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CHARACTERIZATION OF EFFICIENT SOLUTIONS FOR MULTI-OBJECTIVE OPTIMIZATION PROBLEMS INVOLVING STRONG AND GENERALIZED STRONG E-CONVEXITY

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Abstract. In this paper we shall interest with characterization of efficient solutions for special classes of problems. These classes consider strongly E-convexity of involved functions. Sufficient and necessary conditions for a feasible solution to be an efficient or properly efficient solution are obtained.

1. Introduction

The concept of E-convexity of sets and functions was presented by Youness in [7, 8]. This concept was extended to a semi E-convexity by Chen in [4]. E-convexity notation considered only the convex combination of the images of points under E. The authors generalized this notion to the so called strong E-convexity [10]. This generalization took into account the convex combination of points on the segment $[Ex, x + Ex]$, and the points on the segment $[Ey, y + Ey]$. Strong E-convexity was extended to a quasi and pseudo strongly E-convexity in [11].

In this paper, we formulate a multi-objective programming problem which it involves strongly E-convex functions. An efficient solution for considered problem is characterized by weighting, and ε-constraint approaches. In the

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end of the paper, we obtain sufficient and necessary conditions for a feasible solution to be an efficient or properly efficient solution for this kind of problems.

Now let us summarize some definitions of strongly E-convex sets, strongly E-convex functions, generalized strongly E-convex functions; and some results about them.

Definition 1.1. [10] A set $M \subseteq R^n$ is said to be a strongly E-convex set with respect to an operator $E : R^n \to R^n$ if $\lambda(\alpha x + Ex) + (1 - \lambda)(\alpha y + Ey) \in M$ for each $x, y \in M,\ 0 \le \alpha \le 1,$ and $0 \le \lambda \le 1$.

Every strongly E-convex set with respect to an operator $E : R^n \to R^n$ is an E-convex set when $\alpha = 0$. If $M \subseteq R^n$ is strongly E-convex set, then $E(M) \subseteq M$. If $M_1\, and\, M_2$ are a strongly E-convex sets, then $M_1 \bigcap M_2$ is a strongly E-convex set but $M_1 \bigcup M_2$ is not necessarily strongly E-convex set. If $E : R^n \to R^n$ is a linear map, and M_1, $M_2 \subseteq R^n$ are strongly E-convex sets, then $M_1 + M_2$ is a strongly E-convex set.

Definition 1.2. [10] A real valued function $f : M \subseteq R^n \to R$ is said to be a strongly E-convex function on M, with respect to an operator $E : R^n \to R^n$, if M is a strongly E-convex set and, for each $x, y \in M,\ 0 \le \alpha \le 1,\ and\ 0 \le \lambda \le 1$,

$$f(\lambda(\alpha x + Ex) + (1 - \lambda)(\alpha y + Ey)) \le \lambda f(Ex) + (1 - \lambda)f(Ey).$$

If $f(\lambda(\alpha x+Ex)+(1-\lambda)(\alpha y+Ey)) \ge \lambda f(Ex)+(1-\lambda)f(Ey)$, then f is called a strongly E-concave function on M. If the inequality signs in the previous two inequalities are strict, then f is called sharp strongly E-convex and sharp strongly E-concave, respectively.

Every strongly E-convex function, with respect to an operator $E : R^n \to R^n$ is E-convex function when $\alpha = 0$. Let $E_0 : R \to R$, and $E_0(f(x) + t) = f(Ex) + t$, for each nonnegative real number t, then a numerical function f defined on strongly E-convex set $M \subseteq R^n$ is strongly E-convex if and only if its *epi(f)* is strongly $E \times E_0$-convex on $M \times R$. If $(f_i)_{i\in I}$ is a family of numerical functions, which are strongly E-convex and bounded from above, then the numerical function $f(x) = \underset{i\in I}{Sup}\, f_i(x)$ is a strongly E-convex on M. If $f : R^n \to R$ is a differentiable strongly E-convex function, then, for each $x, y \in M$, $(Ex - Ey)\nabla f(Ey) \le f(Ex) - f(Ey)$. For more details about strongly E-convex sets and strongly E-convex functions, see [10, 11].

