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CONVERGENCE THEOREMS OF MODIFIED THREE-STEP ITERATIONS FOR NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

G. S. Saluja

Department of Mathematics, Govt. Nagarjuna P.G. College of Science
Raipur - 492001 (C.G.), India
e-mail: saluja1963@gmail.com

Abstract. In this paper, we study modified three-step iteration scheme and establish some strong and Δ-convergence theorems for nearly asymptotically nonexpansive mappings in the framework of hyperbolic spaces. Our results extend and generalize the previous works given in the current existing literature.

1. Introduction

The class of asymptotically nonexpansive mapping introduced by Goebel and Kirk [7] in 1972, is an important generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self mapping of C has a fixed point. There are number of papers dealing with the approximation of fixed points of asymptotically nonexpansive, asymptotically quasi-nonexpansive mappings, asymptotically nonexpansive type mappings and asymptotically quasi-nonexpansive type mappings in uniformly convex Banach spaces and are studied by many authors using modified Mannn, modified Ishikawa and modified Noor iteration processes (see, e.g. [19, 20, 29], [34]-[37], [43], [46]-[48]).

The concept of Δ-convergence in a general metric space was introduced by Lim [18]. In 2008, Kirk and Panyanak [16] used the notion of Δ-convergence

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introduced by Lim [18] to prove results in the CAT(0) space and analogous of some Banach space which involve weak convergence. Further, Dhompongsa and Panyanak [6] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the existence problem and the Δ-convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive type mapping, generalized asymptotically quasi-nonexpansive mapping, nearly asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping in the intermediate sense, total asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping through Picard, Mann [21], Ishikawa [12], modified S-iteration [2] have been rapidly developed in the framework of CAT(0) space and many papers have appeared in this direction (se, e.g., [1, 5, 6, 13, 22, 30, 39, 40]).

The purpose of this paper is to establish a Δ-convergence and some strong convergence theorems of modified three-step iteration process for nearly asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces which include both uniformly convex Banach spaces and CAT(0) spaces. Our results extend and improve the previous work from the current existing literature.

2. Preliminaries

Let $F(T) = \{x \in K : Tx = x\}$ denotes the set of fixed point of the mapping T. We begin with the following definitions.

Definition 2.1. Let (X, d) be a metric space and K be its nonempty subset. Then $T\colon K \to K$ said to be

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(2) asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(3) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, p) \leq (1 + u_n)d(x, p)$ for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(4) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(5) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$ as

$k \to \infty$;

(6) a sequence $\{x_n\}$ in K is called approximate fixed point sequence for T (AFPS, in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu [32] (see, also [33]).

Definition 2.2. Let K be a nonempty subset of a metric space (X, d) and fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$. A mapping $T\colon K \to K$ said to be nearly Lipschitzian with respect to $\{a_n\}$ if for all $n \geq 1$, there exists a constant $k_n \geq 0$ such that $d(T^n x, T^n y) \leq k_n[d(x, y) + a_n]$ for all $x, y \in K$.

The infimum of the constants k_n for which the above inequality holds is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant of T^n.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be:

(i) nearly nonexpansive if $\eta(T^n) = 1$ for all $n \geq 1$;
(ii) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} \eta(T^n) = 1$.
(iii) nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \geq 1$.

Throughout this paper, we work in the setting of hyperbolic space introduced by Kohlenbach [17]. It is worth noting that they are different from Gromov hyperbolic space [4] or from other notions of hyperbolic space that can be found in the literature (see for example [8, 15, 28]).

A hyperbolic space [17] is a triple (X, d, W) where (X, d) is a metric space and $W\colon X^2 \times [0, 1] \to X$ is such that

(i) $d(u, W(x, y, \alpha)) \leq \alpha\, d(u, x) + (1 - \alpha) d(u, y)$,
(ii) $d\Big(W(x, y, \alpha), W(x, y, \beta)\Big) = |\alpha - \beta|\, d(x, y)$,
(iii) $W(x, y, \alpha) = W(x, y, (1 - \alpha))$,
(iv) $d\Big(W(x, z, \alpha), W(y, w, \beta)\Big) \leq \alpha\, d(x, y) + (1 - \alpha)\, d(z, w)$,

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

The class of hyperbolic spaces in the sense of Kohlenbach [17] contains all normed linear spaces and convex subsets thereof as well as Hadamard manifolds and CAT(0) spaces in the sense of Gromov [10]. An important example of a hyperbolic space is the open unit ball B_H in a real Hilbert space H is as follows.

