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REMARKS ON THE KKM THEORY OF HADAMARD MANIFOLDS AND HYPERBOLIC SPACES

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Abstract. In 2012, Yang and Pu [Indian J. Pure Appl. Math., **43**(2) (2012), 129–144] proved an Browder type fixed point theorem on Hadamard manifolds with strongly geodesic convexity. In this paper, we show that their main results can be obtained from the one in the more general spaces. As applications, we claim that their maximal element theorems, section theorems, Ky Fan type minimax inequality, and equilibrium theorem about non-cooperative games on Hadamard manifold are already obtained in some sense.

1. Introduction

The KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929 is a versatile tool in Nonlinear Analysis and the area associated with it has been rapidly developed. In 1992, the area was called the KKM theory by Park who began, in 2006, to extend it to abstract convex spaces. See [9]-[15] and the references therein.

In 1990, Reich and Shafrir [17] introduced hyperbolic spaces in order to try to develop a theory of nonexpansive iterations in more general infinite-dimensional manifolds than normed vector spaces. This class of metric spaces contains all normed vector spaces and Hadamard manifolds, as well as the Hilbert ball and the Cartesian product of Hilbert balls. Since 2008, Park found that any hyperbolic spaces are G-convex spaces [9] and also particular cases

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of c-spaces [10]-[13]. Actually, in 2010, Park indicated but not concretely that most of key results in the KKM theory can be applied to hyperbolic spaces; see [13], [14].

On the other hand, a number of authors studied some KKM theoretic results on Hadamard manifolds. For example, Nemeth [4] introduced and studied variational inequalities on Hadamard manifolds, and Zhou and Huang [19] introduced a KKM type theorem on Hadamard manifolds with some applications to a mixed variational inequality and a Fan-Browder fixed point theorem. Moreover, in 2012, Colao, Lopez, Marino, and Martin-Marquez [1] developed an equilibrium theory in Hadamard manifolds. Further, Yang and Pu [18] proved a Fan-Browder type fixed point theorem with strongly geodesic convexity on Hadamard manifolds. It is clear that such results are closely related to the KKM theory on hyperbolic spaces.

In fact, Park [16] showed that three key results of [1] can be extended to hyperbolic spaces and are particular ones for abstract convex spaces in the sense of his in [12], [13]. Similarly, most of main theorems in the KKM theory on abstract convex spaces can be applied to hyperbolic spaces and Hadamard manifolds. In this paper, we show that the main result of [18], [19] also can be obtained from the one in abstract convex spaces.

Section 2 devotes to review some preliminary facts on our abstract convex spaces as in [12], [13]. In Section 3, we are concerned with definitions of Hadamard manifolds and other examples of hyperbolic spaces and we show that any of such spaces are KKM spaces, which means that most results in [12], [13] are applicable to them. Section 4 deals with a KKM type theorem on Hadamard manifolds. In Sections 5 and 6, From a Fan-Browder type alternative on abstract convex spaces, we show that the Fan-Browder type fixed point theorem and the maximal element theorem of Yang and Pu can be considered as corollaries of Park's result. A minimax inequality on Hadamard manifolds and the existence of Nash equilibria can be also deduced from a general version on abstract convex spaces. This will be shown in section 7 and 8.

2. Abstract convex spaces

We follow Park's works [12], [13] and the references therein.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Example 2.4. The following are typical examples of KKM spaces. For details, see [13] and the references therein.

(1) An abstract convex space $(X, D; \Gamma)$ is called an H-*space* by Park if $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma)$ is called a *c*-space by Horvath.

(2) A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

   Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space

$$\Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space}$$
$$\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}$$

For the basic results on the KKM theory, the readers can refer [12], [13], [15] and the references therein.

3. Hadamard manifolds and hyperbolic spaces

In this section, we follow Park [16].

