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HOMOCLINIC SOLUTIONS FOR A CLASS OF SECOND ORDER HAMILTONIAN SYSTEMS WITH LOCALLY DEFINED SUBQUADRATIC POTENTIALS

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Abstract. In this paper, we study the existence of infinitely many homoclinic solutions for a class of subquadratic second order Hamiltonian systems with locally defined potentials. By using a critical point theorem related to the symmetric mountain pass lemma, two new criteria for guaranteeing that the systems have infinitely many homoclinic solutions which converge to zero. Recent results in the literature are generalized and significantly improved.

1. Introduction

Consider the following second order Hamiltonian systems:

$$\ddot{u}(t) - L(t)u + W_u(t, u(t)) = 0, \qquad \forall\, t \in \mathbb{R}, \tag{HS}$$

where $u \in \mathbb{R}^N, W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function. A nontrivial solution $u(t)$ of (HS) is said to be homoclinic to zero if $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N), u \neq 0, u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

During the last several decades, the existence and multiplicity of homoclinic solutions of (HS) have been intensively studied in many papers (see for instance

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[1-23] and references therein) via the variational methods. Most of them (see, [1, 2, 4–10]) treated the case where $L(t)$ and $W(t,u)$ are either independent of t or perioddic in t. In this kind of problem, the function $L(t)$ plays an important role. If $L(t)$ is neither autonomous nor periodic in t, the problem is quite different from the ones just described, because of the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [2], many results (see, [10-23]) were obtained for the case where $L(t)$ is neither a constant nor periodic. In particular, Ding [12] studied the non-periodic system. By using the following two conditions of $L(t)$ respectively, he obtained some excellent results.

(L_1) $l(t) \equiv \inf_{u\in\mathbb{R}^N,|u|=1}(L(t)u,u) \to \infty, \quad \text{as} \quad |t| \to \infty.$

(L_2) There exists an $\alpha < 2$ such that $l(t)|t|^{\alpha-2} \to \infty$ as $|t| \to \infty$.

Since then, these conditions have been used extensively (see, [16-19, 23]). Under condition (L_2), Yang and Zhang consider a new superquadratic condition:

$$\frac{W(t,u)}{|u|^2} \to \infty \quad \text{as} \quad |u| \to \infty.$$

Zhang and Liu [19] considered a new subquadratic condition for generic case by using the variant fountain theorem established in [15]. Wei and Wang [17] considered a case that $W(t,u) = F(t,u) + G(t,u)$, where $F(t,u)$ is subquadratic and $G(t,u)$ is superquadratic.

Recently, Zhang and Yuan [20] obtained the existence of one nontrivial homoclinic solution for (HS) by use of a standard minimizing argument under the following condition:

(H_1) $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u,u) \geq \alpha(t)|u|^2$ and $\alpha(t) \to +\infty \quad \text{as} \quad t \to +\infty$.
But they only dealt with a special case where $W(t,u) = a(t)|u|^\gamma$ satisfies the condition that $a : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $a \in L^{\frac{2}{2-\gamma}}(\mathbb{R},\mathbb{R}^+) \cap L^2(\mathbb{R},\mathbb{R}^+)$ and $1 < \gamma < 2$ is a constant. Later Sun et al. [21] improved above result, they studied the existence of infinitely many homoclinic solutions for this special case for $a(t) \in L^{\frac{2}{2-\gamma}}(\mathbb{R},\mathbb{R}^+)$ only. Then Yang et al. [22] studied for $W(t,u) = m(t)|u|^\gamma + d|u|^p$ where $m : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $m \in L^{\frac{2}{2-\gamma}}(\mathbb{R},\mathbb{R}^+)$ and $1 < \gamma < 2,\ d \geq 0,\ p > 2$ are constants.

We notice that in all these papers $W(t,u)$ was always required to satisfy some kind growth conditions at infinity with respect to u. Very recently, Zhang and Chu [23] obtained some excellent results. They studied the existence of infinitely many homoclinic solutions for (HS) in the case where $W(t,u)$ is only locally defined near the origin with respect to u. More precisely, they presented the following assumptions:

(L_3) For some $a > 0$ and $r > 0$ such that
(i) $L \in C^1(\mathbb{R}, \mathbb{R}^{N^2})$ and $|L'(t)u| \leq a|L(t)u|, \quad \forall |t| \geq r$ and $u \in \mathbb{R}^N$,
or
(ii) $L \in C^2(\mathbb{R}, \mathbb{R}^{N^2})$ and $\langle (L''(t) - aL(t))u, u\rangle \leq 0, \quad \forall |t| \geq r$ and $u \in \mathbb{R}^N$,
where $L'(t) = (d/dt)L(t)$ and $L''(t) = (d^2/dt^2)L(t)$.

(W_1) $W \in C^1(\mathbb{R} \times B_\delta(0), \mathbb{R})$ is even in u and $W(t,0) \equiv 0$, where $B_\delta(0)$ is the ball in $\mathbb{R}^N$ centered at 0 with radius $\delta > 0$;

(W_2) there exist constants $c_1 > 0$ and $1/2 \leq \nu \in (1/(3-\alpha), 1)$ such that
$$|W_u(t,u)| \leq c_1|u|^\nu, \quad \forall (t,u) \in \mathbb{R} \times B_\delta(0);$$

(W_3)
$$\lim_{|u|\to 0} \frac{W(t,u)}{|u|^2} = \infty \quad \text{uniformly for} \quad t \in \mathbb{R};$$

(W_4) $2W(t,u) - W_u(t,u) \cdot u > 0$ for all $t \in \mathbb{R}$ and $u \neq 0$.

Then, they obtained the following theorem.

Theorem A. *Suppose that (L_2), (L_3) and $(W_1) - (W_4)$ are satisfied. Then problem (HS) possesses a sequence of homoclinic solutions $\{u_n\}$ such that $\max_{t\in\mathbb{R}} |u_n(t)| \to 0$ as $n \to \infty$.*

As the authors pointed out, the condition (W_4) was crucial in the proof of Theorem A.