Definition 1.3. [11] A real valued function $f : M \subseteq R^n \to R$ is said to be quasi strongly E-convex function on M, with respect to an operator $E : R^n \to R^n$, if M is strongly E-convex set and, for each $x, y \in M,\ 0 \le \alpha \le$

1, *and* $0 \leq \lambda \leq 1$,

$$f(\lambda(\alpha x + Ex) + (1-\lambda)(\alpha y + Ey)) \leq \max\{f(Ex), f(Ey)\}.$$

If $f(\lambda(\alpha x + Ex) + (1-\lambda)(\alpha y + Ey)) \geq \min\{f(Ex), f(Ey)\}$, then f is called a quasi strongly E-concave function on M. If the inequality signs in the previous two inequalities are strict, then f is called strictly quasi strongly E-convex and strictly quasi strongly E-concave respectively.

Every quasi strongly E-convex function, with respect to an operator $E : R^n \to R^n$ is a quasi E-convex function when $\alpha = 0$. If $f : R^n \to R$ is a strongly E-convex function on a strongly E-convex set $M \subseteq R^n$, then f is a quasi strongly E-convex function on M. If $f : R^n \to R$ is a differentiable quasi strongly E-convex function at $y \in M$. Then $(Ex - Ey)\nabla f(Ey) \leq 0$, for each $x \in M$ [11].

Definition 1.4. [11] A real valued function $f : M \subseteq R^n \to R$ is said to be a pseudo strongly E-convex function on M, with respect to an operator $E : R^n \to R^n$, if M is strongly E-convex set and, there exists a strictly positive function $b : R^n \times R^n \to R$ such that

$$f(Ex) < f(Ey) \Rightarrow f(\lambda(\alpha x+Ex)+(1-\lambda)(\alpha y+Ey)) \leq f(Ey)+\lambda(\lambda-1)b(x,y)$$

for all $x, y \in M$, $0 \leq \alpha \leq 1$, *and* $0 \leq \lambda \leq 1$.

Every strongly E-convex function $f : R^n \to R$ on a strongly E-convex set M is pseudo strongly E-convex function on M. If $f : R^n \to R$ be a differentiable pseudo strongly E-convex function at $y \in M$, then, for each $x \in M$

$$f(Ex) < f(Ey) \;\Rightarrow\; (Ex - Ey)\nabla f(Ey) < 0,$$

or

$$(Ex - Ey)\nabla f(Ey) \geq 0 \Rightarrow f(Ex) \geq f(Ey).$$

2. Preliminaries

Let $E : R^n \to R^n$ be a mapping, $f_j : R^n \to R$, $j = 1, 2, ..., k$, and $g_i : R^n \to R$, $i = 1, 2, ..., m$ be differentiable real valued strongly E-convex functions on R^n. A multi-objective strongly E-convex programming problem is formulated as follows:

$$(P) \qquad \begin{array}{l} \text{Min } f_j(x), \\ \text{subject to} \\ x \in M = \{x \in R^n : g_i(x) \leq 0\}, \end{array}$$

Definition 2.1. A feasible solution x^* for (P) is an efficient solution for (P) if and only if there is no other feasible *x,* for (P), such that for some $i \in \{1, 2, ..., k\}$,

$$f_i(x) < f_i(x^*), f_j(x) \leq f_j(x^*), \quad \textit{for all } j \neq i.$$

Definition 2.2. An efficient solution x^* for (P) is called a properly efficient solution for (P) if there exists a scalar $q > 0$ such that for each i, $i = 1, 2, ..., k$, and each $x \in M$ satisfying $f_i(x) < f_i(x^*)$, there exists at least one $j \neq i$ with $f_j(x) > f_j(x^*)$, and

$$[f_i(x) - f_i(x^*)]/[f_j(x^*) - f_j(x)] \leq q.$$

Lemma 2.1. [11] Let $E : R^n \to R^n$ be a map and let $M \subseteq E(R^n)$. If $M' = \{y \in R^n : (g \circ E)(y) \leq 0\}$, then $E(M') = M$.