In 1990, Reich and Shafrir [17] introduced hyperbolic spaces in order to try to develop a theory of nonexpansive iterations in more general infinite-dimensional manifolds than normed vector spaces:

Definition 3.1. ([17]) Let (X, ρ) be a metric space and $\mathbb{R}$ the real line. We say that a map $c : \mathbb{R} \to X$ is a *metric embedding* of $\mathbb{R}$ into X if

$$\rho(c(s), c(t)) = |s - t|$$

for all real s and t. The image of a metric embedding is called a *metric line.* The image of a real interval $[a, b] := \{t \in \mathbb{R} \mid a \leq t \leq b\}$ under such a map is called a *metric segment.*

Assume that (X, ρ) contains a family M of metric lines, such that for each pair of distinct points x and y in X there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment denoted by $[x, y]$ joining x and y. For each $0 \leq t \leq 1$ there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1 - t)\rho(x, y).$$

This point is denoted by $(1 - t)x \oplus ty$.

We say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho(y, z)$$

for all $x,\ y$ and z in X.

Example 3.2. ([17]) The following are examples of hyperbolic spaces:

(1) All normed vector spaces.
(2) All Hadamard manifoldds, that is, all finite-dimensional connected, simply connected, complete Riemannian manifolds of constant (non-positive sectional) curvature.
(3) The Hilbert ball equipped with the hyperbolic metric.
(4) Arbitrary product of hyperbolic spaces.

In the class of hyperbolic spaces, the concept of convexity can be defined as follows:

Definition 3.3. ([17]) A subset C of a hyperbolic space X is said to be *convex* if, for each pair of points x and y in C, the metric segment $[x, y]$ is also contained in C. The *closed convex hull* of a subset D of X is the intersection of all closed convex subsets of X which contains D.

In the class of Hadamard manifolds, the authors [4], [18], [19] define more special concepts of convexity as follows:

Definition 3.4. A set K of an Hadamard manifold M is said to be *geodesic convex* if for any $p, q \in K$, the geodesic joining p to q is contained in K, that is, for any $p, q \in K$, $\exp_p(t \exp_p^{-1} q) \in K$ for all $t \in [0, 1]$.

A set K of an Hadamard manifold M is said to be *strongly geodesic convex* if for any given $o \in M$ and for any $p, q \in K$, $\exp_o((1-t)\exp_o^{-1} p + t \exp_o^{-1} q) \in K$ for all $t \in [0, 1]$.

Since any geodesic is a metric segment and the point o can be given by p, a strongly geodesic convex set in an Hadamard manifold is a geodesic convex set hence a convex set in a hyperbolic space.

In previous works of Park, he noted that any hyperbolic spaces are G-convex spaces [9] and also particular cases of c-spaces [12]-[15]. This can be strengthened as follows:

Definition 3.5. The *convex hull* $\operatorname{co} D$ of a subset D of a hyperbolic space X is the intersection of all convex subsets of X which contains D.

Lemma 3.6. *Any convex subset Y of a hyperbolic space $X = (X, \rho, M)$ can be made into a c-space $(X; \Gamma)$ and hence a KKM space.*

Proof. For any $A \in \langle Y \rangle$, let $\Gamma_A = \Gamma(A) = \operatorname{co} A$. Then it is easily seen to be contractible. Therefore $(X; \Gamma)$ is a c-space in the sense of Horvath and hence, a KKM space by Park's KKM theory.

We have the following:

Convex subset of an Hadamard manifold
$\Longrightarrow$ Convex subset of a hyperbolic space
$\Longrightarrow$ c-space $\Longrightarrow$ H-space.

In view of Lemma 3.5, all results in [14], [15] hold for any convex subset of a hyperbolic spaces. In the following sections, we give some examples of this fact in [1]. □

4. The KKM theorem on hyperbolic spaces

Some key results in [1], [18], [19] can be extended to hyperbolic spaces instead of Hadamard manifolds by applying Park's KKM theory of abstract convex spaces in [14], [15].

