach-Mazur distances satisfying $d(X, Z) \leq d(X, Y) d(Y, Z)$, then we can get $d(X_1, X_2) \leq d_1^{-1} d_2 l_1^{-1} l_2$. Hence, our proof is complete. □

Remark 5.4. *Suppose X, Y are two n-dimensional real normed spaces. Theorem 5.2 gives a new method to calculate $d(X, \mathbb{R}^n)$. It improves the calculation of* $d(X, \mathbb{R}^n) = \inf\{\|T\|\|T^{-1}\| |T : \mathbb{R}^n \mapsto X\}$ *by* $d(X, \mathbb{R}^n) = \|T_0\|\|T_0^{-1}\|$. *It is also helpful to estimate $d(X, Y)$. And, people can change the $d(X, \mathbb{R}^n)$ by renorming.*

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