Let us now formulate the problem:

$$(PE) \qquad \begin{array}{l} Min\ (f_j \circ E)(y), \\ subiect\ to \\ y \in M' = \{y \in R^n : (g_i \circ E)(y) \leq 0\}. \end{array}$$

Denote X and Y the sets of all efficient and properly efficient solutions for problem (P) respectively; and X' and Y' the sets of all efficient and properly efficient solutions for problem (PE) respectively.

Lemma 2.2. [9] If $x^* \in X$, then there is at least one element $y^* \in M'$, $x^* = E(y^*)$, and $y^* \in X'$.

Lemma 2.3. [9] $E(y^*) \in X$ for each $y^* \in X'$.

Remark 2.1. From the two Lemmas 2.2 and 2.3, we obtain *X=E (X')*.

Lemma 2.4. If $x^* \in Y$, then there is at least one element $y^* \in M'$, $x^* = E(y^*)$, and $y^* \in Y'$.

Proof. Let $x^* \in Y$. Then, from Lemma 2.2, there is at least one element $y^* \in X'$, such that $x^* = E(y^*)$. Let $y^* \notin Y'$. Then there is $y' \in M'$ such that for any i, $f_i(Ey') < f_i(Ey^*)$, there exists at least one $j \neq i$ with $f_j(Ey') > f_j(Ey^*)$, and $f_i(Ey') - f_i(Ey^*) > q[f_j(Ey^*) - f_j(Ey')]$, thus there is $x' \in M$, *x'=E(y')* such that $f_i(x') - f_i(x^*) > q[f_j(x^*) - f_j(x')]$ which contradicts $x^* \in Y$. Hence $y^* \in Y'$. □

Lemma 2.5. $E(y^*) \in Y$ for each $y^* \in Y'$.

Proof. $y^* \in Y'$ implies $y^* \in X'$ and hence from Lemma 2.3 $E(y^*) \in X$. Let $E(y^*) \notin Y$. Then there is $x' \in M$ such that for any i, $f_i(x') < f_i(Ey^*)$, there exists at least one $j \neq i$ with $f_j(x') > f_j(Ey^*)$, and $f_i(x') - f_i(Ey^*) > q[f_j(Ey^*) - f_j(x')]$. From Lemma 2.1, there is at least one element $y' \in M'$, such that *x'=E(y')*, and $f_i(Ey') - f_i(Ey^*) > q[f_j(Ey^*) - f_j(Ey')]$ which contradicts $y^* \in Y'$. Hence $E(y^*) \in Y$. □

Remark 2.2. Lemmas 2.4 and 2.5 imply $Y = E(Y')$.

Lemma 2.6. [11] Let $E : R^n \to R^n$ and $M = \{x \in R^n : g(x) \le 0\}$. If $g : R^n \to R^m$ is a quasi strongly E-convex function and $M \subseteq E(R^n)$, then M is a convex set.

Corollary 2.1.[11] Let $E : R^n \to R^n$. If $g : R^n \to R^m$ is a strongly E-convex function on R^n and $M \subseteq E(R^n)$, $M = \{x \in R^n : g(x) \le 0\}$. Then M is a convex set.

Lemma 2.7. If $f : R^n \to R^k$ is a strongly E-convex vector valued function on a strongly E-convex $M' \subseteq R^n$, and $M \subseteq E(R^n)$, then the set

$$A = \bigcup_{x \in M} A(x) = \{z : z \in R^k, z > f(x) - f(x^*)\},\ x \in M$$

is a convex set.

Proof. Let $x^1, x^2 \in M$. Then, from Lemma 2.1, $\exists\, y^1, y^2 \in M'$ such that $x^1 = Ey^1$, $x^2 = Ey^2$. Since f is a strongly E-convex function on M', for $z^1, z^2 \in A$, $0 \le \alpha \le 1$, and $0 \le \lambda \le 1$, we have

$$\lambda z^1 + (1-\lambda)z^2 > \lambda[f(x^1) - f(x^*)] + (1-\lambda)[f(x^2) - f(x^*)],$$

which can be rewritten

$$\begin{aligned} \lambda z^1 + (1-\lambda)z^2 &> \lambda f(Ey^1) + (1-\lambda)f(Ey^2) - f(x^*) \\ &\ge f(\lambda(\alpha y^1 + Ey^1) + (1-\lambda)(\alpha y^2 + Ey^2)) - f(x^*) \\ &\ge f(\lambda Ey^1 + (1-\lambda)Ey^2) - f(x^*) \\ &= f(\lambda x^1 + (1-\lambda)x^2) - f(x^*). \end{aligned}$$