The following is given as Theorem 3 of [15]:

Theorem 4.1. (Generalized partial KKM principle) *Let* $(E, D; \Gamma)$ *be a partial KKM space and* $G : D \multimap E$ *a map such that*

(1) *G is closed-valued;*
(2) *G is a KKM map (that is, $\Gamma_A \subset G(A)$ for all $A \in \langle D \rangle$); and*
(3) *there exists a nonempty compact subset K of E such that*

$$K = \bigcap \{G(z) \mid z \in M\}$$

for some $M \in \langle D \rangle$.

Then $K \cap \bigcap \{G(z) \mid z \in D\} \neq \emptyset$.

Since any Hadamard manifold is a hyperbolic space which is a partial KKM space, Theorem 4.1 is applicable to hyperbolic spaces or Hadamrd manifolds. Hence, we immediately have the following form of the KKM theorem:

Theorem 4.2. *Let M be a hyperbolic space and $K \subset M$ a convex subset. Let $G : K \multimap K$ be a multimap such that, for each $x \in K$, $G(x)$ is closed. Suppose that*

(i) *there exists $x_0 \in K$ such that $G(x_0)$ is compact;*
(ii) $\forall\, x_1, \ldots, x_m \in K,\ co\{x_1, \ldots, x_m\} \subset \bigcup_{i=1}^{m} G(x_i)$.

Then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Remark 4.3. (1) In Theorem 3.1 of [19], Theorem 4.2 for Hadamard manifolds with a KKM map $G : K \multimap M$ and compact K was proved. They could deduce only the finite intersection property of map-values of G.
(2) In Theorem 3.2 of [19], Theorem 4.2 for Hadamard manifolds with a KKM map $G : K \multimap M$ and compact K was proved. Here, the compactness is redundant.
(3) In Lemma 3.1 of [1], Theorem 4.2 for Hadamard manifolds was provided with almost two page proof.

5. Fan-Browder type theorems on hyperbolic spaces

In Park's previous works [9], [10], [13], he obtained some characterizations of (partial) KKM spaces and one of them is closely related to the Fan-Browder fixed point theorem. We need the following for an abstract convex space $(E, D; \Gamma)$:

The Fan-Browder fixed point property. *For any maps* $S : E \multimap D$ *and* $T : E \multimap E$ *satisfying*

(1) $S^{-}(x)$ *is open for each* $x \in E$;
(2) *for each* $x \in E$, $\mathrm{co}_{\Gamma} S(x) \subset T(x)$; *and*
(3) $E = S^{-}(M) = \bigcup_{y \in M} S^{-}(y)$ *for some* $M \in \langle D \rangle$,

T *has a point* $\bar{x} \in E$ *such that* $\bar{x} \in T(\bar{x})$.

The (partial) KKM principle implies the preceding property as follows:

Theorem 5.1. *A partial KKM space* $(E, D; \Gamma)$ *satisfies the Fan-Browder fixed point property.*

Proof. Let $(E, D; \Gamma)$ be a partial KKM space and $S : E \multimap D$, $T : E \multimap E$ be maps satisfying (1)-(3). Suppose that $x \notin T(x)$ for all $x \in E$. Define a closed-valued map $F : D \multimap E$ by $F(y) := E \setminus S^{-}(y)$ for each $y \in D$. Since $E = S^{-}(M)$ and

$$\bigcap_{y \in M} F(y) = \bigcap_{y \in M} (E \setminus S^{-}(y)) = E \setminus \bigcup_{y \in M} S^{-}(y) = \emptyset,$$

$\{F(y)\}_{y \in D}$ does not have the finite intersection property. Hence F is not a KKM map. Therefore, $\Gamma(N) \not\subset F(N)$ for some $N \in \langle D \rangle$. Let $x_0 \in \Gamma(N) \setminus F(N)$. Then $x_0 \in \mathrm{co}_{\Gamma} N$ and $x_0 \notin F(N) = \bigcup_{y \in N} (E \setminus S^{-}(y))$. Hence $x_0 \in S^{-}(y)$ or $y \in S(x_0)$ for all $y \in N$, that is, $N \subset S(x_0)$. Then

$$x_0 \in \mathrm{co}_{\Gamma}(N) \subset \mathrm{co}_{\Gamma} S(x_0) \subset T(x_0).$$

