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INEQUALITIES OF THE JENSEN TYPE FOR A NEW CLASS OF CONVEX FUNCTIONS

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Abstract. In this paper, we prove some Jensen-type inequalities for a newly introduced class of convex functions. Our results generalize and extend some existing results in literature. Some applications for Schur's inequality are also included.

1. Introduction

Convex functions and its inequalities have been found to play important roles in different areas of Mathematics, Science and Technology. Many interesting inequalities in literature are derived from convex functions (see [3]-[5]). Some previous studies have employed different classes of convex functions to prove new inequalities (see [14], [17]).

2. Preliminaries

Throughout this paper, I and J will be used to denote intervals of real numbers.

Definition 2.1. ([14]) A function $f : [a, b] \to R$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \tag{2.1}$$

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for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Definition 2.2. ([6]) A function $f : I \to R$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}. \tag{2.2}$$

This class of convex functions was first introduced in [6] by Godunova and Levin and has gained attention in literature (see [5], [10], [11]).

Definition 2.3. ([2]) Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense), or that f belongs to the class K_s^2, if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y), \tag{2.3}$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

The class K_s^2 was first introduced by Breckner in [2] and some of its properties given in [7].

Definition 2.4. ([5]) We say that $f : I \to R$ is a P-function, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I$, $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{2.4}$$

Definition 2.5. ([18]) Let $h : J \to R$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to R$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y). \tag{2.5}$$

Definition 2.6. ([18]) A function $f : I \to R$ is said to be super-multiplicative if

$$f(xy) \ge f(x)f(y), \tag{2.6}$$

for all $x, y \in I$.

If the inequality (2.6) is reversed, then f is sub-multiplicative.

Definition 2.7. ([18]) If $f, g : I \to R$ are functions such that for $x, y \in I$ the following inequality holds

$$(f(x) - f(y))(g(x) - g(y)) \ge 0,$$

then f and g are said to be similarly ordered.

In this paper, using an analytical technique we obtain some interesting Jensen-type inequalities with respect to our new class of convex functions. Our results generalize those already given in literature.

3. Main results

We now define a new class of convex functions.

Definition 3.1. Let $h : J \to (0, \infty)$, $s \in [0, 1]$, $t \in (0, 1)$, $m \in (0, 1]$ and ϕ be a given real-valued function. Then, $f : I \to [0, \infty)$ is an m-ϕ_{h-s} convex function if for all $x, y \in I$,

$$f(t\phi(x) + m(1-t)\phi(y)) \\ \leq \left(\frac{h(t)}{t}\right)^{-s} f(\phi(x)) + \left(\frac{1}{m}\frac{h(1-t)}{(1-t)}\right)^{-s} f(\phi(y)). \tag{3.1}$$

Observe that:

(i) If $m = 1$, $s = 0$ and $\phi(x) = x$, then $f \in P(I)$.
(ii) If $m = 1$, $h(t) = t^{\frac{s}{s+1}}$ and $\phi(x) = x$, then $f \in SX(h, I)$.
(iii) If $m = 1$, $s = 1$, $h(t) = 1$ and $\phi(x) = x$, then $f \in SX(I)$ i.e., f is convex.
(iv) If $m = 1$, $h(t) = 1$ and $\phi(x) = x$ then f is *Breckner s-convex* or s-convex in the second sense (see [2]).
(v) If $m = 1$, $h(t) = t^2$ and $\phi(x) = x$ then f is *s-Godunova-Levin* function (see [3], [4]).
(vi) If $m = 1$, then $f \in SX(\varphi_{h-s}, I)$ (see [12]).
(vii) If $h(t) = t^2$ and $s = 1$, then f is said to be an *m-Godunova-Levin* function (see [13]).

If the inequality in (3.1) is reversed, then f is m-ϕ_{h-s} concave, that is, $f \in m$-$SV(\phi_{h-s}, I)$. In what follows, m-$SX(\phi_{h-s}, I)$ will be used to denote the collection of all m-ϕ_{h-s} convex functions.

We give some properties and examples of our new class of convex functions.