Obviously, there some functions satisfying (H_1) but do not satisfy (L_2) even though L is positive definite, for example, let
$$L(t) = \ln(|t| + 2)I_n,$$
where I_n is the unit matrix of order n. Then L satisfies (H_1), but L does not satisfy (L_2) since for any $\alpha < 2$, $l(t)|t|^{\alpha-2} \to 0$ as $|t| \to \infty$. We else note that (W_4) is really unnecessary to ensure that (HS) possesses a sequence of homoclinic solutions. This condition is the decisive one in Theorem A to guarantee that 0 is the only critical point of the related functional Φ such that $\Phi(u) = 0$, and this thereby support that $\max_{t\in\mathbb{R}} |u_n(t)| \to 0$ as $n \to \infty$,

so that the cut-off function trick can play a perfect role. It is obvious that this restriction is very stringent, for example, see Lemma 2.8 in section 2. Motivated by the above fact, in this paper, our aim is to generalize and improve some results in the references that we have mentioned. We consider (HS) under (H_1) instead of $(L_2), (L_3)$. For this case, because of the lack of compactness of the Sobolev embedding, we have to impose some integrability on the coefficient function to overcome this difficulty and need to adapt (W_2) to the following condition:

(W_2')
$$|W_u(t,u)| \leq b(t)\ |u|^{\gamma-1}, \quad \forall\ (t,u) \in \mathbb{R} \times B_\delta(0),$$

where $b : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $b \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$ and $1 < \gamma < 2$ is a constant.

We have the following result.

Theorem 1.1. *Suppose that (H_1) and $(W_1), (W_2'), (W_3)$ are satisfied. Then (HS) possesses a sequence of homoclinic solutions $\{u_n\}$ such that*

$$\lim_{n\to\infty} u_n(t) = 0.$$

We note that if $W(t,u) = a(t)|u|^\mu$ such that $a : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function with $\inf_{t\in\mathbb{R}} a(t) = 0$ and $1 < \mu < 2$, then $W(t,u)$ does not satisfy condition (W_3) due to the fact that $\inf_{t\in\mathbb{R}} a(t) = 0$, hence Theorem A and Theorem 1.1 do not adapt for this case. In order to include the case where $W(t,u) = a(t)|u|^\mu$ or $W(t,u) = m(t)|u|^\mu + d|u|^p$, we present the following hypothesis:

(W_3')
$$W(t,u) \geq a(t)|u|^\mu, \quad \forall\ (t,u) \in \mathbb{R} \times B_\delta(0),$$

where $a(t) : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $\inf_{t\in\mathbb{R}} a(t) = 0,\ \sup_{t\in\mathbb{R}} a(t) = A_0 < +\infty$, and $1 < \mu < 2$ is a constant.

We propose the following result.

Theorem 1.2. *Suppose that (H_1) and $(W_1), (W_2'), (W_3')$ are satisfied. Then (HS) possesses a sequence of homoclinic solutions $\{u_n\}$ such that*

$$\max_{t\in\mathbb{R}} |u_n(t)| \to 0 \quad as \quad n \to \infty.$$

Remark 1.3. In our Theorems 1.1 and 1.2, there is no assumption on $W(t,u)$ for u large enough and hence the potential is only locally defined near the origin with respect to u. It is easy to check that W will satisfy $(W_1), (W_2'), (W_3')$ provided that it satisfies all the conditions (H_2') and (H_2'') in theorems in [22],

(H_2') in Theorem 1.2 in [21]. We do not apparently need the condition (W_4) which is a crucial condition in the proof of Theorem 1.1 in [23]. The highlights in this paper are that we do not have to impose integrability on the coefficient function $a(t)$ in Theorem 1.2 and have taken off condition (W_4) in our main results. To the best of our knowledge, there is little literature concerning the existence of infinitely many homoclinic solutions for (HS) in this situation. So our results are significant improvement for some related results in the literature (see, e.g., [12, 17, 19-23]).

Remark 1.4. As in paper [23], the tools are also the cut-off function technique and variational methods. We will first modify $W(t,u)$ for u outside a neighborhood of the origin 0 in Theorems 1.1 and 1.2 respectively to get a modified system. Then we show that the modified system possesses a sequence of homoclinic solutions, which converges to zero in L^∞ norm. Consequently, we obtain infinitely many homoclinic solutions for the original Hamiltonian system (HS).

2. Preliminaries

In order to establish our results via the critical point theory, we firstly describe some properties of the space on which the variational associated with the modified system of (HS) is defined with respect to Theorems 1.1 and 1.2 respectively.

Let

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} (\dot{u}(t) \cdot \dot{v}(t) + L(t)u(t) \cdot v(t))dt < +\infty \right\}.$$

Then the space E is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}} (\dot{u}(t) \cdot \dot{v}(t) + L(t)u(t) \cdot v(t))dt$$

and the corresponding norm $\|u\|^2 = \langle u, u \rangle$. Denote by E^* its dual space with the associated operator norm $\|\cdot\|_{E^*}$. Note that $E \subset H^1(\mathbb{R}, \mathbb{R}^N) \subset L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, +\infty]$ with the embedding being continuous. Therefore, there exists a constant $\beta_p > 0$ such that

$$\|u\|_p \le \beta_p \|u\|, \quad \forall\, u \in E, \quad p \ge 2. \tag{2.1}$$

Here $L^p(\mathbb{R}, \mathbb{R}^N)$ and $H^1(\mathbb{R}, \mathbb{R}^N)$ denote the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norms

$$\|u\|_p := \left(\int_{\mathbb{R}} |u(t)|^p dt \right)^{1/p} \quad \text{and} \quad \|u\|_{H^1} := (\|u\|_2^2 + \|\dot{u}\|_2^2)^{1/2}$$

respectively. $L^\infty(\mathbb{R},\mathbb{R}^N)$ is the Banach space of essentially bounded functions equipped with the norm

$$\|u\|_\infty := \operatorname{ess\,sup}\{|u(t)| : t \in \mathbb{R}\}.$$

Lemma 2.1. ([3], Lemma 1) *If L satisfies (H_1), then E is compactly embedded in $L^2 \equiv L^2(\mathbb{R},\mathbb{R}^N)$.*