This contradicts the non-existence of fixed points of T. □

Theorem 5.2. *Let* $(X; \Gamma)$ *be a nonempty compact partial KKM space. Suppose that* $F : X \multimap X$ *and* $H : X \multimap X$ *are two set-valued maps with the following conditions:*

(i) *for any* $x \in X$, $\mathrm{co}_{\Gamma} H(x) \subset F(x)$;
(ii) *for any* $x \in X$, *there exists* $y \in X$ *such that* $x \in \mathrm{int} H^{-1}(y)$.

Then there exists $x^* \in X$ *such that* $x^* \in F(x^*)$.

Proof. Let $S^{-} = \mathrm{int} H^{-}$, $T = F$ and apply the above theorem. □

Corollary 5.3. *Let* X *be a nonempty convex compact subset of a hyperbolic space* M. *Suppose that* $F : X \multimap X$ *and* $H : X \multimap X$ *are two set-valued maps with the following conditions:*

(i) *for any* $x \in X$, $\mathrm{co}H(x) \subset F(x)$*;*
(ii) *for any* $x \in X$, $H^{-1}(y)$ *is open in* X*.*

Then there exists $x^* \in X$ *such that* $x^* \in F(x^*)$*.*

Corollary 5.4. *Let* X *be a nonempty convex compact subset of a hyperbolic space* M*. Suppose that* $F : X \multimap X$ *is a set-valued map with the following conditions:*

(i) *for any* $x \in X$, $F(x)$ *is nonempty convex in* X*;*
(ii) *for any* $x \in X$, $F^{-1}(y)$ *is open in* X*.*

Then there exists $x^* \in X$ *such that* $x^* \in F(x^*)$*.*

Remark 5.5. (1) In Theorem 4.2 of [19], Corollary 5.4 for Hadamard manifolds was proved.
(2) In Theorem 3.1 and Corollary 3.1-2 of [18], Theorem 5.2 and Corollary 5.3-4 for Hadamard manifolds with strongly geodesic convexity of $F(x)$ were proved.

6. Maximal element theorems on hyperbolic spaces

The Fan-Browder type fixed point theorem have an alternative result. We need the following result which prove the existence of maximal element on abstract convex space $(E, D; \Gamma)$:

Theorem 6.1. *Let* $(X; \Gamma)$ *be a compact partial KKM space. Suppose that* $S : X \multimap X$ *and* $T : X \multimap X$ *are two set-valued mappings with the following conditions:*

(i) $x \notin \mathrm{co}_\Gamma T(x)$ *for all* $x \in X$*; and*
(ii) *if* $S(x) \neq \emptyset$*, then there exists* $y \in X$ *such that* $x \in \mathrm{int}T^{-1}(y)$*.*

Then there exists $x_0 \in X$ *such that* $S(x_0) = \emptyset$*.*

Proof. Suppose the contrary that $S(x) \neq \emptyset$ for all $x \in X$. Then by Theorem 5.2, there exists $x^* \in X$ such that $x^* \notin \mathrm{co}_\Gamma T(x^*)$ which contradicts the first condition. □

Corollary 6.2. *Let* X *be a nonempty convex compact subset of hyperbolic space* M*. Suppose that* $A : X \multimap X$ *and* $B : X \multimap X$ *are two set-valued mappings with the following conditions:*

(i) *for each* $x \in X$, $x \notin G\mathrm{co}B(x)$*;*
(ii) *if* $A(x) \neq \emptyset$*, then there exists* $y \in X$ *such that* $x \in \mathrm{int}B^{-1}(y)$*.*

Then there exists $x^* \in X$ *such that* $A(x^*) = \emptyset$*.*

Corollary 6.3. *Let X be a nonempty convex compact subset of hyperbolic space M. Suppose that $A : X \multimap X$ and $B : X \multimap X$ are two set-valued mappings with the following conditions:*

