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# ON THE LOCATION OF ZEROS OF A POLYNOMIAL WITH RESTRICTED COEFFICIENTS

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**Abstract.** If the coefficients of a polynomial $P(z)$ are in decreasing order, then all the zero of $P(z)$ lie in $|z| \leq 1$. The aim of this paper is to present an interesting generalization of this result with less restrictive conditions on the coefficients, which among other things generalize some other known results as well. .

## 1. Introduction

A well known result of Eneström and Kakeya (see[4, 5]) in theory of the distribution of zeros of polynomials is the following:

**Theorem 1.1** (Eneström-Kakeya). *If* $P(z) = \sum_{j=1}^{n} a_j z^j$ *is a polynomial of degree* $n$ *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

*then* $P(z)$ *has its* $n$ *zeros in* $|z| \leq 1$.

In literature there exits several generalizations and extensions of this result.Aziz and Zargar [1] relaxed the hypothesis of Theorem 1.1 and proved the following extension of the theorem:

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**Theorem 1.2.** *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* $n$ *such that for some* $k \geq 1$,

$$ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

*then all the zeros of* $P(z)$ *lie in* $|z+k-1| \leq k$.

Shah and Liman [6] proved the following extensions of Theorem 1.2:

**Theorem 1.3.** *If* $P(z) = \sum_{j=1}^{n} a_j z^j$ *is a polynomial of degree* $n$ *with complex coefficients such that for some real* $\beta$,

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2 \ \ , j = 0, 1, \cdots, n,$$

*and for some* $k \geq 1$,

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

*then all the zeros of* $P(z)$ *lie in*

$$|z+k-1| \leq \frac{1}{|a_n|}\left[(k|a_n| - |a_0|)(\sin\alpha + \cos\alpha) + |a_0| + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|\right].$$

In this paper we wish to generalize Theorem 1.3 to a larger class of polynomials under less restrictive conditions on the coefficients. In fact, we prove

**Theorem 1.4.** *If* $P(z) = \sum_{j=1}^{n} a_j z^j$ *is a polynomial of degree* $n$ *with complex coefficients such that for some real* $\beta$,

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2 \ \ , j = 0, 1, \cdots, n,$$

*and for* $k \geq 1$, $0 < \tau \leq 1$,

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq \tau|a_0|,$$

*then all the zeros of* $P(z)$ *lie in*

$$|z+k-1| \leq \frac{1}{|a_n|}\left[k|a_n|(\sin\alpha + \cos\alpha) + 2|a_0| + 2\sin\alpha \sum_{j=1}^{n-1} |a_j| - \tau|a_0|(\cos\alpha - \sin\alpha + 1)\right].$$

**Remark 1.** For $\tau = 1$, Theorem 1.4 reduces to Theorem 1.3 by noting that for $0 \le \alpha \le \pi/2$, $1 - \cos\alpha + \sin\alpha \le 2\sin\alpha$. For $\alpha = \beta = 0$ and $\tau = 1$, it reduces to Theorem 1.2.

Applying Theorem 1.1 to the polynomial $P(tz)$, we obtain the following :

**Corollary 1.5.** *If* $P(z) = \sum_{j=1}^{n} a_j z^j$ *is a polynomial of degree* $n$ *with complex coefficients such that for some real* $\beta$,

$$|\arg a_j - \beta| \le \alpha \le \pi/2 \ \ , j = 0, 1, \cdots, n,$$

*and for* $k \ge 1$, $t > 0$, $0 < \tau \le 1$,

$$kt^n|a_n| \ge t^{n-1}|a_{n-1}| \ge \cdots \ge t|a_1| \ge \tau|a_0|,$$

*then all the zeros of* $P(z)$ *lie in*

$$|z + kt - t| \le \frac{1}{|a_n|}\left[k|a_n|(\sin\alpha + \cos\alpha) + 2\frac{|a_0|}{t^n} + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - \frac{\tau|a_0|}{t^n}(\cos\alpha - \sin\alpha + 1)\right].$$

**Remark 2.** For $\tau = 1$, Corollary 1.5 reduces to a result of Shah and Liman ([6, Cor. 1]) and for $\alpha = \beta = 0$ and $\tau = 1$, it reduces to a result of Aziz and Muhammad (see [2]).