Now we first modify $W(t,u)$ for u outside a neighborhood of the origin 0 to get $\widetilde{W}(t,u)$ for Theorem 1.1. We choose $0<\sigma<1/(4\beta_2^2)$. By (W_3), there exists a constant $0<r<\delta/2$ such that

$$W(t,u) \geq \sigma|u|^2, \quad \forall\, t\in\mathbb{R} \;\text{ and }\; |u|\leq 2r.$$

Let

$$\widetilde{W}(t,u) = \rho(|u|)W(t,u) + \sigma(1-\rho(|u|))|u|^2, \;\; \forall\,(t,u)\in\mathbb{R}\times\mathbb{R}^N, \tag{2.2}$$

where $\rho \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ is a cut-off function satisfying $\rho(t)=1$ for $0\leq t\leq r$, $\rho(t)=0$ for $t\geq 2r$ and $-2/r\leq \rho'(t)<0$ for $r<t<2r$.

We then consider the following modified Hamiltonian system

$$\ddot{u}(t) - L(t)u + \widetilde{W}_u(t,u(t)) = 0, \;\; \forall\, t\in\mathbb{R} \tag{$\widetilde{\text{HS}}$}$$

and show by variational methods that above system possesses a sequence homoclinic solutions, which converges to zero in L^∞ norm. Consequently, we obtain infinitely many homoclinic solutions for the original Hamiltonian system (HS).

Henceforth, we always assume that c (or C) and c_i (or C_i) stand for generic positive constants.

For later use, we give in the following lemmas some properties of $\widetilde{W}(t,u)$.

Lemma 2.2. *Assume that $(W_1),(W_2'),(W_3)$ hold. Then $\widetilde{W}$ possesses the following properties.*
(1) *There exists a constant c_1 such that*

$$|\widetilde{W}_u(t,u)| \leq c_1(b(t)\;|u|^{\gamma-1} + |u|), \quad \forall\,(t,u)\in\mathbb{R}\times\mathbb{R}^N. \tag{2.3}$$

(2) *There exists a constant c_2 such that*

$$0 \leq \widetilde{W}(t,u) \leq c_2\; b(t)\;|u|^\gamma + \sigma\;|u|^2, \quad \forall\,(t,u)\in\mathbb{R}\times\mathbb{R}^N. \tag{2.4}$$

Proof. By (2.2), direct computation shows that

$$\widetilde{W}_u(t,u) = \rho(|u|)W_u(t,u) + \left[\rho'(|u|)\left(\frac{W(t,u)}{|u|} - \sigma\;|u|\right) + 2\sigma(1-\rho(|u|))\right]u,$$

for all $t \in \mathbb{R}$ and $u \neq 0$. Besides, it is easy to check by (W_1) and (W_2') that

$$\widetilde{W}_u(t,0) \equiv 0, \quad \forall\, t \in \mathbb{R}$$

and $(W_1), (W_2')$ imply

$$|W(t,u)| \leq c\ b(t)\ |u|^\gamma, \quad \forall\ (t,u) \in \mathbb{R} \times B_\delta(0).$$

Hence we have

$$\begin{aligned} |\widetilde{W}_u(t,u)| &\leq (1+4c)b(t)\ |u|^{\gamma-1} + 6\sigma\ |u| \\ &\leq c_1(b(t)\ |u|^{\gamma-1} + |u|), \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N, \end{aligned}$$

where $c_1 = \max\{1+4c, 6\sigma\}$, and (2.3) holds. (2.4) is trivial. This completes the proof. □

Lemma 2.3. *Suppose that $(H_1), (W_1), (W_2'), (W_3)$ are satisfied. If $u_k \rightharpoonup u$ (weakly) in E, then $\widetilde{W}_u(t,u_k) \to \widetilde{W}_u(t,u)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$.*

Proof. Assume that $u_k \rightharpoonup u$ in E, by Lemma 2.1, $u_k \to u$ in L^2 and $u_k \to u$ a.e. in $\mathbb{R}$, passing to a subsequence if necessary. Then we have $\|u_k - u\|_2 \to 0$, as $k \to \infty$, hence there exists a constant N, for all $k > N$, we have $\|u_k - u\|_2 \leq 1$. By (2.3), there holds

$$\begin{aligned} &|\widetilde{W}_u(t,u_k) - \widetilde{W}_u(t,u)|^2 \\ &\leq c_1^2(b(t)|u_k|^{\gamma-1} + b(t)\ |u|^{\gamma-1} + |u_k| + |u|)^2 \\ &\leq 4c_1^2 b^2(t)(|u_k|^{2\gamma-2} + |u|^{2\gamma-2}) + 4c_1^2(|u_k|^2 + |u|^2) \\ &\leq Cb^2(t)(|u_k - u|^{2\gamma-2} + |u|^{2\gamma-2}) + C(|u_k - u|^2 + |u|^2), \end{aligned}$$

for some constant $C > 0$. By Lemma 2.1 again, $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^N)$, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_2 < +\infty,$$

this implies that $u_k(t) \to u(t)$ for almost every $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |u_k - u| = \omega(t) \in L^2(\mathbb{R}, \mathbb{R}).$$

Therefore, we obtain

$$\begin{aligned} &|\widetilde{W}_u(t,u_k) - \widetilde{W}_u(t,u)|^2 \\ &\leq C|b(t)|^2(|\omega(t)|^{2\gamma-2} + |u|^{2\gamma-2}) + C|a(t)|^2(|\omega(t)|^{2\mu-2} + |u|^{2\mu-2}). \end{aligned}$$

Which combining (2.1) yields that

$$\begin{aligned}\int_{\mathbb{R}} |\widetilde{W}_u(t,u_k) - \widetilde{W}_u(t,u)|^2 dt &\le C\ \|b\|^2_{\frac{2}{2-\gamma}}\ (1+\|u\|_2^{2\gamma-2}) + C\ (1+\|u\|_2^2)\\ &\le C_1(1+\|u\|^{2\gamma-2}+\|u\|^2),\end{aligned}$$

for $k > N$ and some constant $C_1 > 0$. By using the Lebesgue Dominated Convergence Theorem, the lemma is proved. □

From above proof it is obvious that $\widetilde{W}_u(t,u) \in L^2(\mathbb{R},\mathbb{R}^N)$ for all $u \in E$.