Proposition 3.2. *If $f, g \in m$-$SX(\phi_{h-s}, I)$, $\phi : \mathbb{R} \to \mathbb{R}$ and $c \geq 0$, $c \in \mathbb{R}$, then $f + g$ and cf are both m-ϕ_{h-s} convex.*

Proof. $f, g \in m$-$SX(\phi_{h-s}, I)$ implies

$$f(\lambda\phi(a) + m(1-\lambda)\phi(b)) \\ \leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} f(\phi(a)) + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b)) \tag{3.2}$$

and

$$\begin{aligned} &g(\lambda\phi(a)+m(1-\lambda)\phi(b)) \\ &\leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} g(\phi(a)) + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} g(\phi(b)). \end{aligned} \tag{3.3}$$

Clearly,

$$\begin{aligned} &(f+g)(\lambda\phi(a)+m(1-\lambda)\phi(b)) \\ &\leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} [(f+g)(\phi(a))] + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} [(f+g)(\phi(b))] \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &cf(\lambda\phi(a)+m(1-\lambda)\phi(b)) \\ &\leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} cf(\phi(a)) + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} cf(\phi(b)). \end{aligned} \tag{3.5}$$

□

We have just shown that addition and scalar multiplication holds for our newly defined class of convex functions so that m-$SX(\phi_{h-s}, I)$ is a linear space for $c \geq 0$. Observe that, m-$SX(\phi_{h-s}, I)$ satisfies the properties (and even more) enjoyed by many known classes of convex function in literature.

Proposition 3.3. *Let f be a non-negative m-convex function on I. If h is a non-negative function such that*

$$h(\lambda) \leq \lambda^{1-\frac{1}{s}}, \quad s \in (0,1], \quad \lambda \in (0,1),$$

then $f \in m$-$SX(\phi_{h-s}, I)$.

Proof. Since m-convex functions generalizes convex functions then Proposition 3.3 implies that all convex functions are examples of our newly defined class of convex function provided that the condition $h(\lambda) \leq \lambda^{1-\frac{1}{s}}$ is satisfied. In particular, an example of such $h(\lambda)$ is $h(\lambda) = \lambda^k$ for $k \geq 1 - \frac{1}{s}$, $s \in (0,1]$.

Now,

$$\begin{aligned} &f(\lambda\phi(a)+m(1-\lambda)\phi(b)) \\ &\leq \lambda f(\phi(a)) + m(1-\lambda) f(\phi(b)) \\ &\leq \left(\frac{h(\lambda)}{\lambda}\right)^{-s} f(\phi(a)) + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b)). \end{aligned} \tag{3.6}$$

Therefore, $f \in m$-$SX(\phi_{h-s}, I)$. □

Similarly, if h has the property that $h(\lambda) \geq \lambda^{1-\frac{1}{s}}$ and for $\lambda \in (0,1)$, then any non-negative concave function f belongs to the class m-$SV(\phi_{h-s}, I)$ i.e., is m-ϕ_{h-s} concave.

This new notion of m-$\phi(x)_{h-s}$ convex functions generalizes quite a number of classes of convex functions which exist in literature. The immediate implication of this is that m-$\phi(x)_{h-s}$ convex functions provide us with many inequalities which generalize and extend the Jensen-type inequalities for the classes of convex functions that already have been considered by several researchers.

Theorem 3.4. *Let $f \in m\text{-}SX(\phi_{h-s}, I)$,*

(i) *If g is a linear function, then $f \circ g$ is m-$\phi_{h-s}(I)$ convex.*
(ii) *If f is increasing and g is m-convex on I such that $\phi(a), \phi(b) \in I$, then $f \circ g$ is m-$\phi_{h-s}(I)$ convex.*

Proof. Since f is m-$\phi_{h-s}(I)$ convex then,

$$\begin{aligned} & f\left(\lambda\phi(x) + m(1-\lambda)\phi(y)\right) \\ & \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(\phi(x)) + \left(m\frac{(1-\lambda)}{h(1-\lambda)}\right)^s f(\phi(y)). \end{aligned} \tag{3.7}$$

Now,

$$\begin{aligned} & f \circ g\left(\lambda\phi(a) + m(1-\lambda)\phi(b)\right) \\ & = f\left(\lambda g(\phi(a)) + m(1-\lambda)g(\phi(b))\right) \\ & \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f \circ g(\phi(a)) + \left(m\frac{(1-\lambda)}{h(1-\lambda)}\right)^s f \circ g(\phi(b)). \end{aligned} \tag{3.8}$$

and (i) is proved.