(i) *for each $x \in X$, $A(x) \subset B(x)$;*
(ii) *for each $x \in X$, $x \notin G\mathrm{co}B(x)$;*
(iii) *for each $y \in X$, $B^{-1}(y)$ is open in X.*

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

Corollary 6.4. *Let X be a nonempty convex compact subset of hyperbolic space M. Suppose that $A : X \multimap X$ is a set-valued mapping with the following conditions:*

(i) *for any $x \in X$, $x \notin G\mathrm{co}A(x)$;*
(ii) *if $A(x) \neq \emptyset$, then there exists $y \in X$ such that $x \in \mathrm{int}A^{-1}(y)$.*

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

Remark 6.5. In Theorem 3.2 and Corollary 3.3-4 of [18], Corollary 6.2-4 for Hadamard manifolds with strongly geodesic convexity of X were proved.

7. A minimax inequality on hyperbolic spaces

Let $X = \prod_{i=1}^n X_i$, $X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$ and $\Gamma = \prod_{i=1}^n \Gamma_i$ be the componentwise multimap. For $A \subset X$, $x_i \in X_i$, $x_{-i} \in X_{-i}$, define $A(x_i)$ and $A(x_{-i})$ as follows:

$$A(x_i) := \{x_{-i} \in X_{-i} | (x_i, x_{-i}) \in A\}, \quad A(x_{-i}) := \{x_i \in X_i | (x_i, x_{-i}) \in A\}.$$

The following section theorem of von Neumann-Fan type already proved by Park (XXII) of [15] using collective fixed point theorem. We give another proof using Fan-Browder type fixed point theorem.

Theorem 7.1. (Section theorem) *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ satisfies the partial KKM principle and, for each i, let A_i and B_i are subsets of X satisfying the following:*

(i) *for each $y \in X$, $B_i(y) := \{x \in X \mid (x_{-i}, y_i) \in B_i\}$ is open; and*
(ii) *for each $x \in X$, $\emptyset \neq \mathrm{co}_\Gamma B_i(x) \subset A_i(x) := \{y \in X \mid (x_{-i}, y_i) \in A_i\}$.*

Then we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

Proof. Define maps $T, S : X \multimap X$ by $T(x) := \bigcap_{i=1}^n A_i(x)$ and $S(x) := \bigcap_{i=1}^n B_i(x)$ for $x \in X$. From (ii), we have

$$\mathrm{co}_\Gamma S(x) = \mathrm{co}_\Gamma \left(\bigcap_{i=1}^n B_i(x) \right) \subset \bigcap_{i=1}^n \mathrm{co}_\Gamma B_i(x) \subset \bigcap_{i=1}^n A_i(x) = T(x)$$

for each $x \in X$. For each $x \in E$ and each i, there exists a $y^{(i)} \in B_i(x)$ by (ii), or $(x_{-i}, y_i^{(i)}] \in B_i$. Hence, we have $(y_1^{(1)}, \ldots, y_n^{(n)}) \in \bigcap_{i=1}^n B_i(x)$. This shows $S(x) \neq \emptyset$. Moreover, $S^-(y) = \bigcap_{i=1}^n B_i(y)$ is open for each $y \in E$ by (i). Since X is compact, it is covered by a finite number of $S^-(y)$'s. Hence, all the requirements of Theorem 5.2 are satisfied. Therefore, there exists an $x^0 \in T(x^0) = \bigcap_{i=1}^n A_i(x^0)$, that is, $x^0 \in A_i$ for all i. □

Corollary 7.2. *For each $i = 1, 2, \cdots, n$, let X_i be a nonempty convex compact subset of hyperbolic spaces M_i . Let $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ be $2n$ subsets of X such that*

(i) *for each i and any $x_{-i} \in X_{-i}$, $G\mathrm{co}B_i(x_{-i}) \subset A_i(x_{-i})$;*
(ii) *for each i and any $x_{-i} \in X_{-i}$, there exists $y_i \in X_i$ such that*

$$x_{-i} \in \mathrm{int}B_i(y_i).$$

Then $\bigcap_{i=1}^n A_i \neq \emptyset$.

