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FINITE DIMENSIONALITY OF THE ATTRACTOR FOR A SEMILINEAR WAVE EQUATION WITH NONLINEAR BOUNDARY DISSIPATION

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Abstract. This paper deals with the long-time behavior of a semilinear wave equation with nonlinear boundary dissipation. By estimating energy function, we prove that the fractal dimension of the global attractor for the dynamical system is finite.

1. Introduction

Let $\Omega \subset R^3$ be a bounded connected open set with a smooth boundary Γ. The exterior normal on Γ is denoted by ν. We consider the following equation

$$u_{tt} - \Delta u + \varepsilon u_t + f(u) = 0 \quad in\ Q = [0, \infty) \times \Omega \tag{1.1}$$

subject to the boundary condition

$$\partial_\nu u + u = -g(u_t) \quad in\ \Sigma = [0, \infty) \times \Gamma \tag{1.2}$$

and the initial condition:

$$u(0) = u_0 \ \text{ and } u_t(0) = u_1.$$

Here f and g are nonlinear functions subject to the following assumption,$0 \leq \varepsilon \ll 1$ is a small parameter.

Assumption 1. (H1) $f \in C^2(R)$ such that $|f''(s)| \leq c(1+|s|^{1-\delta})$ for all $s \in R$ and for some $c > 0$ and $\delta > 0$.

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(H2) $\lim_{|s|\to\infty} s^{-1}f(s) > -\lambda$, where λ is the best constant in the Poincare type inequality:$\int_\Omega |\nabla u|^2dx + \int_\Gamma |u|^2dS \geq \lambda \int_\Omega |u|^2dx$.

(g) $g \in C^1(R)$ is an increasing function, $g(0) = 0$ and there exist two positive constants m_1 and m_2 such that $m_1 \leq g'(s) \leq m_2$ for all $|s| \geq 1$.

This problem of long time behavior of solutions to (1.1)-(1.2) was the subject of a recent paper [2]. In [2] it was shown, in particular,that all finite energy solutions are attracted by a global compact attractor,whose structure is determined by the unstable manifolds emanating from stationary solutions.

Fereisel[4] dealt either with one-dimensional wave equations or with equations subjected to severe restrictions (linear bound) imposed on the semilinear terms [8]. Most recently, there has been a renewed interest in this problem and positive results for semilinear wave equations in dimension higher than one with interior dissipation have been established by Prazak [7].

Igor Chueshov [1] has shown under the Assumption 1.1 the fractal dimension the global attractor of the (1.1) is finite when $\varepsilon = 0$. His methods can be extended to $0 < \varepsilon \leq 1$. In this paper we will use the methods to study the (1.1). Our main results read as follows:

Theorem 1. Under the Assumption 1.1 with $g'(0) > 0$, the fractal dimension, the global attractor of the dynamical system (1.1) is finite when $0 < \varepsilon < \varepsilon_0$ where ε_0 is a constant.

In order to describe the decay rates to equilibrium, we introduce a concave, strictly increasing, continuous function $h : \mathrm{R}^+ \to \mathrm{R}^+$ which captures the behavior of $g(s)$ at the origin possessing the properties

$$h(0) = 0 \quad \text{and} \quad s^2 + g^2(s) \leq h(sg(s)) \quad \text{for } |s| \leq 1.$$

Given function h we define

$$H_0(s) = h\left(\frac{s}{c_3}\right),\ G_0(s) = c_1(I + H_0)^{-1}(c_2 s),\ Q(s) = s - (I + G_0)^{-1}(s)$$

where c_1 and c_2 are positive constants depending on $m_1,\ m_2$ and c_3 is proportional to the measure of $\Gamma \times (0, T)$.

2. Proof of Theorem 1

We recall that the fractal dimension $\dim_f M$ of a compact set M can be defined by the formula [3]

$$\dim_f M = \limsup_{\varepsilon\to 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $N(M,\varepsilon)$ is the minimal number of closed sets of diameter 2ε which cover M.