We now prove (ii). Since g is m-convex then

$$g(\lambda\phi(a) + m(1-\lambda)\phi(b)) \leq \lambda g(\phi(a)) + (1-\lambda)g(\phi(b)).$$

Now,

$$\begin{aligned} & f \circ g(\lambda\phi(a) + m(1-\lambda)\phi(b)) \\ & \leq f(\lambda g(\phi(a)) + m(1-\lambda)g(\phi(b))) \\ & \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f \circ g(\phi(a)) + \left(m\frac{(1-\lambda)}{h(1-\lambda)}\right)^s f \circ g(\phi(b)). \end{aligned} \tag{3.9}$$

□

Remark 3.5. Theorem 3.4 generalizes Theorems 1 and 3 recently obtained by Ardic and Özdemir (2012) with appropriate choices of $h(\lambda)$ and s. Infact, with $h(\lambda) = \lambda^2$, $m = 1$ and $s = 1$, Theorem 3 of [1] is obtained and with $h(\lambda) = 1$, we recapture Theorem 1 in [1].

Theorem 3.6. *Let $0 \in I$ and $\phi(x)$ be an identity function. Then, the following holds:*

(i) *If* $f \in m\text{-}SX(\phi_{h-s}, I)$, $f(0) = 0$ *and* h *is super-multiplicative, then the inequality*

$$f(\alpha x + m\beta y) \leq \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(x) + \left(\frac{1}{m}\frac{h(\beta)}{\beta}\right)^{-s} f(y), \tag{3.10}$$

holds for all $x, y \in I$ *and* $\alpha, \beta \geq 0$ *such that* $\alpha + \beta \leq 1$.

(ii) *Let* h *be non-negative function with* $\left(\frac{h(\alpha)}{\alpha}\right)^s > (1 + m^s)$ *for some* $\alpha \in (0, \frac{1}{2})$, $m \in (0, 1]$, $s \in (0, 1]$. *If* f *is a non-negative function satisfying* (3.10) *for all* $x, y \in I$, *and* $\alpha, \beta > 0$ *with* $\alpha + \beta \leq 1$, *then* $f(0) = 0$.

(iii) *If* $f \in m\text{-}SV(\phi_{h-s}, I)$, $f(0) = 0$ *and* h *is sub-multiplicative, then the inequality*

$$f(\alpha x + m\beta y) \geq \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(x) + \left(\frac{1}{m}\frac{h(\beta)}{\beta}\right)^{-s} f(y), \tag{3.11}$$

holds for all $x, y \in I$ *and* $\alpha, \beta > 0$ *such that* $\alpha + \beta \leq 1$.

(iv) *Let* h *be a non-negative function with* $\left(\frac{h(\alpha)}{\alpha}\right)^s > (1 + m^s)$ *for some* $\alpha \in (0, \frac{1}{2})$, $m \in (0, 1]$, $s \in (0, 1]$. *If* f *is a non-negative function satisfying* (3.11) *for all* $x, y \in I$ *and* $\alpha, \beta > 0$ *with* $\alpha + \beta \leq 1$, *then* $f(0) = 0$.

Proof. We first prove (i). The proof is trivial for $\alpha, \beta > 0$, $\alpha + \beta = 1$, since (3.10) reduces to (3.1) by Definition 3.1. Let $\alpha, \beta > 0$, $\alpha + \beta = r < 1$ and let a and b be numbers such that $a = \frac{\alpha}{r}$ and $b = \frac{\beta}{r}$, then $a + b = 1$ and

$$\begin{aligned}
f(\alpha x + m\beta y) &= f(arx + mbry) \\
&\leq \left(\frac{h(a)}{a}\right)^{-s} f(rx) + \left(\frac{1}{m}\frac{h(b)}{b}\right)^{-s} f(ry) \\
&\leq \left(\frac{h(a)}{a}\right)^{-s} \left[\left(\frac{h(r)}{r}\right)^{-s} f(x) + \left(\frac{h(1-r)}{1-r}\right)^{-s} f(0)\right] \\
&\quad + \left(\frac{1}{m}\frac{h(b)}{b}\right)^{-s} \left[\left(\frac{h(r)}{r}\right)^{-s} f(y) + \left(\frac{h(1-r)}{1-r}\right)^{-s} f(0)\right] \\
&\leq \left(\frac{h(ar)}{ar}\right)^{-s} f(x) + \left(\frac{1}{m}\frac{h(br)}{br}\right)^{-s} f(y) \\
&= \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(x) + \left(\frac{1}{m}\frac{h(\beta)}{\beta}\right)^{-s} f(y).
\end{aligned}$$

