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BALL CONVERGENCE FOR A NINTH ORDER NEWTON-TYPE METHOD FROM QUADRATURE AND A DOMIAN FORMULAE IN A BANACH SPACE

Ioannis K. Argyros[1] and Santhosh George[2]

[1]Department of Mathematical Sciences, Cameron University
Lawton, OK 73505, USA
e-mail: iargyros@cameron.edu

[2]Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, 757 025, India
e-mail: sgeorge@nitk.ac.in

Abstract. We present a local convergence analysis of a ninth order Newton-type method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. We only use hypotheses on the first Fréchet-derivative. The local convergence analysis in [7, 11, 12, 24] used hypotheses up to the second Fréchet derivative although only the first derivative appears in this method. Hence, the application of the methods is extended under less computational cost. This work also provides computable convergence ball and computable error bounds. Numerical examples are also provided in this study.

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Using mathematical modeling, many problems in computational sciences and other disciplines can be expressed as a nonlinear equation (1.1) [2, 5, 12, 22]. Closed form solutions of

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these nonlinear equations exist only for few special cases which may not be of much practical value. Therefore solutions of these nonlinear equations (1.1) are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods [1]-[24]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1]-[24].

We study the local convergence analysis of methods defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(x_n)^{-1}F(y_n), \end{aligned} \tag{1.2}$$

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - F'(x_n)^{-1}F(y_n), \\ x_{n+1} &= x_n - \left[\frac{1}{6}F'(x_n) + \frac{2}{3}F'\left(\frac{x_n+z_n}{2}\right) + \frac{1}{6}F'(x_n)\right]^{-1} F(x_n), \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - F'(x_n)^{-1}F(y_n), \\ w_n &= z_n - F'(z_n)^{-1}F(z_n), \\ x_{n+1} &= w_n - F'(z_n)^{-1}F(w_n), \end{aligned} \tag{1.4}$$

where x_0 is an initial point. Methods (1.2), (1.3) and (1.4) are based on quadrature and Adomian decomposition formulae and have convergence order 3 [2, 5], 4 [11, 12] and 9 [24], respectively when $X = Y = \mathbb{R}^m$. Moreover, the convergence of the first two methods was shown under hypotheses reaching up to the third Fréchet derivative of operator F whereas the convergence of method (1.4) was shown under hypotheses up to ninth Fréchet derivative (although only the first Fréchet derivative appears in these methods). The hypotheses on the Fréchet derivatives limit the applicability of these methods. As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We have that

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x$$

and

$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Then, obviously, function F''' is unbounded on D. Notice also that the proofs of convergence use Taylor expansions. In the present study, we study the local convergence of these methods using hypotheses only on the first Fréchet derivative taking advantage of the Lipschitz continuity of the first Fréchet - derivative. This way, we expand the applicability of these methods.Notice also that our results are presented in the more general setting of a Banach space.

The rest of the paper is organized as follows. In Section 2 and Section 3 we present, respectively the local convergence of method (1.4) and method (1.3). The convergence of method (1.2) is presented in Section 3 as a special case of method (1.3). Finally the numerical examples are presented in the concluding Section 4.

2. Local convergence analysis

We present the local convergence analysis of method (1.4) in this section. Let $L_0 > 0$, $L > 0$, $M \geq 1$ be given parameters. It is convenient for the local convergence analysis that follows to introduce some functions and parameters. Define function g_1 on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)}$$

and parameter

$$r_1 = \frac{2}{2L_0 + L} < \frac{1}{L_0}. \tag{2.1}$$

Then, we have that $g_1(r_1) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$. Moreover, define functions g_2 and h_2 on the interval $[0, \frac{1}{L_0})$ by

$$g_2(t) = \left(1 + \frac{M}{1 - L_0 t}\right) g_1(t)$$

and

$$h_2(t) = g_2(t) - 1.$$

We have that $h_2(0) = -1 < 0$ and $h_2(r_1) = g_1(r_1) + \frac{M}{1-L_0 r_1} g_1(r_1) - 1 = \frac{M}{1-L_0 r_1} > 0$, since $g_1(r_1) = 1$ and $1 - L_0 r_1 > 0$. It follows from the intermediate value theorem that function h_2 has zeros in the interval $(0, r_1)$. Denote by r_2

the smallest such zero. Furthermore, define functions g_3 and h_3 on the interval $[0, \frac{1}{L_0})$ by