Throughout the paper we shall use the energy functional

$$\begin{aligned} E(w(t)) &= \frac{1}{2}\int_\Omega |\nabla w(t)|^2 + \frac{1}{2}\int_\Omega |w_t(t)|^2 + \frac{1}{2}\int_\Gamma |w(t)|^2 \\ &\equiv \frac{1}{2} \parallel (w(t), w_t(t)) \parallel_{\mathcal{F}}^2, \end{aligned} \tag{2.1}$$

which corresponds to a linear version ($f \equiv 0$ and $g \equiv 0$,$\varepsilon \equiv 0$) of problem (1.1) and (1.2), $\mathcal{F} \equiv H^1(\Omega) \times L^2(\Omega)$ which is often referred to as the finite energy space.

The key inequality in our considerations is the following inequality.

Lemma 1. Let $u(t)$ and $v(t)$ be two solutions to (1.1) possessing the properties

$$\parallel (u(t), u_t(t)) \parallel_{\mathcal{F}}^2 \leq R \quad \text{and} \quad \parallel (v(t), v_t(t)) \parallel_{\mathcal{F}}^2 \leq R \quad \text{for all} \quad t \geq 0,$$

with some constant $R > 0$. Denote $z(t) \equiv u(t) - v(t)$. Then ,under the Assumption 1 and $g'(0) > 0$,there exist positive constants C_1, C_2 and β such that

$$E(z(t)) \leq C_1 e^{-\beta t} E(z(0)) + C_2 \int_0^t e^{-\beta(t-s)} \parallel z(s) \parallel_{L^2(\Omega)}^2 ds \tag{2.2}$$

for all $t \geq 0$.

Remark. Since the global attractor(we write $\mathcal{A}$) is an invariant set we know that the solutions $(u(t), u_t(t))$ and $(v(t), v_t(t))$ corresponding to the initial data (u_0, u_1), $(v_0, v_1) \in \mathcal{A}$ will stay in $\mathcal{A}$. Moreover, (2.2) implies that

$$E(z(t)) \leq C_1 e^{-\beta t} E(z(0)) + \frac{C_2}{\beta} \max_{s\in[0,t]} \parallel z(s) \parallel_{L^2(\Omega)}^2 ds \quad for \ all \quad t \geq 0.$$

This inequality will be crucial in the proof of the theorem.

To prove Lemma 1 we need the following observability inequality which is valid without the restriction $g'(0) > 0$.

Lemma 2. Let $T > T_0 \equiv 4(r + \frac{1}{\sqrt{\lambda}})$, where r is the radius of a minimal ball in R^n containing Ω and λ is the constant from (H2).Assume that two solutions $u(t)$ and $v(t)$ to problem (1.1) possess the property

$$\parallel (u(t), u_t(t)) \parallel_{\mathcal{F}} \leq R \quad \text{and} \quad \parallel (v(t), v_t(t)) \parallel_{\mathcal{F}} \leq R \quad for \ \ all \quad t \in [0,T]$$

with some constant $R > 0$. Then, under the Assumption 1 there exist positive constants $C_1(T)$ and $C_2(R,T)$ such that for $z(t) \equiv u(t) - v(t)$ we have the relations

$$E(z(T))+\int_0^T E(z(t))dt \le C_2\int_0^T \parallel z\parallel^2_{L^2(\Omega)} dt$$
$$+C_1(I+H_0)\left(\int_0^T \langle g(u_t(t))-g(v_t(t)),z_t(t)\rangle\, dt\right), \tag{2.3}$$

where $H_0(s)=Th(s/T)$ and $h(s)$ is defined in the beginning. If $g'(0)>0$, then (2.3)hold with $H_0(s)\equiv s$ and

$$E(z(T))\le C_1(E(z(0))-E(z(T)))+C_2\int_0^T \parallel z\parallel^2_{L^2(\Omega)} dt. \tag{2.4}$$

Proof. We will denote different constant by C_i pointing out their dependence on the parameters when it becomes important.