We now prove (ii). Suppose $f(0) \neq 0$, then $f(0) > 0$. By setting $x = y = 0$ in (3.10), we get

$$f(0) \leq \left(\frac{h(\alpha)}{\alpha}\right)^{-s} f(0) + \left(\frac{1}{m}\frac{h(\beta)}{\beta}\right)^{-s} f(0). \tag{3.12}$$

Again, we set $\alpha = \beta$, where $\alpha \in (0, \frac{1}{2})$ and dividing both sides of the inequality (3.12) by $f(0)$ to obtain $\left(\frac{h(\alpha)}{\alpha}\right)^{-s} \geq (1 + m^s)^{-1}$, for all $\alpha \in (0, \frac{1}{2})$. This contradicts the assumption that $\left(\frac{h(\alpha)}{\alpha}\right)^{s} > (1+m^s)$, so $f(0)$ must be 0. Hence the proof of (ii). The proofs of (iii) and (iv) follow by similar arguments and are thus omitted here. □

Theorem 3.7. *Let f and g be similarly ordered on I for all $a, b \in I$. If $f \in SX(\phi_{h_1 - s}, I)$ and $g \in SX(\phi_{h_2 - s}, I)$ such that*

$$\left(\frac{h(\lambda)}{\lambda}\right)^{-s} + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} \leq c^{-s},$$

for all $\lambda \in (0, 1)$, $m \in (0, 1]$ with $h(\lambda) = \max\{h_1(\lambda), h_2(\lambda)\}$ and c is a fixed positive number. Then, $fg \in SX(\phi_{ch-s}, I)$.

Proof. Since f and g are similarly ordered, then

$$(f(\phi(a)) - f(\phi(b)))(g(\phi(a)) - g(\phi(b))) \geq 0, \quad \forall\, \phi(a), \phi(b) \in I.$$

Implying,

$$f(\phi(a))g(\phi(a)) + f(\phi(b))g(\phi(b)) \geq f(\phi(a))g(\phi(b)) + f(\phi(b))g(\phi(a)).$$

Now,

$$\begin{aligned}
&fg(\lambda\phi(a) + (1-\lambda)\phi(b)) \\
&= f(\lambda\phi(a) + (1-\lambda)\phi(b))g((\lambda\phi(a) + (1-\lambda)\phi(b))) \\
&\leq \left[\left(\frac{h_1(\lambda)}{\lambda}\right)^{-s} f(\phi(a)) + \left(\frac{1}{m}\frac{h_1(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b))\right] \\
&\quad \times \left[\left(\frac{h_2(\lambda)}{\lambda}\right)^{-s} g(\phi(a)) + \left(\frac{1}{m}\frac{h_2(1-\lambda)}{1-\lambda}\right)^{-s} g(\phi(b))\right] \\
&\leq \left(\frac{h(\lambda)}{\lambda}\right)^{-2s} fg(\phi(a)) + \left(\frac{1}{m}\frac{h(\lambda)h(1-\lambda)}{\lambda(1-\lambda)}\right)^{-s} f(\phi(a))g(\phi(b)) \\
&\quad + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\frac{h(\lambda)}{\lambda}\right)^{-s} f(\phi(b))g(\phi(a)) + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-2s} fg(\phi(b))
\end{aligned}$$

$$\leq \left(\frac{h(\lambda)}{\lambda}\right)^{-2s} fg(\phi(a)) + \left(\frac{1}{m}\frac{h(\lambda)}{\lambda}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(a))g(\phi(a))$$
$$+ \left(\frac{1}{m}\frac{h(\lambda)}{\lambda}\frac{h(1-\lambda)}{1-\lambda}\right)^{-s} f(\phi(b))g(\phi(b)) + \left(\frac{1}{m}\frac{h(1-\lambda)}{1-\lambda}\right)^{-2s} fg(\phi(b))$$
$$\leq \left(\frac{ch(\lambda)}{\lambda}\right)^{-s} fg(\phi(a)) + \left(\frac{1}{m}\frac{ch(1-\lambda)}{1-\lambda}\right)^{-s} fg(\phi(b)).$$

Hence, $fg \in m\text{-}SX(\phi_{ch-s}, I)$. This completes the proof. □

Remark 3.8. Theorem 3.7 extends the following result of Varoŝanec on $SX(h, I)$ i.e., the class of $h-$convex functions (see [18]).