Remark 7.3. (1) Theorem 7.1 generalizes historically well-known intersection theorems due to von Neumann, Fan, Bielawski, Kirk et al., and Park; see [7], [9].
(2) In Theorem 3.5 of [18], Corollary 7.2 for Hadamard manifolds with strongly geodesic convexity of M were proved.

Theorem 7.4. (Minimax inequality) *Let $(X;\Gamma)$ be a nonempty compact partial KKM space, and let $f : X \times X \to \mathbb{R}$ be a real-valued function such that*

(i) *for each $x \in X$, $y \mapsto f(x, y)$ is quasi-concave on X;*
(ii) *for each $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous on X;*
(iii) *$f(x, x) \leq 0$ for all $x \in X$.*

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for all $y \in X$.

Proof. By $F(x) := \{y \in X | f(x, y) > 0\}$, we can define convex set-valued map $F : X \multimap X$.

Suppose for any $x \in X$, there is $y \in X$ such that $f(x, y) > 0$. This implies that $F(x)$ is non-empty. The condition (ii) shows $F^-(y)$ is open for all $y \in X$. Using Theorem 5.2 with $H = F$, we can find a fixed point x^* of F, that is, a point satisfying $f(x^*, x^*) > 0$ which contradicts condition (iii). Therefore, for some $x^* \in X$, we have $f(x^*, y) \leq 0$ for all $x \in X$. □

Corollary 7.5. *Let X be a nonempty convex compact subset of hyperbolic space M, and let $f : X \times X \to \mathbb{R}$ be a real-valued function such that*

(i) *for each $x \in X$, $y \mapsto f(x, y)$ is quasi-concave on X;*
(ii) *for each $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous on X;*
(iii) *$f(x, x) \leq 0$ for all $x \in X$.*

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for all $y \in X$.

Remark 7.6. In Theorem 3.6 of [18], Corollary 7.5 for Hadamard manifolds with strongly geodesic convexity of M were proved.

8. Nash equilibrium points on hyperbolic spaces

In [14], Park proved the following generalized Nash equilibrium theorem using the preceding section theorem. We give its proof for the completeness.

Theorem 8.1. (Generalized Nash equilibrium theorem) *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i, g_i : X = X_{-i} \times X_i \to \mathbb{R}$ be real functions such that*

(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*
(ii) *for each $x^i \in X^i$, $x_i \mapsto f_i(x_{-i}, x_i)$ is quasiconcave on X_i;*
(iii) *for each $x^i \in X^i$, $x_i \mapsto g_i(x_{-i}, x_i)$ is u.s.c. on X_i; and*
(iv) *for each $x_i \in X_i$, $x^i \mapsto g_i(x_{-i}, x_i)$ is l.s.c. on X^i.*

Then there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i(\hat{x}_{-i}, y_i) \quad \textit{for all } i.$$

Proof. For any $\varepsilon > 0$, we define

$$A_{\varepsilon,i} = \left\{ x \in X \mid f_i(x) > \max_{y_i \in X_i} g_i(x_{-i}, y_i) - \varepsilon \right\},$$

$$B_{\varepsilon,i} = \left\{ x \in X \mid g_i(x) > \max_{y_i \in X_i} g_i(x_{-i}, y_i) - \varepsilon \right\}$$

for each i. Then

(1) for each $x_{-i} \in X_{-i}$, $B_{\varepsilon,i}(x^i) \subset A_{\varepsilon,i}(x^i)$;
(2) for each $x_{-i} \in X_{-i}$, $A_{\varepsilon,i}(x_{-i})$ is Γ_i-convex;
(3) for each $x_{-i} \in X_{-i}$, $B_{\varepsilon,i}(x_i) \neq \emptyset$ since $x_i \mapsto g_i(x_{-i}, x_i)$ is u.s.c. on the compact space X_i; and
(4) for each $x_i \in X_i$, $B_{\varepsilon,i}(x_i)$ is open since $x^i \mapsto g_i(x_{-i}, x_i)$ is l.s.c. on X^i.