Our starting point is the following identity:

$$\frac{1}{2}\int_{Q_T}(|z_t|^2+|\nabla z|^2)=-\int_{Q_T}(f(u)-f(v))(\widetilde{h}\cdot\nabla z+z)$$
$$-\int_\Omega z_t(\widetilde{h}\cdot\nabla z+z)|_0^T-\int_{Q_T}\varepsilon z_t(\widetilde{h}\cdot\nabla z+z)+\int_{\sum_T}\Big[\partial_\nu z(\widetilde{h}\cdot\nabla z+z)$$
$$+\frac{1}{2}((\widetilde{h}\cdot\nu)(|z_t|^2-|\nabla z|^2)\Big], \tag{2.5}$$

where $\widetilde{h}=x-x_0$ for some $x_0\in R^3$ and $Q_T=(0,\ T)\times\Omega$ and $\sum_T=(0,T)\times\Gamma$.Here T is a positive constant which will be determined later in the proof. We choose $x_0\in R^3$ such that $\sup\limits_{x\in\Omega}|\widetilde{h}(x)|=r$, where r is the radius of a minimal ball in R^3 containing Ω.

In the following we will transform (2.5) to an energy inequality.

At first we estimate the second term in (2.5). By

$$f(u)-f(v)=(u-v)\int_0^1 f'(su+(1-s)v)ds=z\int_0^1 f'(su+(1-s)v)ds,$$

the Hölder's inequality, and the embedding $H^s(\Omega)\subset L^p(\Omega)$ for $s=\frac{3}{2}-\frac{3}{p}$ and $p\ge 2$, we can get that

$$\parallel f(u(t))-f(v(t))\parallel_{L^2(\Omega)}\le C(R,\Omega,c,\delta)\cdot\parallel z(t)\parallel_{L^{\frac{6}{1+\delta}}(\Omega)},\ t\in[0,T], \tag{2.6}$$

for some $\delta>0$ and

$$-\int_\Omega (f(u(t))-f(v(t)))\widetilde{h}\cdot\nabla z(t)$$
$$\le C(\varepsilon)r^2\parallel f(u(t))-f(v(t))\parallel^2_{L^2(\Omega)}+\varepsilon\parallel\nabla z(t)\parallel_{L^2(\Omega)},$$

for any $\varepsilon > 0$.By Hölder's inequality, and the embedding $H^s(\Omega) \subset L^p(\Omega)$ for $s = \frac{3}{2} - \frac{3}{p}$ and $p \geq 2$,

$$-\int_\Omega (f(u(t)) - f(v(t)))\widetilde{h} \cdot z \leq C(R, \Omega, c, \delta) \parallel z(t) \parallel_{L^{\frac{6}{1+\delta}}(\Omega)}, \ t \in [0, T],$$

So we can conclude

$$-\int_{Q_T} (f(u(t)) - f(v(t)))\widetilde{h} \cdot z \leq C \int_0^T \parallel z(t) \parallel_{L^{\frac{6}{1+\delta}}(\Omega)} dt.$$

It is easy to see that $|\int_\Omega z_t(\widetilde{h} \cdot \nabla z + z)| \leq (r + \frac{1}{\sqrt{\lambda}})E(z(t))$.

Hence, according to (2.5) and the definition of the energy functional (2.2) yield

$$\begin{aligned}\int_0^T E(z(t))dt \quad &\leq C_1(r) \left(\parallel z_t \parallel^2_{L^2(\sum_T)} + \parallel \nabla z \parallel^2_{L^2(\sum_T)} + \parallel z \parallel^2_{L^2(\sum_T)}\right) \\ &+ \left(r + \tfrac{1}{\sqrt{\lambda}}\right)[E(z(T)) + E(z(0))] + \left(r + \tfrac{1}{\sqrt{\lambda}}\right)\varepsilon \textstyle\int_0^T E(z(t))dt \\ &+ C_2(r, R, \varepsilon) \textstyle\int_0^T \parallel z(t) \parallel^2_{L^{\frac{6}{1+\delta}}(\Omega)} dt + \int_0^T \parallel \nabla z \parallel^2_{L^2(\Omega)} dt. \quad (2.7)\end{aligned}$$