If we define $\lambda^{\frac{s}{s+1}} = ch(\lambda)$, $\phi(x) = x$ and set $m = 1$ in Theorem 3.7, the Corollary 3.9 follows immediately.

Corollary 3.9. *Let f and g be similarly ordered functions on I, $\forall\, x, y \in I$. If $f \in SX(h_1, I)$ and $g \in SX(h_2, I)$ such that $h(\lambda) + h(1-\lambda) \leq c$ for all $\lambda \in (0, 1)$ with $h(\lambda) = \max\{h_1, h_2\}$ and c a fixed positive number. Then the product fg belongs to $SX(ch, I)$.*

Theorem 3.10. *Let $h : J \to R$ be a non-negative super-multiplicative and let $f : I \to R$ be a function such that $f \in m\text{-}SX(\phi_{h-s}, I)$ where $\phi(x) = x$. Then, for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ and $x_3 - x_1$, $x_3 - x_2$, $x_2 - x_1 \in J$, the following holds:*

$$\begin{aligned} &[(x_3 - x_2)(x_2 - x_1)h(x_3 - x_2)]^{-s} f(x_1) \\ &\quad - [(x_3 - x_2)(x_2 - x_1)h(x_3 - x_1)]^{-s} f(x_2) \\ &\quad + \left[\frac{1}{m}(x_3 - x_1)(x_3 - x_2)h(x_2 - x_1)\right]^{-s} f(x_3) \geq 0. \end{aligned} \tag{3.13}$$

Proof. Since $f \in m\text{-}SX(\phi_{h-s}, I)$, then it is easy to see that
$$\frac{x_3 - x_2}{x_3 - x_1}, \frac{x_2 - x_1}{x_3 - x_1} \in J \quad \text{and} \quad \frac{x_3 - x_2}{x_3 - x_1} + \frac{x_2 - x_1}{x_3 - x_1} = 1.$$
Also,
$$\begin{aligned} (h(x_3 - x_2))^{-s} &= \left(h\left(\frac{x_3 - x_2}{x_3 - x_1}(x_3 - x_1)\right)\right)^{-s} \\ &\geq \left(h\left(\frac{x_3 - x_2}{x_3 - x_1}\right) h(x_3 - x_1)\right)^{-s}. \end{aligned}$$
Similarly,
$$(h(x_2 - x_1))^{-s} \geq \left(h\left(\frac{x_2 - x_1}{x_3 - x_1}\right) h(x_3 - x_1)\right)^{-s}.$$

Setting $\alpha = \frac{x_3-x_2}{x_3-x_1}$, $x = x_1$, $my = x_3$, $\beta = \frac{x_2-x_1}{x_3-x_1}$ in (2.1) such that $f\left(\frac{x_3}{m}\right) \leq f(x_3)$, we have $x_2 = \alpha x + m\beta y$ and

$$f(x_2) \leq \left(\frac{h\left(\frac{x_3-x_2}{x_3-x_1}\right)}{\frac{x_3-x_2}{x_3-x_1}}\right)^{-s} f(x_1) + \left(\frac{1}{m}\frac{h\left(\frac{x_2-x_1}{x_3-x_1}\right)}{\frac{x_2-x_1}{x_3-x_1}}\right)^{-s} f\left(\frac{x_3}{m}\right) \tag{3.14}$$

$$\leq \left(\frac{\frac{h(x_3-x_2)}{h(x_3-x_1)}}{\frac{x_3-x_2}{x_3-x_1}}\right)^{-s} f(x_1) + \left(\frac{1}{m}\frac{\frac{h(x_2-x_1)}{h(x_3-x_1)}}{\frac{x_2-x_1}{x_3-x_1}}\right)^{-s} f(x_3). \tag{3.15}$$

Multiplying both sides of (3.15) by $\left[\frac{h(x_3-x_1)}{x_3-x_1}\right]^{-s}$ and rearranging, gives (3.13). □

This result has several implications for the Schur's inequality (interested reader can see [9] and references therein).