$$g_3(t) = \frac{Lg_2(t)^2 t}{2(1 - L_0 t)}$$

and

$$h_3(t) = g_3(t) - 1.$$

We have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It follows that function h_3 has zeros in the interval $(0, \frac{1}{L_0})$ denoted by r_3. Notice that $h_3(r_2) = \frac{Lr_2}{2(1-L_0 r_2)} - 1 < 0$, since $r_2 < r_1$ and $g_2(r_2) = 1$, so that $r_2 < r_3$. Finally, define functions g_4 and h_4 on the interval $[0, \frac{1}{L_0})$ by

$$g_4(t) = \left(1 + \frac{M}{1 - L_0 t}\right) g_3(t)$$

and

$$h_4(t) = g_4(t) - 1.$$

We have that $h_4(0) = -1 < 0$ and $h_4(r_3) = \frac{M}{1-L_0 r_3} > 0$, as since $M \geq 1$ and $1 - L_0 r_3 > 0$. Denote by r the smallest zero of function h_4 on the interval $[0, r_3)$. Then, for each $t \in [0, r)$

$$0 \leq g_1(t) < 1, \tag{2.2}$$

$$0 \leq g_2(t) < 1, \tag{2.3}$$

$$0 \leq g_3(r) < 1 \tag{2.4}$$

and

$$0 \leq g_4(t) < 1. \tag{2.5}$$

Let $U(v, \rho)$ and $\overline{U}(v, \rho)$ denote the open and closed ball in X, respectively, with center $v \in X$ and of radius $\rho > 0$.

Next, we present the local convergence analysis of method (1.4) using the preceeding notation.

Theorem 2.1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$ and $M \geq 1$ such that for each $x, y \in D$*

$$F(x^*) = 0,\ F'(x^*)^{-1} \in L(Y, X), \tag{2.6}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|, \tag{2.7}$$

$$\|F'(x^*)^{-1}(F(x) - F(y))\| \leq L\|x - y\|, \tag{2.8}$$

$$\|F'(x^*)^{-1} F'(x)\| \leq M \tag{2.9}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.10}$$

where r is defined above Theorem 2.1. Then, the sequence $\{x_n\}$ generated by method (1.4) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.11}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.12}$$

$$\|w_n - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.13}$$

and

$$\|x_{n+1} - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.14}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, if there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T)$.

Proof. We shall show estimates (2.11)–(2.14) using mathematical induction. Using (2.1), (2.7) and the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$ we get that

$$\|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.15}$$

It follows from (2.15) and the Banach Lemma on invertible operators [2, 5, 17] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r}. \tag{2.16}$$

Hence, y_0 and z_0 are well defined by method (1.4) for $n = 0$. (2.1), (2.2), (2.8) and (2.16) that

$$\begin{aligned}
\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1} \\
&\quad \times [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)(x_0 - x^*)]\|d\theta \\
&\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned} \tag{2.17}$$

which shows (2.11) for $n = 0$ and $y_0 \in U(x^*, r)$. Notice that for each $\theta \in [0, 1]$ $\|x^* + \theta(x_0 - x^*)\| = \theta\|x_0 - x^*\| < r$. That is $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. We can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))d\theta. \tag{2.18}$$

Then, using (2.9) and (2.17) we get that

$$\begin{aligned} \|F'(x^*)^{-1}F(x_0)\| &= \left\| \int_0^1 F'(x^* + \theta(x_0 - x^*))d\theta \right\| \\ &\leq M\|x_0 - x^*\| \end{aligned} \tag{2.19}$$

and

$$\|F'(x^*)^{-1}F(y_0)\| \leq M\|y_0 - x^*\|. \tag{2.20}$$

Using second substep of method (1.4), (2.3), (2.16), (2.17) and (2.20) we have in turn that

$$\begin{aligned} \|z_0 - x^*\| &\leq \|y_0 - x^*\| + \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\ &\leq \|y_0 - x^*\| + \frac{M\|y_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \\ &\leq \left(1 + \frac{M}{1 - L_0\|x_0 - x^*\|}\right)\|y_0 - x^*\| \\ &\leq \left(1 + \frac{M}{1 - L_0\|x_0 - x^*\|}\right) g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.21}$$

which shows (2.12) for $n = 0$ and $z_0 \in U(x^*, r)$. Then, as in (2.16) we also have by (2.3) that

$$\begin{aligned} \|F'(z_0)^{-1}F'(x^*)\| &\leq \frac{1}{1 - L_0\|z_0 - x^*\|} \\ &\leq \frac{1}{1 - L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|} \\ &\leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \end{aligned} \tag{2.22}$$