Multiplying the differential equation for z by z_t and integrating by parts result in the following energy identity

$$\begin{aligned}&E(z(t)) + \int_s^t \langle g(u_t(\tau)) - g(v_t(\tau)), z_t(\tau)\rangle\, d\tau + \varepsilon \int_s^t (z_t(\tau), z_t(\tau))d\tau \\ &+ \int_s^t \langle f(u(\tau)) - f(v(\tau)), z_t(\tau)\rangle\, d\tau = E(z(s)) \quad for\ 0 \leq s \leq t. \qquad (2.8)\end{aligned}$$

Since the second term and the third on the left side are non-negative, we can also write

$$E(z(t)) \leq E(z(s)) - \int_s^t \langle f(u(\tau)) - f(v(\tau)), z_t(\tau)\rangle\, d\tau \quad for\ 0 \leq s \leq t. \quad (2.9)$$

Therefore using (2.6), the embedding $H^1(\Omega) \subset L^6(\Omega) \subset L^{\frac{6}{1+\delta}}(\Omega)$ and Grownwall's lemma we obtain from (2.9) that

$$E(z(t)) \leq E(z(s)) \cdot e^{a_R(t-s)} \quad \text{for all } \ 0 \leq s \leq t, \qquad (2.10)$$

where the constant $a_R > 0$ depends on Ω, c and δ.By (g) ,without loss of generality, we can suppose that $0 \leq g'(s) \leq m_2$ for all $s \in R$. Therefore we derive from (2.9),(2.8) and (g) we have the following result

$$\begin{aligned} E(z(s)) &\leq E(z(t)) + m_2 \int_s^t \int_\Gamma |z_t(\tau)|^2 d\tau \\ &+ \int_s^t (2\varepsilon \parallel z_t(\tau) \parallel^2_{L^2(\Omega)} + C(\varepsilon, R) \parallel z(\tau) \parallel^2_{L^{\frac{6}{1+\delta}}(\Omega)}) d\tau, \end{aligned} \quad (2.11)$$

for any $\varepsilon > 0$ and for all $0 \leq s \leq t$. In a same way from (2.8) we have

$$\begin{aligned} \int_s^t \langle \; g(u_t(\tau)) - g(v_t(\tau)), z_t(\tau) \rangle \, d\tau &\leq E(z(s)) - E(z(t)) \\ + \int_s^t (2\varepsilon \parallel z_t(\tau) \parallel^2_{L^2(\Omega)} &+ C(\varepsilon, R) \parallel z(\tau) \parallel^2_{L^{\frac{6}{1+\delta}}(\Omega)}) d\tau, \end{aligned} \quad (2.12)$$

for any $\varepsilon > 0$ and for all $0 \leq s \leq t$.

The estimate of the tangential derivative $\nabla_\tau z$ on $\sum_T$ is given in Lemma 7.2 in Lasiecka and Iriggiani (1992) below [6].For $0 < \alpha < T/2$ and $\eta \in (0, 1/2)$ there exists a constant $C = C(\alpha, \eta, T, \Omega)$ such that

$$\int_\alpha^{T-\alpha} \int_\Gamma |\nabla_\tau z(t)|^2 dt \leq C \Bigg(\parallel \partial_\nu z \parallel^2_{L^2(\sum_T)} + \parallel z_t \parallel^2_{L^2(\sum_T)} + \parallel z \parallel^2_{H^{\frac{1}{2+\eta}}(Q_T)} + \parallel f(u) - f(v) \parallel^2_{H^{-\frac{1}{2+\eta}}(Q_T)} \Bigg).$$

By (2.6) the last term on the right-hand side can be estimated in the following

$$\begin{aligned} &\parallel f(u) - f(v) \parallel^2_{H^{-\frac{1}{2+\eta}}(Q_T)} \\ \leq \; & C \parallel f(u) - f(v) \parallel^2_{L^2(Q_T)} \leq C(R) \int_0^T \parallel z(t) \parallel^2_{L^{\frac{6}{1+\delta}}(\Omega)} dt \end{aligned} \quad (2.13)$$