Let B_H be the open unit ball in H. Then

$$k_{B_H}(x,y) = arg\,tanh(1-\sigma(x,y))^{1/2},$$

where

$$\sigma(x,y) = \frac{\Big(1-\|x\|^2\Big)\Big(1-\|y\|^2\Big)}{|1-\langle\, x, y\,\rangle|^2}$$

for all x, $y \in B_H$, defines a metric on B_H (also known as Kobayashi distance). The open unit ball B_H together with this metric is coined as Hilbert ball. Since (B_H, k_{B_H}) is a unique geodesic space, so one can define a convexity mapping W for the corresponding hyperbolic space (B_H, k_{B_H}, W).

A metric space (X, d) is called a convex metric space introduced by Takahashi in [45] if it satisfies only (i). A subset K of a hyperbolic space X is *convex* if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

A hyperbolic space (X, d, W) is said to be uniformly convex [44] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that $d(W(x, y, \frac{1}{2}), u) \le (1-\delta)r$ whenever $d(x, u) \le r$, $d(y, u) \le r$ and $d(x, y) \ge \varepsilon\, r$.

A mapping $\eta\colon (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

Let K be a nonempty subset of hyperbolic space X. Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(., \{x_n\})\colon X \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in X\}$. The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \Big\{x \in X : r(x, \{x_n\}) \le r(y, \{x_n\})\Big\} \quad \text{for any } y \in K.$$

The set of all asymptotic center of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [44] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [16]. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Recall that Δ-convergence coincides with weak convergence in Banach space with Opial's property [26].

In the sequel we need the following lemmas.

Lemma 2.3. ([14]) *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some b, $c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 2.4. ([14]) *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 2.5. ([46]) *Let $\{p_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of non-negative numbers satisfying the inequality*

$$p_{n+1} \le (1 + q_n)p_n + r_n, \quad \forall\, n \ge 1.$$

If $\sum_{n=1}^\infty q_n < \infty$ and $\sum_{n=1}^\infty r_n < \infty$, then $\lim_{n\to\infty} p_n$ exists.

First, we define the modified three-step iteration scheme in hyperbolic space as follows.

Let K be a nonempty closed convex subset of a hyperbolic space X and $T\colon K \to K$ be a nearly asymptotically nonexpansive mapping. Then, for an arbitrary chosen $x_1 \in K$, we construct the sequence $\{x_n\}$ in K such that

$$\begin{cases} x_{n+1} = W(T^n x_n, T^n y_n, \alpha_n), \\ \quad y_n = W(x_n, T^n z_n, \beta_n), \\ \quad z_n = W(x_n, T^n x_n, \gamma_n), \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in (0,1) is called modified three-step iteration scheme. Iteration scheme (2.1) is independent of modified Noor iteration, modified Ishikawa iteration and modified Mann iteration schemes.

If $\gamma_n = 0$ for all $n \ge 1$, then iteration scheme (2.1) reduces to the following iteration scheme:

$$\begin{cases} x_{n+1} = W(T^n x_n, T^n y_n, \alpha_n), \\ \quad y_n = W(x_n, T^n x_n, \beta_n), \quad n \ge 1, \end{cases} \tag{2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in (0,1) is called modified S-iteration scheme.

Iteration procedures in fixed point theory are lead by the considerations in summability theory. For example, if a given sequence converges, then we don't look for the convergence of the sequence of its arithmetic means. Similarly, if the sequence of Picard iterates of any mapping T converges, then we don't look for the convergence of other iteration procedures.