Now we define the variational functional Φ on E associated with the modified system $(\widetilde{\text{HS}})$ by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}}(|\dot{u}(t)|^2 + L(t)u(t)\cdot u(t))dt - \int_{\mathbb{R}}\widetilde{W}(t,u(t))dt\\ &= \frac{1}{2}\|u\|^2 - \Psi(u),\end{aligned} \tag{2.5}$$

where $\Psi(u) = \int_{\mathbb{R}}\widetilde{W}(t,u(t))dt$. By (2.4), (W_2') and Lemma 2.1, which imply that $\widetilde{W}(t,u) \in L(\mathbb{R},\mathbb{R})$ for all $u \in E$, we know that Φ and Ψ are both well defined.

Lemma 2.4. *Under conditions of Theorem* 1.1*, we have that* $\Psi \in C^1(E,\mathbb{R})$ *and* $\Psi' : E \to E^*$ *is compact, and hence* $\Phi \in C^1(E,\mathbb{R})$*. Moreover,*

$$\Psi'(u)v = \int_{\mathbb{R}}\widetilde{W}_u(t,u(t))\cdot v(t)dt,$$

$$\begin{aligned}\Phi'(u)v &= \int_{\mathbb{R}}(\dot{u}(t)\cdot\dot{v}(t) + L(t)u(t)\cdot v(t))dt - \int_{\mathbb{R}}\widetilde{W}_u(t,u(t))\cdot v(t)dt\\ &= \langle u,v\rangle - \Psi'(u)v,\end{aligned} \tag{2.6}$$

for all $u,v \in E$*, and critical points of* Φ *on* E *belong to* $C^2(\mathbb{R},\mathbb{R}^N)$ *and are homoclinic solutions of* $(\widetilde{HS})$*.*

Proof. For any given $u \in E$, let us define functional $J(u)$ on E for all $u \in E$ as follows:

$$J(u)v = \int_{\mathbb{R}}\widetilde{W}_u(t,u(t))\cdot v(t)dt, \ \ \forall\, v \in E.$$

It is obvious that $J(u)$ is linear and $J(u) \in E^*$ since $\widetilde{W}_u(t,u) \in L^2(\mathbb{R},\mathbb{R}^N)$ and $E \subset L^2(\mathbb{R},\mathbb{R}^N)$. By the Mean Value Theorem and Lemma 2.3, for any

$u, h \in E$, we have

$$\lim_{s\to 0}(\Psi(u+s\ h)-\Psi(u))/s = \lim_{s\to 0}\int_{\mathbb{R}} \widetilde{W}_u(t,u(t)+\theta(t)\ s\ h(t))\cdot h(t)dt$$
$$= \int_{\mathbb{R}} \widetilde{W}_u(t,u(t))\cdot h(t)dt = J(u)h.$$

Hence $D\Psi(u) = J(u)$ is the Gâteaux derivative of Ψ at u.

Now we verify that $D\Psi(u)$ is weakly continuous in u. Let $u_n \rightharpoonup u$ in E, we obtain

$$\begin{aligned}\|D\Psi(u_n)-D\Psi(u)\|_{E^*} &= \sup_{\|v\|=1}|(J(u_n)-J(u))v| \\ &= \sup_{\|v\|=1}\left|\int_{\mathbb{R}}(\widetilde{W}_u(t,u_n(t))-\widetilde{W}_u(t,u(t)))\cdot v(t)dt\right| \\ &\le \sup_{\|v\|=1}\|\widetilde{W}_u(t,u_n)-\widetilde{W}_u(t,u)\|_2\|v\|_2 \\ &\le \beta_2\ \|\widetilde{W}_u(t,u_n)-\widetilde{W}_u(t,u)\|_2 \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Consequently, $D\Psi(u)$ is weakly continuous, therefore $\Psi \in C^1(E,\mathbb{R})$ and $\Psi'(u) = D\Psi(u) = J(u)$ is compact by the weakly continuity of Ψ' since E is a Hilbert space. Hence (2.6) is also verified and $\Phi \in C^1(E,\mathbb{R})$.

Lastly, we check that critical points of Φ on E are homoclinic solutions of $(\widetilde{\text{HS}})$. We know that $E \subset H^1(\mathbb{R},\mathbb{R}^N) \subset C(\mathbb{R},\mathbb{R}^N)$. If $u \in E$ is one critical point of Φ, by (2.6), we have

$$\ddot{u} = L(t)\ u - \widetilde{W}_u(t,u),$$

which yields that $u \in C^2(\mathbb{R},\mathbb{R}^N)$ and satisfies $(\widetilde{\text{HS}})$, i.e., u is a classical solution of $(\widetilde{\text{HS}})$ and it is easy to check that u satisfies $\dot{u}(t) \to 0$ as $|t| \to \infty$. This completes the proof. □

Secondly, we modify $W(t,u)$ for u outside a neighborhood of the origin 0 to get $\widehat{W}(t,u)$ for Theorem 1.2. We choose $0 < \sigma < \min\left\{1, \frac{1}{4A_0\beta_2^2}\right\}$. By (W_3'), choose a constant $0 < r < \delta/2$, we have

$$W(t,u) \ge a(t)\ |u|^\mu \ge \sigma\ a(t)\ |u|^2, \quad \forall\ t\in\mathbb{R} \ \text{ and } \ |u| \le 2r.$$

Let

$$\widehat{W}(t,u) = \rho(|u|)W(t,u) + \sigma\ (1-\rho(|u|))\ a(t)\ |u|^2, \ \forall\ (t,u)\in\mathbb{R}\times\mathbb{R}^N, \quad (2.7)$$

where ρ is the same one as in (2.2).