We derive from (2.11)

$$\begin{aligned} E(z(\alpha)) + E(z(T-\alpha)) &\leq 2E(z(T)) + 2m_2 \parallel z_t \parallel^2_{L^2(\sum_T)} + 4\varepsilon \parallel z_t \parallel^2_{L^2(Q_T)} \\ &+ C(\varepsilon, R) \int_0^T \parallel z(\tau) \parallel^2_{L^{\frac{6}{1+\delta}}(\Omega)} d\tau \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} E(z(\alpha)) &+ E(z(T-\alpha)) \leq 2\alpha E(z(T)) + 2\alpha m_2 \parallel z_t \parallel^2_{L^2(\sum_T)} \\ &+ 4\alpha\varepsilon \parallel z_t \parallel^2_{L^2(Q_T)} + \alpha C(\varepsilon, R) \int_0^T \parallel z(\tau) \parallel^2_{L^{\frac{6}{1+\delta}}(\Omega)} d\tau \end{aligned} \quad (2.15)$$

for any $\alpha \in (0, T/2)$.Therefore using estimate (2.7) for the interval $(\alpha, T - \alpha)$ and also relation (2.13) we obtain (after redefining ε) and setting $c_0 =$

$2\left(\alpha + r + \frac{1}{\lambda}\right)$,

$$\begin{aligned} &\int_0^T E(z(t))dt \leq C_1\{\| z_t \|^2_{L^2(\sum_T)} + \| \partial_\nu z \|^2_{L^2(\sum_T)} + \| z \|^2_{L^2(\sum_T)}\} \\ + \quad & C_2(\varepsilon, R)\left\{\int_0^T \| z(t) \|^2_{L^{\frac{6}{(1+\delta)}}(\Omega)} dt + \| z(t) \|^2_{H^{\frac{2}{(2+\eta)}}(Q_T)}\right\} \\ + \quad & \varepsilon \int_0^T (\| z_t(t) \|^2_{L^2(\Omega)} + \| \nabla z(t) \|^2_{L^2(\Omega)})dt + c_0 E(z(T)). \end{aligned} \tag{2.16}$$

Integrating now inequality (2.9) with $t = T$ with respect to s over the interval $[0, T]$ and using (2.6) yields

$$TE(z(T)) \leq \int_0^T E(z(s))ds + \varepsilon \| z_t \|^2_{L^2(Q_T)} + C(R, T, \varepsilon) \int_0^T \| z(t) \|^2_{L^{\frac{6}{1+\delta}}(\Omega)} dt.$$

Hence,choosing ε small enough, from (2.16) we obtain that

$$\begin{aligned} &TE(z(T)) + \frac{1}{2}\int_0^T E(z(t))dt \\ &\leq C_1\left(\| z_t \|^2_{L^2(\sum_T)} + \| \partial_\nu z \|^2_{L^2(\sum_T)} + \| z \|^2_{L^2(\sum_T)}\right) \\ &+2c_0 E(z(T)) + C_2\left\{\int_0^T \| z(t) \|^2_{L^{\frac{6}{1+\delta}}(\Omega)} dt + \| z \|^2_{H^{\frac{1}{2+\eta}}(Q_T)}\right\}. \end{aligned}$$

Therefore for any $T > T_0 \equiv 4\left(r + \frac{1}{\sqrt{\lambda}}\right)$ we can choose appropriate α and obtain an estimate of the form

$$\begin{aligned} &TE(z(T)) + \int_0^T E(z(t))dt \\ &\leq C_1(T)\left(\| z_t \|^2_{L^2(\sum_T)} + \| \partial_\nu z \|^2_{L^2(\sum_T)} + \| z \|^2_{L^2(\sum_T)}\right) \\ &+C_2(R, T)\left\{\int_0^T \| z(t) \|^2_{L^{\frac{6}{1+\delta}}(\Omega)} dt + \| z \|^2_{H^{\frac{1}{2+\eta}}(Q_T)}\right\}. \end{aligned} \tag{2.17}$$