The three-step iterative approximation problems were studied extensively by Noor [24, 25], Glowinsky and Le Tallec [9], and Haubruge et al. [11]. It has been shown [9] that three-step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

3. Main results

Lemma 3.1. *Let K be a nonempty convex subset of a hyperbolic space X and let $T\colon K \to K$ be a nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \big(\eta(T^n) - 1\big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$.*

Proof. Let $p \in F(T)$ and let $\rho = \sup_{n\in\mathbb{N}} \eta(T^n)$. From (2.1), we have

$$\begin{aligned}
d(z_n, p) &= d(W(x_n, T^n x_n, \gamma_n), p) \\
&\leq (1-\gamma_n) d(x_n, p) + \gamma_n d(T^n x_n, p) \\
&\leq (1-\gamma_n) d(x_n, p) + \gamma_n [\eta(T^n)(d(x_n, p) + a_n)] \\
&\leq (1-\gamma_n)\eta(T^n) d(x_n, p) + \gamma_n \eta(T^n) d(x_n, p) + \gamma_n \eta(T^n) a_n \\
&= \eta(T^n)[(1-\gamma_n) d(x_n, p) + \gamma_n d(x_n, p)] + \gamma_n \eta(T^n) a_n \\
&\leq \eta(T^n) d(x_n, p) + \eta(T^n) a_n. \qquad (3.1)
\end{aligned}$$

Again, using (2.1) and (3.1), we get

$$\begin{aligned}
d(y_n, p) &= d(W(x_n, T^n z_n, \beta_n), p) \\
&\leq (1-\beta_n) d(x_n, p) + \beta_n d(T^n z_n, p) \\
&\leq (1-\beta_n) d(x_n, p) + \beta_n [\eta(T^n)(d(z_n, p) + a_n)] \\
&\leq (1-\beta_n) d(x_n, p) + \beta_n \eta(T^n) d(z_n, p) + \beta_n \eta(T^n) a_n \\
&\leq (1-\beta_n) d(x_n, p) + \beta_n \eta(T^n)[\eta(T^n) d(x_n, p) \\
&\quad + \eta(T^n) a_n)] + \beta_n \eta(T^n) a_n
\end{aligned}$$

$$
\begin{aligned}
&\le \eta(T^n)^2[(1-\beta_n)d(x_n,p)+\beta_n d(x_n,p)] \\
&\quad +\beta_n\eta(T^n)^2 a_n+\beta_n\eta(T^n)a_n \\
&\le \eta(T^n)^2 d(x_n,p)+\rho(1+\rho)a_n. \qquad (3.2)
\end{aligned}
$$

Finally, using (2.1) and (3.2), we get

$$
\begin{aligned}
d(x_{n+1},p) &= d(W(T^n x_n, T^n y_n, \alpha_n),p) \\
&\le (1-\alpha_n)d(T^n x_n,p)+\alpha_n d(T^n y_n,p) \\
&\le (1-\alpha_n)[\eta(T^n)(d(x_n,p)+a_n)]+\alpha_n[\eta(T^n)(d(y_n,p)+a_n)] \\
&= (1-\alpha_n)\eta(T^n)d(x_n,p)+\alpha_n\eta(T^n)d(y_n,p)+\eta(T^n)a_n \\
&\le (1-\alpha_n)\eta(T^n)d(x_n,p)+\eta(T^n)a_n \\
&\quad +\alpha_n\eta(T^n)[\eta(T^n)^2 d(x_n,p)+\rho(1+\rho)a_n] \\
&\le \eta(T^n)^3 d(x_n,p)+(1+\rho+\rho^2)\eta(T^n)a_n \\
&\le \eta(T^n)^3 d(x_n,p)+(1+\rho+\rho^2)\rho\, a_n \\
&= (1+f_n)d(x_n,p)+g_n, \qquad (3.3)
\end{aligned}
$$

where $f_n = \big(\eta(T^n)^3-1\big) = \big(1+\eta(T^n)+\eta(T^n)^2\big)\big(\eta(T^n)-1\big)$ and $g_n = (1+\rho+\rho^2)\rho\, a_n$. Since $\sum_{n=1}^{\infty}\big(\eta(T^n)-1\big)<\infty$ and $\sum_{n=1}^{\infty} a_n<\infty$, it follows that $\sum_{n=1}^{\infty} f_n<\infty$ and $\sum_{n=1}^{\infty} g_n<\infty$. Hence by Lemma 2.5, we get that $\lim_{n\to\infty} d(x_n,p)$ exists. This completes the proof. $\square$

Lemma 3.2. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T\colon K\to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n,\eta(T^n))\}$ such that*

$$\sum_{n=1}^{\infty} a_n<\infty \quad \textit{and} \quad \sum_{n=1}^{\infty}\big(\eta(T^n)-1\big)<\infty.$$

Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F(T)\neq\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[l,m]$ for some $l,m\in(0,1)$. If $d(x,T^n y)\le d(T^n x,T^n y)$ for all x, $y\in K$, then $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

Proof. From Lemma 3.1, we obtain $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F(T)$. Suppose that $\lim_{n\to\infty} d(x_n,p)=r\ge 0$. Since

$$d(T^n x_n,p)\le \eta(T^n)(d(x_n,p)+a_n) \quad \text{for all } n\ge 1,$$

we have

$$\limsup_{n\to\infty} d(T^n x_n,p)\le r.$$

Also (3.2) yields

$$\limsup_{n\to\infty} d(y_n,p)\le r. \qquad (3.4)$$

Hence

$$\limsup_{n\to\infty} d(T^n y_n, p) \le \limsup_{n\to\infty} \eta(T^n)(d(y_n, p) + a_n) \le r. \tag{3.5}$$

Since

$$r = \lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(T^n x_n, T^n y_n, \alpha_n), p),$$

it follows from Lemma 2.3 that

$$\lim_{n\to\infty} d(T^n x_n, T^n y_n) = 0. \tag{3.6}$$

From (2.1) and (3.6), we have

$$\begin{aligned} d(x_{n+1}, T^n x_n) &= d(W(T^n x_n, T^n y_n, \alpha_n), T^n x_n) \\ &\le \alpha_n\, d(T^n x_n, T^n y_n) \\ &\le d(T^n x_n, T^n y_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.7}$$

Hence from (3.6) and (3.7), we have

$$\begin{aligned} d(x_{n+1}, T^n y_n) &\le d(x_{n+1}, T^n x_n) + d(T^n x_n, T^n y_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.8}$$

Now using (3.8), we have

$$\begin{aligned} d(x_{n+1}, p) &\le d(x_{n+1}, T^n y_n) + d(T^n y_n, p) \\ &\le d(x_{n+1}, T^n y_n) + \eta(T^n)(d(y_n, p) + a_n). \end{aligned} \tag{3.9}$$

The inequality (3.9) gives

$$r \le \liminf_{n\to\infty} d(y_n, p). \tag{3.10}$$

Also (3.1) yields

$$\limsup_{n\to\infty} d(z_n, p) \le r. \tag{3.11}$$

Hence

$$\limsup_{n\to\infty} d(T^n z_n, p) \le \limsup_{n\to\infty} \eta(T^n)(d(z_n, p) + a_n) \le r. \tag{3.12}$$

From (3.4) and (3.10), we get

$$r = \lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d(W(x_n, T^n z_n, \beta_n), p). \tag{3.13}$$

Applying Lemma 2.3 in (3.13), we obtain

$$\lim_{n\to\infty} d(x_n, T^n z_n) = 0. \tag{3.14}$$

Now using (3.6) and hypothesis of the theorem $d(x, T^n y) \leq d(T^n x, T^n y)$ for all x, $y \in K$, we get

$$\begin{aligned} d(x_n, T^n x_n) &\leq d(x_n, T^n y_n) + d(T^n x_n, T^n y_n) \\ &\leq d(T^n x_n, T^n y_n) + d(T^n x_n, T^n y_n) \\ &= 2\, d(T^n x_n, T^n y_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.15}$$

Again note that

$$d(x_n, T^n y_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^n y_n). \tag{3.16}$$

From (3.6) and (3.15), we obtain

$$\lim_{n\to\infty} d(x_n, T^n y_n) = 0. \tag{3.17}$$

By uniform continuity of T, $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$ implies that

$$\lim_{n\to\infty} d(Tx_n, T^{n+1} x_n) = 0.$$

From (2.1), (3.15) and (3.16), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(T^n x_n, T^n y_n, \alpha_n), x_n) \\ &\leq (1 - \alpha_n) d(x_n, T^n x_n) + \alpha_n\, d(T^n y_n, x_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.18}$$

Also

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\ &\quad + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\leq \big(1 + \eta(T^{n+1})\big) d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\ &\quad + d(T^{n+1} x_n, Tx_n) + a_{n+1}. \end{aligned} \tag{3.19}$$

The above inequality gives

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.20}$$

This completes the proof. □

We now establish a Δ-convergence and some strong convergence theorems of modified three-step iteration scheme for nearly asymptotically nonexpansive mapping in the framework of uniformly convex hyperbolic space.