Therefore, by applying Theorem 7.1, we have

$$\bigcap_{i=1}^{n} A_{\varepsilon,i} \neq \emptyset \quad \text{for every} \quad \varepsilon > 0.$$

Since X is compact, there exists an $\hat{x} \in X$ such that $f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i(\hat{x}_{-i}, y_i)$ for all i. □

The *n-person non-cooperative game $G\{I, X_i, f_i\}$ on partial KKM spaces* consists of

(1) $I = \{1, 2, \cdots, n\}$ is the set of players;
(2) for each $i \in I$, the nonempty strategy set X_i of the ith player is a partial KKM space; and
(3) for each $i \in I$, the payoff function $f_i : \prod_{i \in I} X_i \to \mathbb{R}$ of the ith player.

A point $x^* = ({x^*}_i, {x^*}_{-i}) \in X = \prod_{i \in I} X_i$ is called *a Nash equilibrium point* if for each $i \in I$, $f_i({x^*}_i, {x^*}_{-i}) = \max_{u_i \in X_i} f_i(u_i, {x^*}_{-i})$.

This concept is a natural extension of the local maxima (for the case $n = 1$, $f = f_1$) and of the saddle points (for the case $n = 2$, $f_1 = -f$, $f_2 = f$).

From Theorem 8.1, we obtain the following form of the Nash equilibrium theorem for abstract convex spaces:

Theorem 8.2. *Let $G\{I, X_i, f_i\}$ be a n-person non-cooperative game on partial KKM spaces satisfying:*

(i) *for each $i \in I$, X_i is compact;*
(ii) *for each $i \in I$, f_i is upper semicontinuous on X;*
(iii) *for each $i \in I$ and each fixed $u_i \in X_i$, $f_i(u_i, \cdot)$ is lower semicontinuous on X^i; and*
(iv) *for each $i \in I$ and each fixed $u_{-i} \in X_{-i}$, $f_i(\cdot, u_{-i})$ is quasi-concave on X_i.*

Then there exists at least one Nash equilibrium point $x^* \in X$.

Remark 8.3. (1) Theorem 8.1 is also proved in (XXIV) of [15] using a Fan type analytic alternative.
(2) For G-convex spaces, Theorem 5 and Corollary 5.1 hold for not-necessarily finite family; see [6].
(3) Theorem 8.2 generalizes well-known equilibrium theorems due to Nash, Fan, Bielawski, Kirk et al., and Park; see [2], [3], [13], [14] and references therein.

Corollary 8.4. *Let $G\{I, X_i, f_i\}$ be a n-person non-cooperative game on hyperbolic spaces satisfying:*

(i) *for each $i \in I$, X_i is convex compact subset of a hyperbolic space M_i;*
(ii) *for each $i \in I$, f_i is upper semicontinuous on X;*
(iii) *for each $i \in I$ and each fixed $u_i \in X_i$, $f_i(u_i, \cdot)$ is lower semicontinuous on X_{-i}; and*
(iv) *for each $i \in I$ and each fixed $u_{-i} \in X_{-i}$, $f_i(\cdot, u_{-i})$ is quasi-concave on X_i.*

Then there exists at least one Nash equilibrium point $x^* \in X$.

Theorem 8.5. *Let $G\{I, X_i, f_i\}$ be a n-person non-cooperative game on partial KKM spaces satisfying:*

(i) *for each $i \in I$, X_i is compact;*
(ii) *for each $i \in I$, $\sum_{i=1}^n f_i$ is upper semicontinuous on X;*
(iii) *for each $y \in X$, $x \mapsto \sum_{i=1}^n f_i(y_i, x_{-i})$ is lower semicontinuous on X; and*
(iv) *for each $x \in X$, $y \mapsto \sum_{i=1}^n f_i(y_i, x_{-i})$ is quasi-concave on X.*

Then there exists at least one Nash equilibrium point $x^* \in X$.