Theorem 3.11. *Let $w_1, w_2, \cdots, w_n$ be positive real numbers and let (a, b) be an interval in I. If $h : (0, \infty) \to R$ is a non-negative super-multiplicative function and $f \in m\text{-}SX(\phi_{h-s}, I)$ where ϕ is an identity function, then for all $x_1, x_2, \cdots, x_n \in (a, b)$*

$$\sum_{i=1}^{n} p_i f(x_i) \leq f(a)\sum_{i=1}^{n} p_i \left(\frac{h\left(\frac{b-x_i}{b-a}\right)}{\frac{b-x_i}{b-a}}\right)^{-s} + f(b)\sum_{i=1}^{n} p_i \left(\frac{1}{m}\frac{h\left(\frac{x_i-a}{b-a}\right)}{\frac{x_i-a}{b-a}}\right)^{-s}, \tag{3.16}$$

where

$$p_i = \left(\frac{h\left(\frac{w_i}{W_n}\right)}{\frac{w_i}{W_n}}\right)^{-s}, \quad W_n = \sum_{i=1}^{n} w_i.$$

Proof. Setting $x_1 = a, x_2 = x_i, x_3 = b$ in (3.14) we obtain

$$f(x_i) \leq \left(\frac{h\left(\frac{b-x_i}{b-a}\right)}{\frac{b-x_i}{b-a}}\right)^{-s} f(a) + \left(\frac{1}{m}\frac{h\left(\frac{x_i-a}{b-a}\right)}{\frac{x_i-a}{b-a}}\right)^{-s} f(b). \tag{3.17}$$

Multiplying both sides of (3.17) by p_i and adding both sides of the resulting inequality for $(i = 1, \cdots, n)$, we obtain (3.16). This completes the proof. □

Theorem 3.12. *Let $f \in m\text{-}SX(\phi_{h-s}, I)$ and $\sum_{i=1}^{n} t_i = T_n = 1$, $t_i \in (0,1)$, $i = 1, 2, \cdots, n$. Then*

$$f\left(\sum_{i=1}^{n} m^{i-1} t_i \phi(x_i)\right) \leq \left[m^{1-i} \frac{h(t_i)}{t_i}\right]^{-s} \sum_{i=1}^{n} f(\phi(x_i)).$$

Proof. Observe that

$$\begin{aligned}
& f\left(\sum_{i=1}^{n} m^{i-1} t_i \phi(x_i)\right) \\
& = f\left(T_{n-1} \sum_{i=1}^{n-1} m^{i-1} \frac{t_i}{T_{n-1}} \phi(x_i) + m^{n-1} t_n \phi(x_n)\right) \\
& \leq \left[\frac{h(T_{n-1})}{T_{n-1}}\right]^{-s} f\left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} m^{i-1} \frac{t_i}{T_{n-2}} \phi(x_i) + m^{n-2} t_{n-1} \phi(x_{n-1})\right) \\
& \quad + \left[\frac{1}{m^{n-1}} \frac{h(t_n)}{t_n}\right]^{-s} f(\phi(x_n)) \\
& \leq \left[\frac{h(T_{n-1})}{T_{n-1}}\right]^{-s} \left[h\left(\frac{T_{n-2}}{T_{n-1}}\right)\right]^{-s} \left[\frac{T_{n-1}}{T_{n-2}}\right]^{-s} f\left(\sum_{i=1}^{n-2} m^{i-1} \frac{t_i}{T_{n-2}} \phi(x_i)\right) \\
& \quad + \left[\frac{1}{m^{n-2}} \frac{h(t_{n-1})}{t_{n-1}}\right]^{-s} f(\phi(x_{n-1})) + \left[\frac{1}{m^{n-1}} \frac{h(t_n)}{t_n}\right]^{-s} f(\phi(x_n)) \\
& \vdots \\
& \leq \left[\frac{1}{m^{i-1}} \frac{h(t_i)}{t_i}\right]^{-s} \sum_{i=1}^{n} f(\phi(x_i)).
\end{aligned}$$

□

Remark 3.13. Set $\phi(x) = x$, $h(t) = 1$, $m = 1$ and $s = 1$ in Theorem 3.12, then we obtain what is known in literature as the discrete version of the Jensen's inequality (see [8]).