Theorem 3.3. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T \colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mappings with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and*

$\sum_{n=1}^{\infty}\big(\eta(T^n)-1\big)<\infty$. *Let* $\{x_n\}$ *be a sequence in* K *defined by* (2.1). *Assume that* $F(T)\neq\emptyset$. *Suppose that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequence in* $[l,m]$ *for some* $l,m\in(0,1)$. *Then* $\{x_n\}$ *is* Δ*-convergent to an element of* $F(T)$.

Proof. Since $\{x_n\}$ is bounded by Lemma 3.1, therefore $\{x_n\}$ has a unique asymptotic center (see, [44]), that is, $A(\{x_n\})=\{x\}$ (say). Let $A(\{y_n\})=\{v\}$. Then by (3.20), $\lim_{n\to\infty}d(y_n,Ty_n)=0$. T is nearly asymptotically nonexpansive mapping with sequence $\{(a_n,\eta(T^n))\}$. By uniform continuity of T

$$\lim_{n\to\infty}d(T^iy_n,T^{i+1}y_n) \;=\; 0 \quad \text{for } i=1,2,\cdots. \tag{3.21}$$

Now we claim that v is a fixed point of T. For this, we define a sequence $\{z_n\}$ in K by $z_m=T^mv$, $m\in\mathbb{N}$. for integers $m,n\in\mathbb{N}$, we have

$$\begin{aligned} d(z_m,y_n) &\le d(T^mv,T^my_n)+d(T^my_n,T^{m-1}y_n)+\cdots+d(Ty_n,y_n)\\ &\le \eta(T^n)(d(v,y_n)+a_m)+\sum_{i=0}^{m-1}d(T^iy_n,T^{i+1}y_n). \end{aligned}\tag{3.22}$$

Then from (3.21) and (3.22), we have

$$r(z_m,\{y_n\})=\limsup_{m\to\infty}d(z_m,y_n)\le\eta(T^m)[r(v,\{y_n\})+a_m].$$

Hence

$$\limsup_{m\to\infty}r(z_m,\{y_n\})\le r(v,\{y_n\}). \tag{3.23}$$

Since $A_K(\{y_n\})=\{v\}$, by definition of asymptotic center $A_K(\{y_n\})$ of a bounded sequence $\{y_n\}$ with respect to $K\subset X$, we have

$$r(v,\{y_n\})\le r(y,\{y_n\}),\quad \forall\, y\in K.$$

This implies that

$$\liminf_{m\to\infty}r(z_m,\{y_n\})\ge r(v,\{y_n\}), \tag{3.24}$$

therefore, from (3.23) and (3.24), we have

$$\lim_{m\to\infty}r(z_m,\{y_n\})=r(v,\{y_n\}).$$

It follows from Lemma 2.4 that $T^mv\to v$. By uniform continuity of T, we have

$$Tv=T(\lim_{m\to\infty}T^mv)=T^{m+1}v=v,$$

which implies that v is a fixed point of T, that is, $v\in F(T)$.

Next, we claim that v is the unique asymptotic center for each subsequence $\{y_n\}$ of $\{x_n\}$. Assume contrarily, that is, $x \neq v$. Since $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 3.1, therefore, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(y_n, v) &< \limsup_{n\to\infty} d(y_n, x) \leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(y_n, v),\end{aligned}$$

a contradiction and hence $x = v$. Since $\{y_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore, $A_K(\{y_n\}) = \{v\}$ for all subsequence $\{y_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T\colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mappings with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \big(\eta(T^n) - 1\big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F(T) \neq \emptyset$ is a closed set. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf_{y\in F(T)} d(x, y)$.*