Lemma 2.5. *Assume that* $(W_1), (W_2'), (W_3')$ *hold. Then* $\widehat{W}$ *possesses the following properties.*

(1) *There exists a constant* c_1 *such that*

$$|\widehat{W}_u(t,u)| \le c_1(b(t)\ |u|^{\gamma-1} + |u|), \quad \forall\ (t,u) \in \mathbb{R}\times\mathbb{R}^N. \tag{2.8}$$

(2) *There exists a constant* c_2 *such that*

$$\widehat{W}(t,u) \le c_2\ b(t)\ |u|^\gamma + A_0\sigma\ |u|^2, \quad \forall\ (t,u) \in \mathbb{R}\times\mathbb{R}^N. \tag{2.9}$$

(3)

$$\widehat{W}(t,u) \ge \sigma\ a(t)\ |u|^2, \quad \forall\ (t,u) \in \mathbb{R}\times\mathbb{R}^N. \tag{2.10}$$

Proof. Note first that $(W_1), (W_2')$ imply

$$|W(t,u)| \le c\ b(t)\ |u|^\gamma, \quad \forall\ (t,u) \in \mathbb{R}\times B_\delta(0).$$

By (2.9), direct computation shows that

$$\begin{aligned}&\widehat{W}_u(t,u)\\ &= \rho(|u|)W_u(t,u) + \left[\rho'(|u|)\left(\frac{W(t,u)}{|u|} - \sigma a(t)|u|\right) + 2\sigma a(t)(1-\rho(|u|))\right] u,\end{aligned}$$

for all $t\in\mathbb{R}$ and $u \ne 0$. Besides, it is easy to check by (W_1) and (W_2') that

$$\widehat{W}_u(t,0) \equiv 0, \qquad \forall\ t\in\mathbb{R}.$$

For $|u| < r$ or $|u| > 2r$, one has

$$|\widehat{W}_u(t,u)| \le b(t)\ |u|^{\gamma-1} + 2\sigma A_0\ |u|,$$

for $r \le |u| \le 2r$, one has

$$\begin{aligned}|\widehat{W}_u(t,u)| &\le b(t)\ |u|^{\gamma-1} + 2/r\ (c\ b(t)\ |u|^{\gamma-1} + \sigma A_0\ |u|)|u| + 2\sigma A_0\ |u|\\ &\le (1+4c)\ b(t)\ |u|^{\gamma-1} + 6\sigma A_0\ |u|.\end{aligned}$$

Thus (2.8) holds by choosing $c_1 = \max\{1+4c, 6\sigma A_0\}$.

By computation as similar as above , (2.9) is trivial. For (2.10), for $|u| \ge 2r$, one has

$$\widehat{W}(t,u) = \sigma\ a(t)\ |u|^2,$$

for $|u| \le 2r$, note that $2r < \delta \le 1$ and $\sigma < 1$, one has

$$\begin{aligned}\widehat{W}(t,u) &\ge \rho(|u|)\ a(t)\ |u|^\mu + \sigma\ (1-\rho(|u|))\ a(t)\ |u|^2\\ &= \rho(|u|)\ a(t)\ |u|^\mu\ (1-\sigma\ |u|^{2-\mu}) + \sigma\ a(t)\ |u|^2\\ &\ge \rho(|u|)\ a(t)\ |u|^\mu\ (1-\sigma) + \sigma\ a(t)\ |u|^2\\ &\ge \sigma\ a(t)\ |u|^2.\end{aligned}$$

This completes the proof. □

We now consider the following modified Hamiltonian system

$$\ddot{u}(t) - L(t)u + \widehat{W}_u(t, u(t)) = 0, \qquad \forall\, t \in \mathbb{R} \tag{$\widehat{\text{HS}}$}$$

and show that above system possesses a sequence homoclinic solutions, which converges to zero in L^∞ norm. Consequently, we obtain infinitely many homoclinic solutions for the primary Hamiltonian system (HS).

Define the variational functional Φ on E associated with the modified system ($\widehat{\text{HS}}$) by

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} (|\dot{u}(t)|^2 + L(t)u(t)\cdot u(t))dt - \int_{\mathbb{R}} \widehat{W}(t, u(t))dt \\ &= \frac{1}{2}\|u\|^2 - \Psi(u). \end{aligned} \tag{2.11}$$

It is obvious from (2.8), (2.9), (W_2') and Lemma 2.1 that $\widehat{W}_u(t,u) \in L^2(\mathbb{R}, \mathbb{R}^N)$ and $\widehat{W}(t,u) \in L(\mathbb{R}, \mathbb{R})$ for all $u \in E$, hence we know that Φ and Ψ are both well defined. Note that (2.3) and (2.8) are same, similarly, we have the following lemmas.