Theorem 3.14. *Let $t_1, \cdots, t_n$ be positive real numbers ($n \geq 2$). If h is a non-negative super-multiplicative function and if $f \in m\text{-}SX(\phi_{h-s}, I)$ where ϕ is an identity function, for $x_1, \cdots, x_n \in I$, then*

$$f\left(\frac{1}{T_n} \sum_{i=1}^{n} m^{i-1} t_i x_i\right) \leq \sum_{i=1}^{n} \left[m^{1-i} \frac{h(\frac{t_i}{T_n})}{\frac{t_i}{T_n}}\right]^{-s} f(x_i), \tag{3.18}$$

where $T_n = \sum_{i=1}^{n} t_i$*. If* h *is sub-multiplicative and* $f \in m\text{-}SV(\phi_{h-s}, I)$*, then the inequality* (3.18) *is reversed.*

Proof. Let us suppose that $f \in SX(\phi_{h-s}, I)$. If $n = 2$, then the inequality (3.18) is equivalent to the definition of ϕ_{h-s} convex functions with $\lambda = \frac{t_1}{T_2}$ and $1-\lambda = \frac{t_2}{T_2}$. Suppose that (3.18) holds for $n-1$, then for n-tuples $(x_1, \cdots, x_n)$ and $(t_1, \cdots, t_n)$, we have,

$$\begin{aligned}
& f\left(\frac{1}{T_n}\sum_{i=1}^{n} m^{i-1} t_i x_i\right) \\
&= f\left(m^{n-1}\frac{t_n}{T_n}x_n + \sum_{i=1}^{n-1} m^{i-1}\frac{t_i}{T_n}x_i\right) \\
&= f\left(m^{n-1}\frac{t_n}{T_n}x_n + \frac{T_{n-1}}{T_n}\sum_{i=1}^{n-1} m^{i-1}\frac{t_i}{T_{n-1}}x_i\right) \\
&\le \left[\frac{1}{m^{n-1}}\frac{h\left(\frac{t_n}{T_n}\right)}{\frac{t_n}{T_n}}\right]^{-s} f(x_n) + \left[\frac{h\left(\frac{T_{n-1}}{T_n}\right)}{\frac{T_{n-1}}{T_n}}\right]^{-s} f\left(\sum_{i=1}^{n-1} m^{i-1}\frac{t_i}{T_{n-1}}x_i\right) \\
&\le \left[\frac{1}{m^{n-1}}\frac{h\left(\frac{t_n}{T_n}\right)}{\frac{t_n}{T_n}}\right]^{-s} f(x_n) \\
&\quad + \left[\frac{h\left(\frac{T_{n-1}}{T_n}\right)}{\frac{T_{n-1}}{T_n}}\right]^{-s} \sum_{i=1}^{n-1}\left[\frac{1}{m^{i-1}}\frac{h\left(\frac{t_i}{T_{n-1}}\right)}{\frac{t_i}{T_{n-1}}}\right]^{-s} f(x_i) \\
&\le \left[m^{1-n}\frac{h\left(\frac{t_n}{T_n}\right)}{\frac{t_n}{T_n}}\right]^{-s} f(x_n) + \sum_{i=1}^{n-1}\left[m^{1-i}\frac{h\left(\frac{t_i}{T_n}\right)}{\frac{t_i}{T_n}}\right]^{-s} f(x_i) \\
&= \sum_{i=1}^{n}\left[m^{1-i}\frac{h\left(\frac{t_i}{T_n}\right)}{\frac{t_i}{T_n}}\right]^{-s} f(x_i).
\end{aligned}$$

□

Remark 3.15. If we choose $h(\lambda) = 1$, $m = 1$ and $s = 1$, then Theorem 3.14 becomes the well known discrete version of the classical Jensen inequality for convex functions (see [8], [16]). If we choose $h(\lambda) = 1$ and $m = 1$, then our result is the Jensen-type inequality for s-convex functions (see [16]). If

$h(\lambda) = \lambda^2$ and $m = 1$, then we obtain the very recent result known as the Jensen-type inequality for s-Godunova-Levin functions (see [15]).