If we take $k = \left|\frac{a_{n-1}}{a_n}\right| \ge 1$ in theorem 1.1, we get the following:

**Corollary 1.6.** *If* $P(z) = \sum_{j=1}^{n} a_j z^j$ *is a polynomial of degree* $n$ *with complex coefficients such that for some real* $\beta$,

$$|\arg a_j - \beta| \le \alpha \le \pi/2 \ \ , j = 0, 1, \cdots, n,$$

*and*

$$|a_n| \ge t^{n-1}|a_{n-1}| \ge \cdots \ge t|a_1| \ge \tau|a_0|, \quad 0 < \tau \le 1,$$

*then all the zeros of $P(z)$ lie in*

$$\left|z+\left|\frac{a_{n-1}}{a_n}\right|-1\right| \le \frac{1}{|a_n|}\Bigg[|a_{n-1}|\,(\sin\alpha+\cos\alpha)+2|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j| - \tau|a_0|\,(\cos\alpha-\sin\alpha+1)\Bigg].$$

## 2. Proof of Theorem

For the proof of Theorem 1.4, we need the following lemma due to Govil and Rahman (see[3]).

**Lemma 2.1.** *If for some $t>0$, $|ta_j|\ge|a_{j-1}|$ and $|\arg a_j-\beta|\le\alpha\le\pi/2$, for some real $\beta$, then*

$$|ta_j-a_{j-1}|\le\Big[(|ta_j|-|a_{j-1}|)\cos\alpha+(|ta_j|+|a_{j-1}|)\sin\alpha\Big].$$

***Proof of Theorem* 1.4**. Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n+a_{n-1})z^n+\cdots+(a_1-a_0)z+a_0\\&=-a_nz^n(z+k-1)+(ka_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}\\&\qquad+\cdots+(a_1-\tau a_0)z+(\tau a_0-a_0)z+a_0.\end{aligned}$$

For $|z|>1$ so that $1/|z|^{n-j}<1$, $0\le j\le n$, we have

$$|F(z)|\ge|z|^n\Bigg[|a_n||z+k-1|-\Bigg\{|ka_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}+\cdots+\frac{|a_2-a_1|}{|z|^{n-2}}+\frac{|a_1-\tau a_0|}{|z|^{n-1}}+\frac{|\tau a_0-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}\Bigg\}\Bigg]$$

$$\geq |z|^n\Bigg[|a_n||z+k-1| - \Bigg\{|ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_2 - a_1| + |a_1 - \tau a_0| + |\tau a_0 - a_0| + |a_0|\Bigg\}\Bigg]$$

$$\geq |z|^n\Bigg[|a_n||z+k-1| - \Big\{(k|a_n| - |a_{n-1}|)\cos\alpha + (k|a_n| + |a_{n-1}|)\sin\alpha\Big\} - \Big\{(|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha\Big\} - \cdots - \Big\{(|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha\Big\} - \Big\{(|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha\Big\} - \Big\{(1-\tau)|a_0| + |a_0|\Big\}\Bigg] \text{(by using the Lemma 2.1)}$$

$$= |z|^n\Bigg[|a_n||z+k-1| - \Big\{k|a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| + 2|a_0| - \tau|a_0|(\cos\alpha - \sin\alpha + 1)\Big\}\Bigg] > 0,$$

if

$$|z+k-1| > \frac{1}{|a_n|}\Big[k|a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| + 2|a_0| - \tau|a_0|(\cos\alpha - \sin\alpha + 1)\Big].$$

This shows that the zeros of $F(z)$ having modulus greater than 1 lie in

$$|z+k-1| \leq \frac{1}{|a_n|}\Big[k|a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| + 2|a_0| - \tau|a_0|(\cos\alpha - \sin\alpha + 1)\Big].$$

But the zeros of $F(z)$ having modulus less than or equal to 1 already satisfy the above inequality. Therefore it follows that all the zeros of $F(z)$ and hence

$P(z)$ lie in the disk

$$|z+k-1| \leq \frac{1}{|a_n|}\Big[k|a_n|(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=1}^{n-1}|a_j| + 2|a_0|-\tau|a_0|(\cos\alpha-\sin\alpha+1)\Big].$$

This completes the proof. □

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