Hence, w_0 and x_1 are well defined by method (1.4) for $n = 0$. Then, by using (2.4), (2.21) and (2.22) as in (2.17) we get that

$$\begin{aligned} \|w_0 - x^*\| &\leq \frac{L\|z_0 - x^*\|^2}{2(1 - L_0\|z_0 - x^*\|)} \\ &\leq \frac{L g_2(\|x_0 - x^*\|)^2\|x_0 - x^*\|^2}{2(1 - L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|)} \\ &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &< \|x_0 - x^*\| < r, \end{aligned} \tag{2.23}$$

which shows (2.13) for $n = 0$ and $w_0 \in U(x^*, r)$. Then, using (2.1), (2.5), (2.20) (for $y_0 = w_0$), (2.22) and (2.23) we obtain that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|w_0 - x^*\| + \frac{M\|w_0 - x^*\|}{1 - L_0\|z_0 - x^*\|} \\ &\leq \left(1 + \frac{M}{1 - L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}\right)\|w_0 - x^*\| \\ &\leq \left(1 + \frac{M}{1 - L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}\right) g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq g_4(\|x_0 - x^*\|)\|x_0 - x^*\| < r, \end{aligned} \tag{2.24}$$

which shows (2.14) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, w_0, x_1 by x_k, y_k, z_k, w_k, x_{k+1} in the preceding estimates we arrive at (2.11)-(2.14). Then, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Finally, to show the uniqueness part, let $Q = \int_0^1 F'(y^* + t(x^* - y^*))dt$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.10), we get that

$$\begin{aligned} \|F'(x^*)^{-1}(Q - F'(x^*))\| &\leq \int_0^1 L_0\|y^* + t(x^* - y^*) - x^*\|dt \\ &\leq \int_0^1 (1 - t)\|x^* - y^*\|dt \\ &\leq \frac{L_0}{2}T < 1. \end{aligned} \tag{2.25}$$

It follows from (2.25) that Q is invertible. Then, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. □

Remark 2.2. (1) In view of (2.7) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ &\leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition (2.9) can be dropped and be replaced by

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$. Moreover, condition (2.8) can be replaced by the condition

$$\begin{aligned} &\|F'(x^*)^{-1}(F'(x^* + \theta(x - x^*)) - F'(x))\| \\ &\leq \bar{L}(1 - \theta)\|x - x^*\| \end{aligned}$$

for each $x, y \in D$, some $\bar{L} \in (0, L]$ and $\theta \in [0, 1]$.
(2) The results obtained here can be used for operators F satisfying autonomous differential equations [2, 5] of the form

$$F'(x) = T(F(x)),$$

where T is a continuous operator. Then, since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$.
(3) The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2, 5].
(4) The parameter r_1 given by (2.1) was shown by us to be the convergence radius of Newton's method [2, 6]

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.26}$$

under the conditions (2.6)– (2.8). It follows from the definition that the convergence radius r of method (1.4) cannot be larger than the convergence radius r_1 of the second order Newton's method (2.26). As already noted in [2, 5] r_1 is at least as large as the convergence ball given by Rheinboldt [21]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \quad as \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [22].
(5) It is worth noticing that method (1.4) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [7, 11, 12, 24]. Moreover, we can compute the computational order of convergence (COC) [2, 5, 13] defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence.

3. Local convergence of method (1.3)

We present the local convergence analysis of method (1.3) in this section. We define functions g_1, g_2, h_2 and parameters r_1 and r_2 as in Section 2. Then, define functions p and h_p on the interval $[0, \frac{1}{L_0})$ by

$$p(t) = \frac{L_0}{2}(1 + g_2(t))t$$

and

$$h_p(t) = p(t) - 1.$$

Then, $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Hence, function h_p has a minimal zero in the interval $(0, \frac{1}{L_0})$ denoted by r_p. Moreover, define functions g_3 and h_3 on the interval $[0, \frac{1}{L_0})$ by