For the boundary terms we can get the result from the proof of [1].Since

$$\partial_\nu z + z = -g(u_t) + g(v_t) = -z_t \int_0^1 g'(su_t + (1 - s)v_t)ds,$$

using the inequality $0 \leq g' \leq m_2$ we have

$$\| z_t \|^2_{L^2(\sum_T)} + \| \partial_\nu z \|^2_{L^2(\sum_T)} \leq (1 + 2m_2^2) \| z_t \|^2_{L^2(\sum_T)} + 2 \| z \|^2_{L^2(\sum_T)}.$$

By Assumption (g) we can get

$$\begin{aligned} & \| z_t \|^2_{L^2(\sum_T)} + \| \partial_\nu z \|^2_{L^2(\sum_T)} \\ \leq \quad & C_0 \int_0^T \langle g(z_t), z_t \rangle \, dt + C_1 \int_{\sum_T^*} |z_t|^2 ds dt + 2 \| z \|^2_{L^2(\sum_T)}, \end{aligned}$$

where$\sum_T^* = \{(x,t) \in \sum_T : |z_t(x,t)| \leq 1\}$.Since $s^2 \leq h(g(s))$ for $|s| \leq 1$,using Jensen's inequality we obtain that

$$\begin{aligned} \| z_t \|^2_{L^2(\sum_T)} \quad + \quad \| \partial_\nu z \|^2_{L^2(\sum_T)} & \leq C \int_0^T (I+h)(\langle g(z_t), z_t \rangle) dt + 2 \| z \|^2_{L^2(\sum_T)} \\ & \leq \quad C \cdot (I + H_0) \left(\int_0^T \langle g(z_t), z_t \rangle \, dt \right) + 2 \| z \|^2_{L^2(\sum_T)} . \qquad (2.18) \end{aligned}$$

Therefore (2.18) and (2.17) yield

$$\begin{aligned} & E(z(T)) + \int_0^T E(z(t)) dt \\ & \leq C_1 (I + H_0) \left(\int_0^T \langle g(u_t(t)) - g(v_t(t)), z_t \rangle \, dt \right) + C_2 A(z), \quad (2.19) \end{aligned}$$

where $C_1 = C_1(T), C_2 = C_2(R,T)$ and $A(z)$ is an abbreviation for a collection of lower-order terms, that is,

$$A(z) = \int_0^T \| z(t) \|^2_{L^{\frac{6}{1+\delta}}(\Omega)} \, dt + \| z \|^2_{H^{\frac{1}{2+\eta}}(Q_T)} + \| z \|^2_{L^2(\sum_T)} .$$

From (2.12) and (2.19) we also have

$$E(z(T)) + \int_0^T E(z(t)) dt \leq C_1 (I + H_0)(E(z(0)) - E(z(T))) + C_2 A(z), \quad (2.20)$$

In the following we will estimate $A(z)$.By the trace theorem we obtain that $\| z(t) \|^2_{L^2(\sum_T)} \leq C \| z(t) \|^2_{H^{\frac{1}{2+\eta}}(\Omega)}$. The interpolation together with the embedding $H^{1-\frac{\delta}{2}}(\Omega) \subset L^{\frac{6}{1+\delta}}(\Omega)$ give us that

$$\begin{aligned} \| z \|^2_{H^{\frac{1}{2}+\eta}(Q_T)} & \leq \quad 2\varepsilon \| z \|^2_{H^1(Q_T)} + C(\varepsilon) \| z \|^2_{L^2(Q_T)} \\ & \leq \quad c\varepsilon \int_0^T E(z(t)) dt + C(\varepsilon) \| z \|^2_{L^2(Q_T)}, \end{aligned}$$

$$\begin{aligned} \int_0^T \| z(t) \|^2_{L^{\frac{6}{1+\delta}}(\Omega)} \, dt & \leq \quad C \int_0^T \| z(t) \|^2_{H^{1-\frac{\delta}{2}}(\Omega)} \, dt \\ & \leq \quad \varepsilon \int_0^T E(z(t)) dt + C(\varepsilon) \| z \|^2_{L^2(Q_T)} . \end{aligned}$$

Thus we can draw the following estimate

$$A(z) \le \varepsilon \int_0^T E(z(t))dt + C(\varepsilon) \parallel z \parallel^2_{L^2(Q_T)}, \tag{2.21}$$

with arbitrary $\varepsilon > 0$ .Hence,(2.19) and (2.20) yield the desired relation.