Lemma 2.6. *Suppose that* $(H_1), (W_1), (W_2'), (W_3')$ *are satisfied. If* $u_k \rightharpoonup u$ *(weakly) in* E*, then* $\widehat{W}_u(t,u_k) \to \widehat{W}_u(t,u)$ *in* $L^2(\mathbb{R}, \mathbb{R}^N)$*.*

Lemma 2.7. *Let* (H_1)*,* (W_1)*,* (W_2') *and* (W_3') *be satisfied. Then* $\Psi \in C^1(E, \mathbb{R})$ *and* $\Psi' : E \to E^*$ *is compact, and hence* $\Phi \in C^1(E, \mathbb{R})$*. Moreover, we have*

$$\Psi'(u)v = \int_{\mathbb{R}} \widehat{W}_u(t, u(t)) \cdot v(t)dt, \tag{2.12}$$

$$\begin{aligned} \Phi'(u)v &= \int_{\mathbb{R}} (\dot{u}(t)\cdot\dot{v}(t) + L(t)u(t)\cdot v(t))dt - \int_{\mathbb{R}} \widehat{W}_u(t, u(t))\cdot v(t)dt \\ &= \langle u, v\rangle - \Psi'(u)v \end{aligned} \tag{2.13}$$

for all $u, v \in E$*, and critical points of* Φ *on* E *belong to* $C^2(\mathbb{R}, \mathbb{R}^N)$ *and are homoclinic solutions of* ($\widehat{HS}$)*.*

In order to study the critical points of Φ, we now recall an abstract critical points theorem due to Kajikiya [25] which related to the symmetric mountain pass lemma, is crucial in proving our main results. Let E be a real Banach space, Σ is the set of subset of E, which is closed and symmetric with respect to 0, i.e.,

$$\Sigma = \{B \subset E : B \text{ is closed and } x \in B \text{ if and only if } -x \in B\}.$$

For any $B \in \Sigma$, the genus η of B is defined by

$$\eta(B) := \inf\{k \in \mathbb{N} : \text{ there is an odd map } \phi \in C(B, \mathbb{R}^k \setminus \{0\})\}.$$

When there does not exist such a finite k, set $\eta(B) = \infty$. Moreover, set $\eta(\emptyset) = 0$. For each $k \in \mathbb{N}$, let $\Sigma_k = \{B \in \Sigma : \eta(B) \geq k\}$.

We state the theorem as a lemma as follows.

Lemma 2.8. ([25], Theorem 1) *Let E be an infinite dimensional Banach space and $\varphi \in C^1(E, \mathbb{R})$ an even functional with $\varphi(0) = 0$. Suppose that φ satisfies*

(1) *φ is bounded from below and satisfies (PS) condition;*
(2) *For each $k \in \mathbb{N}$, there exists a $B_k \in \Sigma_k$ such that $\sup_{u \in B_k} \varphi(u) < 0$.*

Then either (i) *or* (ii) *holds.*

(i) *There exists a critical point sequence $\{w_k\}$ such that $\varphi(w_k) < 0$ and $\lim_{k\to\infty} w_k = 0$.*
(ii) *There exist two critical point sequences $\{w_k\}$ and $\{v_k\}$ such that $\varphi(w_k) = 0$, $w_k \neq 0$, $\lim_{k\to\infty} w_k = 0$, $\varphi(v_k) < 0$, $\lim_{k\to\infty} \varphi(v_k) = 0$ and $\{v_k\}$ converges to a nonzero limit.*

3. Proofs of the main results

After all the preparations above, we are now ready to give the proofs of our main results.

Proof of Theorem 1.1. First we prove that Φ defined in (2.5) is bounded from below. For any $u \in E$, by (2.4), we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widetilde{W}(t, u(t))dt \\ &\geq \frac{1}{2}\|u\|^2 - c_2\ \|b\|^2_{\frac{2}{2-\gamma}}\ \|u\|_2^{\gamma} - \sigma\|u\|_2^2.\end{aligned}$$

By (2.1) and the choice of σ, we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{4}\|u\|^2 - c_2\ \|b\|^2_{\frac{2}{2-\gamma}}\ \|u\|_2^{\gamma} \\ &= \frac{1}{4}\|u\|^2 - c\|u\|^{\gamma}\end{aligned}$$

for some constant $c > 0$, which implies that $\Phi(u)$ is coercive on E and a fortiori, bounded from below.

Next, we prove that Φ satisfies the (PS) condition. Let $\{u_n\} \subset E$ be a (PS) sequence, i.e.,

$$|\Phi(u_n)| \leq C_1 \ \text{ and } \ \Phi'(u_n) \to 0 \ \text{ in } \ E^* \text{ as } \ n \to \infty$$

for some constant $C_1 > 0$. Then $\{u_n\}$ is bounded in E since $\Phi(u)$ is coercive on E. Thus, by passing to a suitable subsequence if necessary and Lemma 2.4, Ψ' is compact, we know that there exists a point $u \in E$ such that $\{u_n\}$

converges weakly to u in E and $\|\Psi'(u_n) - \Psi'(u)\|_{E^*} \to 0$ as $n \to \infty$. Hence by (2.6), we have

$$\begin{aligned}\|u_n - u_m\|^2 &= \langle u_n - u_m, u_n - u_m\rangle = \langle u_n, u_n - u_m\rangle - \langle u_m, u_n - u_m\rangle \\ &= (\Phi'(u_n) - \Phi'(u_m))(u_n - u_m) + (\Psi'(u_n) - \Psi'(u_m))(u_n - u_m) \\ &\le (\|\Phi'(u_n)\|_{E^*} + \|\Phi'(u_m)\|_{E^*})\|u_n - u_m\| \\ &\quad + \|\Psi'(u_n) - \Psi'(u_m)\|_{E^*}\|u_n - u_m\| \\ &\to 0 \quad \text{as} \quad n \to \infty,\ m \to \infty.\end{aligned}$$

This shows that $\{u_n\}$ is a Cauchy sequence in E. By the completeness of E, we know that $\{u_n\}$ strongly converges to u in E proving the (PS) condition.

Furthermore, we claim that for any $k \in \mathbb{N}$, there exists a closed symmetric subset $B_k \subset E$ with the genus $\eta(B_k) = k$ such that $\sup_{u\in B_k} \Phi(u) < 0$.