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 3.1, for all $p \in F(T)$, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. With the help of inequality $1 + x \leq e^x$, $x \geq 0$. For any integer $m \geq 1$, we have from (3.3)

$$\begin{aligned} d(x_{n+m}, p) &\leq (1 + f_{n+m-1}) d(x_{n+m-1}, p) + g_{n+m-1} \\ &\leq e^{f_{n+m-1}} d(x_{n+m-1}, p) + g_{n+m-1} \\ &\leq e^{f_{n+m-1}} [e^{f_{n+m-2}} d(x_{n+m-2}, p) + g_{n+m-2}] + g_{n+m-1} \\ &\leq e^{(f_{n+m-1}+f_{n+m-2})} d(x_{n+m-2}, p) + e^{(f_{n+m-1}+f_{n+m-2})} \times \\ &\quad [g_{n+m-1} + g_{n+m-2}] \\ &\leq \cdots \\ &\leq \left(e^{\sum_{k=n}^{n+m-1} f_k}\right) d(x_n, p) + \left(e^{\sum_{k=n}^{n+m-1} f_k}\right) \sum_{k=n}^{n+m-1} g_k \\ &= R\, d(x_n, p) + R \sum_{k=n}^{n+m-1} g_k, \end{aligned} \tag{3.25}$$

where $R = e^{\sum_{n=1}^{\infty} f_n} < \infty$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence

$\{p_{n_k}\} \subset F(T)$ such that $d(x_{n_k}, p_{n_k}) \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$d(x_{n_k}, p_{n_k}) < \frac{\varepsilon}{4R} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty} g_k < \frac{\varepsilon}{4R}, \tag{3.26}$$

for all $k \geq k_\varepsilon$. For any $m \geq 1$ and for all $n \geq n_{k_\varepsilon}$, by (3.26), we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_{n_k}) + d(x_n, p_{n_k}) \\ &\leq R\, d(x_n, p_{n_k}) + R \sum_{k=n_{k_\varepsilon}}^{\infty} g_k + R\, d(x_n, p_{n_k}) + R \sum_{k=n_{k_\varepsilon}}^{\infty} g_k \\ &= 2R\, d(x_n, p_{n_k}) + 2R \sum_{k=n_{k_\varepsilon}}^{\infty} g_k \\ &< 2R \cdot \frac{\varepsilon}{4R} + 2R \cdot \frac{\varepsilon}{4R} = \varepsilon. \end{aligned} \tag{3.27}$$

This proves that $\{x_n\}$ is a Cauchy sequence in closed subset K a complete hyperbolic space X and so it must converge to a point q in K, that is, $\lim_{n\to\infty} x_n = q$. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives $d(q, F(T)) = 0$. Since $F(T)$ is closed, we have $q \in F(T)$. Thus $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. $\square$

Theorem 3.5. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T\colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mappings with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \big(\eta(T^n) - 1\big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. If T^m for some $m \geq 1$ is semi-compact, then $\{x_n\}$ convergence strongly to a fixed point of T.*

Proof. Suppose T^m for some $m \geq 1$ is semi-compact. By Lemma 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By the uniform continuity of T, we get

$$\begin{aligned} d(x_n, Tx_n) \to 0 \quad &\Rightarrow \quad d(Tx_n, T^2x_n) \to 0 \quad \Rightarrow \quad \cdots \\ &\Rightarrow \quad d(T^i x_n, T^{i+1} x_n) \to 0 \end{aligned}$$

for all $i = 1, 2, 3, \cdots$, it follows that

$$d(x_n, T^m x_n) \leq \sum_{i=0}^{m-1} d(T^i x_n, T^{i+1} x_n) \to 0 \quad \text{as } n \to \infty.$$

Since $d(x_n, T^m x_n) \to 0$ as $n \to \infty$ and T^m is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} T^m x_{n_j} = x \in K$. Note that

$$d(x_{n_j}, x) \leq d(x_{n_j}, T^m x_{n_j}) + d(T^m x_{n_j}, x) \to 0 \quad \text{as } j \to \infty.$$

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we get $x \in F(T)$. Since $\lim_{n\to\infty} d(x_n, x)$ exists by Lemma 3.1 and $\lim_{j\to\infty} d(x_{n_j}, x) = 0$, we conclude that $x_n \to x \in F(T)$. This shows that the sequence $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. $\square$

Senter and Dotson [42] introduced the concept of condition (A) as follows.

Definition 3.6. ([42]) A mapping $T\colon K \to K$ is said to satisfy condition (A) if there exists a non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in K$.

As an application of Theorem 3.3, we establish another strong convergence result employing condition (A).