Proof. We define a function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \sum_{i=1}^{n} [f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})].$$

Then we can verify that

(1) for each $x \in X$, $y \mapsto \phi(x, y)$ is quasi-concave on X by condition (iv);
(2) for each $y \in X$, $x \mapsto \phi(x, y)$ is lower semicontinuous on X by conditions (ii) and (iii); and
(3) $\phi(x, x) \leq 0$ for all $x \in X$.

By minimax inequality [Theorem 7.4], there exists an $x^* \in X$ such that $\phi(x^*, y) \leq 0$ for all $y \in X$. For each $i \in I$ and any $u_i \in X_i$, this inequality for $y = (u_i, x^*_{-i}) \in X$ shows

$$f_i({x^*}_i, {x^*}_{-i}) = \max_{u_i \in X_i} f_i(u_i, {x^*}_{-i})$$

for all $i \in I$. Therefore x^* is a Nash equilibrium. □

Corollary 8.6. *Let $G\{I, X_i, f_i\}$ be a n-person non-cooperative game on hyperbolic spaces satisfying:*

(i) *for each $i \in I$, X_i is strongly geodesic convex compact subset of a hyperbolic space M_i;*
(ii) *for each $i \in I$, $\sum_{i=1}^n f_i$ is upper semicontinuous on X;*

(iii) *for each* $y \in X$, $x \mapsto \sum_{i=1}^{n} f_i(y_i, x_{-i})$ *is lower semicontinuous on* X*; and*

(iv) *for each* $x \in X$, $y \mapsto \sum_{i=1}^{n} f_i(y_i, x_{-i})$ *is quasi-concave on* X*.*

Then there exists at least one Nash equilibrium point $x^* \in X$*.*

Remark 8.7. (1) In Theorem 3.12 of [1], Corollary 8.3 for Hadamard manifolds was proved.
(2) In Theorem 4.1-2 of [18], Corollary 8.3 and 8.5 for Hadamard manifolds with strongly geodesic convexity of M were proved.
(3) Since every finite sum of u.s.c. or l.s.c. or quasi-concave maps also u.s.c. or l.s.c. or quasi-concave respectively, we can prove Theorem 8.2 using Theorem 8.4.

9. The Nemeth fixed point theorem on hyperbolic spaces

As is well-known, Park introduced a large number of generalized fixed point theorems. We follow one of them in Park [11]:

Definition 9.1. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is an abstract convex space with a basis $\mathcal{U}$ of a uniform structure of E.

The following gives a particular subclass or subsets of abstract convex uniform spaces.

Definition 9.2. An abstract convex uniform space $(E \supset D; \Gamma; \mathcal{U})$ is called an *L*Γ*-space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

Theorem 9.3. ([11], Corollary 4.5.) *Let* $(X \supset D; \Gamma; \mathcal{U})$ *be a Hausdorff KKM* $L\Gamma$*-space and* $T : X \multimap X$ *a compact u.s.c. map with nonempty closed* Γ*-convex values. Then* T *has a fixed point.*

Corollary 9.4. *Let* K *be a nonempty compact and convex subset of a hyperbolic space* M*. Then every continuous function* $f : K \to K$ *has a fixed point.*

Proof. In any hyperbolic spaces in the sense of Reich-Shafrir are metric spaces with a metric ρ. Since ρ-balls are ρ-convex, hyperbolic spaces and their ρ-convex subsets are locally convex in this sense. □

Corollary 9.5. ([4], Lemma 1) *Let* K *be a nonempty compact and geodesic convex subset of an Hadamard manifold* M*. Then every continuous function* $f : K \to K$ *has a fixed point.*

Remark 9.6. This is applied to show a variational inequality on Hadamard manifolds in [4].

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