$$g_3(t) = g_1(t) + \frac{LM(1 + g_2(t))t}{2(1 - L_0 t)(1 - p(t))}$$

and

$$h_3(t) = g_3(t) - 1.$$

Then, again we have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to r_p^-$. Denote by r_3 the smallest zero of function h_3 in the interval $(0, r_p)$. Notice that $h_1(r_1) = \frac{M}{1 - L_0 r_1} > 0$, since $g_1(r_1) = 1$ and $1 - L_0 r_1 > 0$. Hence, we have that $r_2 < r_1$. Set

$$r = \min\{r_2, r_3\}. \tag{3.1}$$

Then, we have that for each $t \in [0, r)$

$$0 \le g_1(t) < 1,$$

$$0 \le g_2(t) < 1, \tag{3.2}$$

$$0 \le g_3(t) < 1. \tag{3.3}$$

Next, we present the local convergence analysis of method (1.3) in an analogous way to method (1.4) using the preceeding notation.

Theorem 3.1. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied but* r *is defined by* (3.1)*. Then, the conclusions of Theorem* 2.1 *hold with method* (1.3) *replacing method* (1.4)*.*

Proof. According to the proof of Theorem 2.1 we only need to show using mathematical induction that

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < r. \tag{3.4}$$

We have by (2.8) and (2.12) that

$$\begin{aligned}
&\left\| F'(x^*)^{-1} \left[\frac{1}{6} F'(x_0) + \frac{2}{3} F'\left(\frac{x_0 + z_0}{2} \right) + \frac{1}{6} F'(z_0) - F'(x_0) \right] \right\| \\
&\leq \frac{4}{6} \left\| F'(x^*)^{-1} \left(F'\left(\frac{x_0 + z_0}{2} \right) - F'(x_0) \right) \right\| \\
&\quad + \frac{1}{6} \|F'(x^*)^{-1}(F'(z_0) - F'(x_0))\| \\
&\leq \frac{2}{3} L \left\| \frac{x_0 + z_0}{2} - x_0 \right\| + \frac{1}{6} \|z_0 - x_0\| = \frac{L}{2} \|x_0 - z_0\| \\
&\leq \frac{L}{2} (\|x_0 - x^*\| + \|z_0 - x^*\|) \\
&\leq \frac{L}{2} (1 + g_2(\|x_0 - x^*\|))\|x_0 - x^*\|.
\end{aligned} \tag{3.5}$$

Moreover, we get by (2.7), (2.12) and (3.2) that

$$\begin{aligned}
&\left\| F'(x^*)^{-1} \left[\frac{1}{6} F'(x_0) + \frac{2}{3} F'\left(\frac{x_0 + z_0}{2} \right) + \frac{1}{6} F'(z_0) - F'(x^*) \right] \right\| \\
&\leq \frac{1}{6} \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\
&\quad + \frac{2}{3} \left\| F'(x^*)^{-1} \left(F'\left(\frac{x_0 + z_0}{2} \right) - F'(x^*) \right) \right. \\
&\quad + \frac{1}{6} \|F'(x^*)^{-1}(F'(z_0) - F'(x^*))\| \\
&\leq \frac{L_0}{6} \|x_0 - x^*\| + \frac{4}{6} L_0 \left\| \frac{x_0 + z_0}{2} - x^* \right\| + \frac{L_0}{6} \|x_0 - x^*\| \\
&\leq \frac{L_0}{6} \|x_0 - x^*\| + \frac{1}{3} (\|x_0 - x^*\| + \|z_0 - x^*\|) + \frac{L_0}{6} \|z_0 - x^*\| \\
&= \frac{L_0}{2} (\|x_0 - x^*\| + \|z_0 - x^*\|) \\
&\leq \frac{L_0}{2} (1 + g_2(\|x_0 - x^*\|))\|x_0 - x^*\| \\
&= p(\|x_0 - x^*\|) < 1.
\end{aligned} \tag{3.6}$$

If follows from (3.6) that $\frac{1}{6} F'(x_0) + \frac{2}{3} F'(\frac{x_0+z_0}{2}) + \frac{1}{6} F'(z_0)$ is invertible and

$$\left\|\left(\frac{1}{6}F'(x_0)+\frac{2}{3}F'\left(\frac{x_0+z_0}{2}\right)+\frac{1}{6}F'(z_0)\right)^{-1}F'(x^*)\right\|$$
$$\leq \frac{1}{1-p(\|x_0-x^*\|)}. \tag{3.7}$$