Theorem 3.7. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T\colon K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mappings with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \big(\eta(T^n) - 1\big) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0,1)$. Suppose that T satisfies the condition (A). Then $\{x_n\}$ convergence strongly to a fixed point of T.*

Proof. By Lemma 3.2, we know that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.28}$$

From condition (A) and (3.28), we get

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore we obtain

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

The conclusion now follows from Theorem 3.3. This completes the proof. $\square$

Example 3.8. ([32]) Let $E = \mathbb{R}$, $K = [0,1]$ and $T\colon K \to K$ be a mapping defined by

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ \\ 0, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Here $F(T) = \{\frac{1}{2}\}$. Clearly, T is discontinuous and a non-Lipschitzian mapping. However, it is a nearly nonexpansive mapping and hence nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\} = \{\frac{1}{2^n}, 1\}$. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$d(Tx, Ty) \leq d(x, y) + a_1 \quad \text{for all} \quad x,\, y \in K$$

and

$$d(T^n x, T^n y) \leq d(x, y) + a_n \quad \text{for all} \quad x,\, y \in K \text{ and } n \geq 2,$$

since

$$T^n x = \frac{1}{2} \quad \text{for all} \quad x \in [0,1] \text{ and } n \geq 2.$$

4. Conclusion

1. We prove a Δ-convergence and some strong convergence theorems of modified three-step iteration process which contains modified S-iteration process in the framework of uniformly convex hyperbolic spaces.

2. Theorem 3.3 extends Theorem 3.8 of Agarwal et al. [2] to the case of modified three-step iteration scheme and from uniformly convex Banach space to a uniformly convex hyperbolic space considered in this paper.

3. Theorem 3.3 also extends Theorem 3.3 of Dhompongsa and Panyanak [6] to the case of more general class of nonexpansive mappings which are not necessarily Lipschitzian, modified three-step iteration scheme and from CAT(0) space to a uniformly convex hyperbolic space considered in this paper.

4. Theorem 3.3 also extends Theorem 3.5 of Niwongsa and Panyanak [23] to the case of more general class of asymptotically nonexpansive mappings which are not necessarily Lipschitzian, modified three-step iteration scheme and from CAT(0) space to a uniformly convex hyperbolic space considered in this paper.

5. Our results also extend the corresponding results of Xu and Noor [48] to the case of more general class of asymptotically nonexpansive mappings, modified three-step iteration scheme and from a Banach space to a uniformly convex hyperbolic space considered in this paper.

6. Our results also extend and generalize the corresponding results of [3, 13, 27, 30, 31, 38, 41] for a more general class of non-Lipschitzian mappings, modified three-step iteration scheme and from uniformly convex metric space, CAT(0) space to a uniformly convex hyperbolic space considered in this paper.

References

[1] M. Abbas, Z. Kadelburg and D.R. Sahu, *Fixed point theorems for Lipschitzian type mappings in CAT*(0) *spaces*, Math. Comput. Model., **55** (2012), 1418–1427.

[2] R.P. Agarwal, Donal O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, Nonlinear Convex Anal., **8**(1) (2007), 61–79.

[3] I. Beg, *An iteration scheme for asymptotically nonexpansive mappings on uniformly convex metric spaces*, Nonlinear Anal. Forum, **6** (2001), 27–34.

[4] M.R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Vol. **319** of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany (1999).

[5] S.S. Chang, L. Wang, H.W. Joesph Lee, C.K. Chan and L. Yang, *Demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces*, Appl. Math. Comput., **219**(5) (2012), 2611–2617.

[6] S. Dhompongsa and B. Panyanak, *On $\triangle$-convergence theorem in CAT(0) spaces*, Comput. Math. Appl., **56**(10) (2008), 2572–2579.

[7] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35** (1972), 171–174.

[8] K. Goebel and W.A. Kirk, *Iterations processes for nonexpansive mappings*, Contemp. Math., **21** (1983), 115–123.

[9] R. Glowinski and P. Le Tallec, *"Augemented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics"* Siam, Philadelphia (1989).

[10] M. Gromov, *Hyperbolic groups. Essays in group theory* (S. M. Gersten, ed). Springer Verlag, MSRI Publ., **8** (1987), 75–263.

[11] S. Haubruge, V.H. Nguyen and J.J. Strodiot, *Convergence analysis and applications of the Glowinski Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl., **97** (1998), 645–673.

[12] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44** (1974), 147–150.

[13] S.H. Khan and M. Abbas, *Strong and $\triangle$-convergence of some iterative schemes in CAT(0) spaces*, Comput. Math. Appl., **61**(1) (2011), 109–116.