In the case $g'(0) > 0$ we have $h(s) = c_0 s$ and therefore (2.3) holds with $H(s) = s$. Similarly, (2.20) and (2.21) imply (2.4). □

Proof of Lemma 2. Under the hypotheses and $g'(0) > 0$ it implies from (2.4) that

$$E(z(nT)) \le \frac{C_1}{1+C_1} E(z((n-1)T)) + \frac{C_2}{1+C_1} \int_{(n-1)T}^{nT} \parallel z \parallel^2_{L^2(\Omega)} dt, \tag{2.22}$$

$(n = 1, 2, \cdots)$ for fixed $T > T_0$. We can prove by induction that

$$E(z(nT)) \le \gamma^n E(z(0)) + \frac{C_2}{1+C_1} \sum_{k=1}^{n} \gamma^{n-k} \int_{(k-1)T}^{kT} \parallel z \parallel^2_{L^2(\Omega)} dt \tag{2.23}$$

for all positive integers n, where $\gamma = C_1/(1+C_1)$.It follows from (2.10) that

$$E(z(t)) \le E(z(nT)) \cdot e^{a_R T} \quad \text{for all} \quad nT \le (n-1)T, \quad n = 0, 1, 2, \cdots . \tag{2.24}$$

Set $\beta = \frac{1}{T} \ln \frac{1}{\gamma}$. It is clear that $\gamma^{n-k} \le \frac{1}{\gamma^2} \exp -\beta(t-\tau)$ for $t \le (n+1)T$ and $\tau \ge (k-1)T$. Therefore (2.2) follows from (2.23) and (2.24).

In order to prove the theorem, we need the following theorem 2 [1]:

Theorem 2. Let X be a separable Hilbert space and A be a bounded closed set in X. Assume that there exists a mapping $V : A \mapsto X$ such that

(1) $A \subset VA$.

(2)V is Lipschitz on A , i.e, there exists $L > 0$ such that

$$\parallel Va_1 - Va_2 \parallel \le L \parallel a_1 - a_2 \parallel \quad \text{for all } a_1, a_2 \in A.$$

(3)There exist compact seminorms $n_1(x)$ and $n_2(x)$ on X such that

$$\parallel Va_1 - Va_2 \parallel \le \eta \parallel a_1 - a_2 \parallel + K \cdot [n_1(a_1 - a_2) + n_2(Va_1 - Va_2)],$$

for all $a_1, a_2 \in A$, where $0 < \eta < 1$ and $K > 0$ are constants.

Then A is a compact set in X of a finite fractal dimension with the estimate for the following dimension

$$\dim_f A \le d \cdot \ln\left(1 + \frac{4(1+L)}{\alpha}\right) \cdot \left[\ln \frac{1}{\eta + \delta + \alpha \widetilde{K}}\right]^{-1}.$$

In a special case when the seminorms n_i have the form $n_i(v) = \| P_i v \|$, where P_1 and P_2 are finite-dimensional orthoprojections , we have that

$$\dim_f A \leq (\dim P_1 + \dim P_2 \cdot \ln\left(1 + \frac{8(1+L)\sqrt{2}K}{1-\eta}\right) \cdot \left[\ln \frac{2}{1+\eta}\right]^{-1}.$$

Proof of Theorem 1. Set the space $X = \mathcal{F} \times H_1(Q_T)$ equipped with the norm

$$\| U \|_X = \| \nabla u_0 \|^2_{L^2(\Omega)} + \| u_0 \|^2_{L^2(\Gamma)} + \| u_1 \|^2_{L^2(\Omega)} + \int_0^T E(v(t))dt,$$

where $U = (u_0, u_1, v)$,$T > 0$ is a constant to be determined later. On the space X we define a seminorm

$$n_T(U) = \max_{0 \leq t \leq T} \| v(t) \|_{L^2(\Omega)},$$

then $n_T(U)$ is a compact seminorm on X by the compactness of the imbedding [9].Consider in the space X the set

$$\mathcal{A}_T = \{U \equiv (u_0, u_1, u(t) \text{for } t \in [0,T]) : (u_0, u_1) \subset \mathcal{A}\},$$

where $u(t)$ is the solution to (1.1) with initial data $u(0) = u_0, u_t(0) = u_1$ and $\mathcal{A}$ is an attractor. We define the operator $V_T : \mathcal{A} \mapsto X$ by the formula

$$V_T : (u_0, u_1, u(t)) \mapsto (u(T), u_t(T), u(T+t)) = (s(T)(u_0, u_1), u(T+t)).$$