Corollary 3.16. *Under the conditions of Theorem* 3.14*,*

(i) *if* $T_n = 1$*, then*

$$f\left(\sum_{i=1}^{n} m^{i-1} t_i x_i\right) \leq \sum_{i=1}^{n} \left[m^{1-i} \frac{h(t_i)}{t_i}\right]^{-s} f(x_i), \tag{3.19}$$

(ii) *if* $t_1 = t_2 = \cdots = t_n$*, then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} m^{i-1} t_i x_i\right) \leq \left[\frac{h\left(\frac{1}{n}\right)}{\frac{1}{n}}\right]^{-s} \sum_{i=1}^{n} m^{1-i} f(x_i), \tag{3.20}$$

(iii) *if* h *is sub-multiplicative and* $f \in m\text{-}SV(\phi_{h-s}, I)$*, then the inequalities* (3.19) *and* (3.20) *are reversed.*

Corollary 3.17. *Under the conditions of Corollary* 3.16*, if* $h(\lambda) = 1$*, then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} m^{i-1} t_i x_i\right) \leq \frac{1}{n^s} \sum_{i=1}^{n} m^{1-i} f(x_i). \tag{3.21}$$

Theorem 3.18. *Let* $h : [0,1] \to \mathbb{R}$ *be a super-multiplicative function,* $m \in (0,1]$*,* $s \in [0,1]$*,* $n \geq 2$*. If* $f : [0, \frac{a}{m^{n-1}}] \to \mathbb{R}$ *belongs to* $m\text{-}SX(\phi_{h-s}, I)$ *where* ϕ *is an identity function, then for each* $x_i \in [0,a]$ *and* $t_i \geq 0$ $(i = 1, 2, \cdots, n)$*,*

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} m^{1-i} \left[\frac{h\left(\frac{t_i}{T_n}\right)}{\frac{t_i}{T_n}}\right]^{-s} f\left(\frac{x_i}{m^{i-1}}\right), \tag{3.22}$$

where $T_n = \sum_{i=1}^{n} t_i$*. If* h *is sub-multiplicative and* $f \in m\text{-}SV(\phi_{h-s}, I)$*, where* $I = [0, \frac{a}{m^{n-1}}]$*, then* (3.22) *above is reversed.*

Proof. Putting $y_i = \frac{x_i}{m^{i-1}}$, $i = 1, 2, \cdots, n$ and using (3.11) to obtain

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) = f\left(\frac{1}{T_n}\sum_{i=1}^{n} m^{i-1} t_i\, y_i\right) \leq \sum_{i=1}^{n} m^{1-i} \left[\frac{h\left(\frac{t_i}{T_n}\right)}{\frac{t_i}{T_n}}\right]^{-s} f(y_i)$$

$$= \sum_{i=1}^{n} m^{1-i} \left[\frac{h\left(\frac{t_i}{T_n}\right)}{\frac{t_i}{T_n}}\right]^{-s} f\left(\frac{x_i}{m^{i-1}}\right).$$

This completes the proof. □

Corollary 3.19. *For $m \in (0,1]$, $s \in [0,1]$ and $n \geq 2$, the assertion $f \in m$-$SX(\phi_{1-s}, I)$ is valid if and only if $\forall x_i \in [0,a]$ and $t_i > 0$ $(i = 1, 2, \cdots, n)$, then*

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} m^{1-i}\left(\frac{t_i}{T_n}\right)^s f\left(\frac{x_i}{m^{i-1}}\right),$$

where $T_n = \sum_{i=1}^{n} t_i$.

Corollary 3.20. *Under the conditions of Theorem* 3.18*,*

(i) *if $T_n = 1$, then*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} m^{1-i}\left[\frac{h(t_i)}{t_i}\right]^{-s} f\left(\frac{x_i}{m^{i-1}}\right), \tag{3.23}$$

(ii) *if $t_1 = t_2 = \cdots = t_n$, then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \left[\frac{h\left(\frac{1}{n}\right)}{\frac{1}{n}}\right]^{-s} \sum_{i=1}^{n} m^{1-i} f\left(\frac{x_i}{m^{i-1}}\right), \tag{3.24}$$

(iii) *if h is sub-multiplicative and $f \in m$-$SV(\phi_{h-s}, I)$ where $I = [0, \frac{a}{m^{n-1}}]$, then the inequalities* (3.23) *and* (3.24) *are reversed.*

Corollary 3.21. *Under the assumptions of Corollary* 3.20*, if $h(\lambda) = 1$, then*

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n^s}\sum_{i=1}^{n} m^{1-i} f\left(\frac{x_i}{m^{i-1}}\right).$$

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