[14] A.R. Khan, H. Fukhar-ud-din and M.A.A. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., **2012** (2012), 12 pages.

[15] W.A. Kirk, *Krasnoselskii's iteration process in hyperbolic space*, Numer. Funct. Anal. Optim., **4** (1982), 371–381.

[16] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68** (2008), 3689–3696.

[17] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Amer. Math. Soc., **357** (2005), 89–128.

[18] T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., **60** (1976), 179–182.

[19] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl., **259** (2001), 1–7.
[20] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl., **259** (2001), 18–24.
[21] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.
[22] B. Nanjaras and B. Panyanak, *Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces*, Fixed Point Theory Appl., **2010** (2010), Art. ID 268780.
[23] Y. Niwongsa and B. Panyanak, *Noor iterations for asymptotically nonexpansive mappings in CAT(0) spaces*, Int. J. Math. Anal., **4**(13) (2010), 645–656.
[24] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251**(1) (2000), 217–229.
[25] M.A. Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl., **255** (2001), 589–604.
[26] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597.
[27] M.O. Osilike and S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. and Computer Modelling, **32** (2000), 1181–1191.
[28] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal.: TMA, Series A, Theory Methods, **15**(6) (1990), 537–558.
[29] B.E. Rhoades, *Fixed point iteration for certain nonlinear mappings*, J. Math. Anal. Appl., **183** (1994), 118–120.
[30] A. Şahin and M. Başarir, *On the strong convergence of a modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space*, Fixed Point Theory Appl., **2013**(12), (2013).
[31] A. Şahin and M. Başarir, *On the strong convergence of SP-iteration on* $CAT(0)$ *space*, Journal Ineql. Appl., **2013**(311), (2013).
[32] D.R. Sahu, *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces*, Comment. Math. Univ. Carolinae, **46**(4) (2005), 653–666.
[33] D.R. Sahu and I. Beg, *Weak and strong convergence for fixed points of nearly asymptotically nonexpansive mappings*, Int. J. Mod. Math., **3** (2008), 135–151.
[34] G.S. Saluja, *Strong convergence theorem for two asymptotically quasi-nonexpansive mappings with errors in Banach space*, Tamkang J. Math., **38**(1) (2007), 85–92.
[35] G.S. Saluja, *Strong convergence theorems of three step iterative sequences with errors for asymptotically quasi nonexpansive type nonself mappings*, Advan. Appl. Maths. Anal., **2**(1) (2007), 57–69.
[36] G.S. Saluja, *Convergence result of* (L,α)*-uniformly Lipschitz asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces*, Jñānābha **38** (2008), 41–48.
[37] G.S. Saluja, *Approximating fixed points of Noor iteration with errors for asymptotically quasi-nonexpansive mappings*, Funct. Anal. Approx. Comput., **1**(1) (2009), 31–48.
[38] G.S. Saluja, *Strong convergence of Noor iteration for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces*, Acta Math. Vietnamica, **37**(3) (2012), 341–355.
[39] G.S. Saluja, *On the convergence of modified S-iteration process for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces*, Funct. Anal. Approx. Comput., **6**(2) (2014), 29–38.

[40] G.S. Saluja and J.K. Kim, *On the convergence of a modified S-iteration process for asymptotically quasi-nonexpansive type mappings in a CAT(0) space*, Nonlinear Funct. Anal. Appl., **19**(3) (2014), 329–339.

[41] G.S. Saluja, *Strong and Δ-convergence of new three-step iteration process in CAT(0) spaces*, Nonlinear Anal. Forum, **20** (2015), 43–52.

[42] H.F. Senter and W.G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **44** (1974), 375–380.

[43] N. Shahzad and A. Udomene, *Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces*, Fixed Point Theory and Applications, Vol. 2006, Article ID 18909, Pages 1–10.

[44] T. Shimizu and W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*, Topol. Methods Nonlinear Anal., **8**(1) (1996), 197–203.

[45] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Semin. Rep., **22** (1970), 142–149.

[46] K.K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., **178** (1993), 301–308.

[47] K.K. Tan and H.K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **122** (1994), 733–739.

[48] B.L. Xu and M.A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **267**(2) (2002), 444–453.