We shall show that all conditions of Theorem 2 are satisfied. For (1),this follows from the invariance property of the attractor $\mathcal{A}$ which is equivalent to $V_T \mathcal{A}_T = \mathcal{A}_T$. As for (2),$V_T$ is Lipschitz continuous on $\mathcal{A}_T$. In order to show this statement we will work with two solutions $u(t)$ and $v(t)$ to the problem (1.1).We set $U_1 = (u_0, u_1, u(t))$, $U_2 = (v_0, v_1, v(t))$ and $z(t) = u(t) - v(t)$ and observe that

$$\begin{aligned} \frac{1}{2} \| U_1 - U_2 \|^2_X &= E(z(0)) + \int_0^T E(z(t))dt, \text{ and} \\ \frac{1}{2} \| V_T U_1 - V_T U_2 \|^2_X &= E(z(T)) + \int_T^{2T} E(z(t))dt, \end{aligned} \tag{2.25}$$

From (2.10),we can get

$$\int_T^{2T} E(z(s))ds \leq e^{a_R T} \int_0^T E(z(s))ds.$$

We have the Lipschitz property of V_T with $L = e^{a_R T/2}$ when we combine the above ineqality and formula (2.10).

Integrating the inequality in the Remark over the interval $[T, 2T]$ obtain

$$\int_T^{2T} E(z(t))dt \leq C_1 e^{-\beta T} E(z(0)) + C_2 T \max_{0\leq\tau\leq 2T} \| z(\tau) \|^2_{L^2(\Omega)}, \qquad (2.26)$$

where C_1 and C_2 do not depend on T. Combining the last inequality and the inequality in Remark, we can get

$$E(z(T) + \int_T^{2T} E(z(t))dt \leq C_1 e^{-\beta T} E(z(0)) + C_2 \max_{0\leq\tau\leq 2T} \| z(\tau) \|^2_{L^2(\Omega)} . (2.27)$$

Since

$$\max_{0\leq\tau\leq 2T} \| z(\tau) \|^2_{L^2(\Omega)} \leq \max_{0\leq\tau\leq T} \| z(\tau) \|^2_{L^2(\Omega)} + \max_{0\leq\tau\leq T} \| z(\tau + T) \|^2_{L^2(\Omega)},$$

accounting for the definitions of V_t and the norms in X, relation (2.27) can be written in the following form

$$\| V_T U_1 - V_T U_2 \|^2_X \leq \eta_T \| U_1 - U_2 \|^2_X + K \cdot [\eta_T(U_1 - U_2) + \eta_T(V_T U_1 - V_T U_2)]$$

for all $U_1,\ U_2 \in \mathcal{A}_T$, where $\eta_T = C_1 e^{-\beta T}$. We can select T large enough such that $\eta_T < 1$.

Hence, all the assumption of Theorem 2 are satisfied. It implies that $\mathcal{A}_T$ is a compact set in X of finite fractal dimension. Let $\mathcal{P} : X \to \mathcal{F}$ be the operator defined by the fromula

$$\mathcal{P} : (u_0, U_1, v(t)) \to (u_0, u_1).$$

Since $\mathcal{A} = \mathcal{P}\mathcal{A}_T$ and $\mathcal{P}$ is obviously Lipschitz continuous, we have that

$$\dim^{\mathcal{F}}_{frac} \mathcal{A} = \dim^X_{frac} \mathcal{A}_T < \infty$$

where $\dim^Y_{frac}$ stands for fractal dimension of a set in the space Y.

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