Hence, x_1 is well defined by the last substep of method (1.3) for $n=0$. Then, using (2.11), (2.16), (2.19), (4.1), (3.5), (3.7) and the third substep of method (1.3) for $n=0$, we get in turn that

$$\begin{aligned}
&x_1-x^*\\
&=x_0-x^*-F'(x_0)^{-1}F(x_0)+F'(x_0)^{-1}F(x_0)\\
&\quad-\left(\frac{1}{6}F'(x_0)+\frac{2}{3}F'\left(\frac{x_0+z_0}{2}\right)+\frac{1}{6}F'(z_0)\right)^{-1}F(x_0)\\
&=y_0-x^*+\left[F'(x_0)^{-1}-\left(\frac{1}{6}F'(x_0)+\frac{2}{3}F'\left(\frac{x_0+z_0}{2}\right)+\frac{1}{6}F'(z_0)\right)^{-1}\right]F(x_0),
\end{aligned}$$

so,

$$\begin{aligned}
&\|x_1-x^*\|\\
&\leq\|y_0-x^*\|+\|F'(x_0)^{-1}F'(x^*)\|\\
&\quad\times\left\|\left(\frac{1}{6}F'(x_0)+\frac{2}{3}F'\left(\frac{x_0+z_0}{2}\right)+\frac{1}{6}F'(z_0)\right)^{-1}F'(x^*)\right\|\\
&\quad\times\left\|F'(x^*)^{-1}\left[\frac{1}{6}F'(x_0)+\frac{2}{3}F'\left(\frac{x_0+z_0}{2}\right)+\frac{1}{6}F'(z_0)-F'(x_0)\right]\right\|\\
&\quad\times\|F'(x^*)^{-1}F(x_0)\|\\
&\leq g_1(\|x_0-x^*\|)\|x_0-x^*\|\\
&\quad+\frac{LM(1+g_2(\|x_0-x^*\|))\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)(1-p(\|x_0-x^*\|))}\\
&=g_3(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r,
\end{aligned}$$

which shows (3.4) for $n=0$ and $x_1\in U(x^*,r)$. The rest of the proof follows as the proof of Theorem 2.1. □

Remark 3.2. (a) Comments for method (1.3) can follow immediately as in Remark 2.2.

(b) In order to present the corresponding results for method (1.2), we simply restrict to the definition of functions g_1, g_2 and parameters r_1

and r_2. Moreover, we define

$$r = r_2. \tag{3.8}$$

Hence, in view of the proof of Theorem 3.1, we arrive at

Theorem 3.3. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied but with* r *defined by* (3.8). *Then, the conclusions of Theorem* 3.1 *hold* (*except* (2.13)) *but with method* (1.2) *replacing method* (1.3).

4. Numerical examples

We present numerical examples in this section.

Example 4.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that we get $L_0 = e - 1, L = e, M = 2$. The parameters are given in Table 1

Table 1. Comparison Table

method (1.4)	method (1.3)	method (1.2)
$r_1 = 0.3249$	$r_1 = 0.3249$	$r_1 = 0.3249$
$r_2 = 0.1486$	$r_2 = 0.1486$	$r_2 = 0.1486$
$r_3 = 0.0014$	$r_3 = 0.1000$	
$r = 0.0003$	$r = 0.1000$	$r = 0.1486$

Table 1

Example 4.2. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 2$. The parameters are given in Table 2

TABLE 2. Comparison Table

method (1.4)	method (1.3)	method (1.2)
$r_1 = 0.0045$	$r_1 = 0.0045$	$r_1 = 0.0045$
$r_2 = 0.0023$	$r_2 = 0.0023$	$r_2 = 0.0023$
$r_3 = 0.0637$	$r_3 = 0.0015$	
$r = 0.0204$	$r = 0.0015$	$r = 0.0023$

Table 2

Example 4.3. Let $X = Y = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ and be equipped with the max norm. Let $D = \overline{U}(0, 1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{4.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15, M = 2$. The parameters are given in Table 3

TABLE 3. Comparison Table

method (1.4)	method (1.3)	method (1.2)
$r_1 = 0.0667$	$r_1 = 0.0667$	$r_1 = 0.0667$
$r_2 = 0.0292$	$r_2 = 0.0292$	$r_2 = 0.0292$
$r_3 = 0.7405$	$r_3 = 0.0196$	
$r = 0.3167$	$r = 0.0196$	$r = 0.0292$

Table 3

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