Fix k, let E_k be a k-dimensional subspace of E, therefore there exists a constant $K > 0$ such that

$$\|u\|^2 \le K\ \|u\|_2^2, \quad \forall\, u \in E_k.$$

In view of (W_3) and (2.2), there exists a constant $0 < \omega < r$ such that

$$\widetilde{W}(t, u) = W(t, u) \ge K\ |u|^2, \quad \forall\, t \in \mathbb{R} \quad \text{and} \quad |u| \le \omega.$$

Set $\Omega(u) = \{t \in \mathbb{R} : |u(t)| \le \omega\}$ for $u \in E$. By (2.1), for any $u \in E_k$ with $\|u\| \le \omega/\beta_\infty$, there holds $\|u\|_\infty \le \omega$, hence $\text{meas}(\mathbb{R} \setminus \Omega(u)) = 0$ and

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widetilde{W}(t, u(t))dt = \frac{1}{2}\|u\|^2 - \int_{\Omega(u)} \widetilde{W}(t, u(t))dt \\ &\le \frac{1}{2}\|u\|^2 - \int_{\Omega(u)} K\ |u(t)|^2 dt = \frac{1}{2}\|u\|^2 - K\ \|u\|_2^2 \\ &\le \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2.\end{aligned}$$

Let $B_k = \{u \in E_k : \|u\| = \omega/\beta_\infty\}$, then $\sup_{u\in B_k} \Phi(u) < 0$ and by Borsuk-Ulam Theorem, $\eta(B_k) = k$. This proves the claim.

It follows from above steps that the functional Φ defined in (2.5) satisfies the conditions (1) and (2) in Lemma 2.8. Obviously, Φ is even and $\Phi(0) = 0$. By Lemma 2.8, there exists a sequence of nontrivial critical points $\{u_n\}$ of Φ satisfying $\Phi(u_n) \le 0$ for all $n \in \mathbb{N}$ and $u_n \to 0$ in E as $n \to \infty$. By virtue of (2.1), we further have $u_n \to 0$ in $L^\infty(\mathbb{R}, \mathbb{R}^N)$ as $n \to \infty$. Therefore, for n large enough, there are solutions of (HS). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We prove that the conditions (1) and (2) in Lemma 2.8 are available. First we verify that the functional Φ defined in (2.11) is bounded from below and satisfies (PS) condition.

For any $u \in E$, by (2.9), we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widetilde{W}(t, u(t))dt \\ &\geq \frac{1}{2}\|u\|^2 - c_2 \ \|b\|^2_{\frac{2}{2-\gamma}} \ \|u\|_2^\gamma - \sigma A_0 \ \|u\|_2^2.\end{aligned}$$

By (2.1) and the choice of σ, we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{4}\|u\|^2 - c_2 \ \|b\|^2_{\frac{2}{2-\gamma}} \ \|u\|_2^\gamma \\ &= \frac{1}{4}\|u\|^2 - c \ \|u\|^\gamma\end{aligned}$$

for some constant $c > 0$, which implies that $\Phi(u)$ is coercive on E and a fortiori, bounded from below.

We shall prove that Φ satisfies the (PS) condition. Let $\{u_n\} \subset E$ be a (PS) sequence, i.e.,

$$|\Phi(u_n)| \leq C_1 \ \text{ and } \ \Phi'(u_n) \to 0 \ \text{ in } \ E^* \text{ as } \ n \to \infty$$

for some constant $C_1 > 0$. Then $\{u_n\}$ is bounded in E by above proof. Thus, we know that there exists a point $u \in E$ such that $\{u_n\}$ converges weakly to u in E and $\|\Psi'(u_n) - \Psi'(u)\|_{E^*} \to 0$ as $n \to \infty$. Hence we have

$$\begin{aligned}\|u_n - u\|^2 &= (\Phi'(u_n) - \Phi'(u))(u_n - u) + (\Psi'(u_n) - \Psi'(u))(u_n - u) \\ &\leq \|\Phi'(u_n)\|_{E^*}\|u_n - u\| - \Phi'(u)(u_n - u) \\ &\quad + \|\Psi'(u_n) - \Psi'(u)\|_{E^*}\|u_n - u\| \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

This means that $\{u_n\}$ strongly converges to u in E proving the (PS) condition.

Next we verify that for any $k \in \mathbb{N}$, there exists a closed symmetric subset $B_k \subset E$ with the genus $\eta(B_k) = k$ such that $\sup_{u \in B_k} \Phi(u) < 0$.

Recall that

$$\widehat{W}(t, u) \geq \sigma \ a(t) \ |u|^2, \quad \forall \ (t, u) \in \mathbb{R} \times \mathbb{R}^N.$$

Due to $\inf_{t \in \mathbb{R}} a(t) = 0$, there is no constant $C > 0$ such that

$$\widehat{W}(t, u) \geq \sigma \ a(t) \ |u|^\gamma \geq C \ |u|^\gamma$$

for any $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. Therefore we have to find another way to overcome the difficulty. Fix k, let E_k be a k-dimensional subspace of E. We claim that there exists $\varepsilon > 0$ such that

$$m(\{t \in \mathbb{R} : a(t)\ |u(t)|^2 \geq \varepsilon\ \|u\|^2\}) \geq \varepsilon, \quad \forall\, u \in E_k \setminus \{0\}, \tag{3.1}$$

where $m(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}$. If not, for any $n \in \mathbb{N}$, there exists $u_n \in E_k \setminus \{0\}$ such that

$$m\left(\left\{t \in \mathbb{R} : a(t)\ |u_n(t)|^2 \geq \frac{1}{n}\ \|u_n\|^2\right\}\right) < \frac{1}{n}.$$

Set $v_n(t) = u_n(t)/\|u_n\| \in E_k \setminus \{0\}$, then $\|v_n(t)\| = 1$ and

$$m\left(\left\{t \in \mathbb{R} : a(t)\ |v_n(t)|^2 \geq \frac{1}{n}\right\}\right) < \frac{1}{n}. \tag{3.2}$$

Passing to a subsequence if necessary, we may assume that $v_n \to v_0$ in E_k since the unit sphere of E_k is compact. Evidently, $\|v_0\| = 1$. By Lemma 2.1, we have $v_n \to v_0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ and the equivalence of the norms on E_k. Hence we have

$$\int_{\mathbb{R}} a(t)\ |v_n(t) - v_0(t)|^2 dt \leq A_0 \int_{\mathbb{R}} |v_n(t) - v_0(t)|^2 dt \to 0, \quad \text{as} \quad n \to \infty. \tag{3.3}$$

It is easy to see that there exist $\xi_1, \xi_2 > 0$ such that

$$m(\{t \in \mathbb{R} : a(t)\ |v_0(t)|^2 \geq \xi_1\}) \geq \xi_2. \tag{3.4}$$

In fact, if not, for any positive integer n, we have

$$m\left(\left\{t \in \mathbb{R} : a(t)\ |v_0(t)|^2 \geq \frac{1}{n}\right\}\right) = 0.$$

It implies that

$$\int_{\mathbb{R}} a(t)\ |v_0(t)|^4 dt \leq \frac{1}{n}\ \|v_0\|_2^2 \leq \frac{\beta_2^2}{n}\|v_0\|^2 = \frac{\beta_2^2}{n} \to 0, \text{ as } n \to \infty.$$

Hence $v_0 = 0$ which is a contradiction.

Let

$$\Lambda_0 = \{t \in \mathbb{R} : a(t)\ |v_0(t)|^2 \geq \xi_1\}, \qquad \Lambda_n = \left\{t \in \mathbb{R} : a(t)\ |v_n(t)|^2 < \frac{1}{n}\right\}$$

and $\Lambda_n^c = \mathbb{R} \setminus \Lambda_n = \left\{t \in \mathbb{R} : a(t)\ |v_n(t)|^2 \geq \frac{1}{n}\right\}$. By (3.2) and (3.4), we have

$$\begin{aligned} m(\Lambda_n \cap \Lambda_0) &= m(\Lambda_0 \setminus (\Lambda_n^c \cap \Lambda_0)) \\ &\geq m(\Lambda_0) - m(\Lambda_n^c \cap \Lambda_0) \\ &\geq \xi_2 - \frac{1}{n} \end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} a(t)\ |v_n(t)-v_0(t)|^2dt &\geq \int_{\Lambda_n\cap\Lambda_0} a(t)\ |v_n(t)-v_0(t)|^2dt\\ &\geq \frac{1}{2}\int_{\Lambda_n\cap\Lambda_0} a(t)\ |v_0(t)|^2dt - \int_{\Lambda_n\cap\Lambda_0} a(t)\ |v_n(t)|^2dt\\ &\geq \left(\frac{1}{2}\xi_1-\frac{1}{n}\right)\ m(\Lambda_n\cap\Lambda_0)\\ &\geq \left(\frac{1}{2}\xi_1-\frac{1}{n}\right)\ \left(\xi_2-\frac{1}{n}\right)\ \rightarrow\ \frac{1}{2}\xi_1\ \xi_2>0,\end{aligned}$$

as $n\to\infty$. This is in contradiction to (3.3). Therefore (3.1) holds. For the ε given in (3.1), let

$$\Lambda(u)=\{t\in\mathbb{R}: a(t)\ |u(t)|^2\geq\varepsilon\|u\|^2\},\quad \forall\, u\in E_k\setminus\{0\}.$$

In view of (W_3'), for any $K>0$, there exists a constant $0<\omega<r$ such that $|u|^\mu\geq K|u|^2$ for all $|u|\leq\omega$. Take $K=1/\varepsilon^2$ and $0<\omega\leq\min\{r,K^{\frac{1}{\mu-2}}\}$, by (2.7), we have

$$\widehat{W}(t,u)=W(t,u)\geq a(t)\ |u|^\mu\geq K\ a(t)\ |u|^2$$

for all $t\in\mathbb{R}$ and $|u|\leq\omega$. Set $\Omega(u)=\{t\in\mathbb{R}:|u(t)|\leq\omega\}$ for $u\in E$. By (2.1), for any $u\in E_k\setminus\{0\}$ with $\|u\|\leq\omega/\beta_\infty$, there holds $\|u\|_\infty\leq\omega$, hence $\mathrm{meas}(\mathbb{R}\setminus\Omega(u))=0$ and

$$\begin{aligned}\Psi(u)&=\int_{\mathbb{R}}\widehat{W}(t,u(t))dt=\int_{\Omega(u)}\widehat{W}(t,u(t))dt=\int_{\Omega(u)}W(t,u(t))dt\\ &\geq\int_{\Omega(u)}a(t)\ |u(t)|^\mu dt\geq\int_{\Omega(u)}K\ a(t)\ |u(t)|^2dt\\ &=\int_{\mathbb{R}}K\ a(t)\ |u(t)|^2dt\geq K\ \int_{\Lambda(u)}a(t)\ |u(t)|^2dt\geq K\ \varepsilon\ \|u\|^2\ m(\Lambda(u))\\ &\geq K\ \varepsilon^2\ \|u\|^2=\|u\|^2.\end{aligned}$$

Therefore, for $u\in E_k$ with $\|u\|=\omega/\beta_\infty>0$, we have

$$\Phi(u)=\frac{1}{2}\|u\|^2-\Psi(u)\leq-\frac{1}{2}\|u\|^2=-\frac{\omega^2}{2\beta_\infty^2}<0.$$

Let $B_k=\{u\in E_k:\|u\|=\omega/\beta_\infty\}$, then $\sup_{u\in B_k}\Phi(u)<0$ and $\eta(B_k)=k$.

It follows from above steps that the functional Φ defined in (2.11) satisfies the conditions (1) and (2) in Lemma 2.8. Obviously, Φ is even and $\Phi(0)=0$. By Lemma 2.8, there exists a sequence of nontrivial critical points $\{u_n\}$ of Φ satisfying $\Phi(u_n)\leq 0$ for all $n\in\mathbb{N}$ and $u_n\to 0$ in E as $n\to\infty$. By virtue of (2.1), we further have $u_n\to 0$ in $L^\infty(\mathbb{R},\mathbb{R}^N)$ as $n\to\infty$. Therefore, for n

large enough, there are solutions of (HS). This completes the proof of Theorem 1.2. □

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