Since M is convex set upon Corollary 2.1, $\lambda z^1 + (1-\lambda)z^2 \in A$. Hence A is a convex set. □

For a feasible point $x^* \in M'$, we denote $I(x^*)$ as the index set for binding constraints at x^*, i.e.

$$I(x^*) = \{i : (g_i \circ E)x^* = 0\}.$$

3. Characterizing Efficient Solutions by Weighting Approach

A weighting approach is one of the common approach for characterize efficient solutions of multi-objective programming problems [9]. In the following we shall characterize an efficient solution for a multi-objective strongly E-convex programming problem (P) in term of an optimal solution of the following scalar problem:

$$(\mathrm{P}_w) \qquad Min \sum_{j=1}^{k} w_j f_j(x), \quad subiect\ to \quad x \in M,$$

where $w_j \geq 0,\ j = 1, 2, ..., k,\ \sum_{j=1}^{k} w_j = 1,\ f_j,\ j = 1, 2, ..., k$ are strongly E-convex on M' and M' is a strongly E-convex set.

Theorem 3.1. If $\bar{x} \in M$ is an efficient solution for the problem (P), then there exist $\bar{w}_j \geq 0,\ j = 1, 2, ..., k,\ \sum_{j=1}^{k} \bar{w}_j = 1$ such that $\bar{x}$ is an optimal solution for problem (P_w).

Proof. Let $\bar{x} \in M$ be an efficient solution for problem (P). Then the system $f_j(x) - f_j(\bar{x}) < 0,\ j = 1, 2, ..., k$ has no solution $x \in M$. Upon Lemma 2.7 thus applying the generalized Gordan theorem [5], there exist $p_j \geq 0,\ \ j = 1, 2, ..., k$ such that $p_j[f_j(x) - f_j(\bar{x})] \geq 0,\ j = 1, 2, ..., k$, and $\frac{p_j}{\sum_{j=1}^{k} p_j} f_j(x) \geq \frac{p_j}{\sum_{j=1}^{k} p_j} f_j(\bar{x})$.

Denote $w_j = \frac{p_j}{\sum_{j=1}^{k} p_j}$, then $w_j \geq 0,\ j = 1, 2, ..., k,\ \sum_{j=1}^{k} w_j = 1$, and $\sum_{j=1}^{k} w_j f_j(\bar{x}) \leq \sum_{j=1}^{k} w_j f_j(x)$. Hence $\bar{x}$ is an optimal solution for problem (P_w). □

Theorem 3.2. [3] If $\bar{x} \in M$ is an optimal solution for ($\mathrm{P}_{\bar{w}}$) corresponding to $\bar{w}_j$, then $\bar{x}$ is an efficient solution for problem (P) if either one of the following two conditions holds:
(i) $\bar{w}_j > 0$, for all $j = 1, 2, ..., k$; or (ii) $\bar{x}$ is the unique solution of ($\mathrm{P}_{\bar{w}}$).

4. Characterizing Efficient Solutions by ε-Constraint Approach

In this section, our basic aim is to characterized an efficient solution for problem (P) by ε-constraint approach. To characterize an efficient solution for problem (P) by ε-constraint approach let us scalarize problem (P) to become in the form:

$$\mathrm{P}_q(\varepsilon) \qquad \begin{array}{l} Min\ (f_q \circ E)(x), \\ subiect\ to\ \ x \in M', \\ (f_j \circ E)(x) \leq \varepsilon_j, \quad j = 1, 2, ..., k,\ j \neq q,\ \varepsilon_j \in R. \end{array}$$

where $f_j,\ j = 1, 2, ..., k$ are strongly E-convex on M' and M' is strongly E-convex set .

Theorem 4.1. If $\bar{x} \in M$ is an efficient solution for problem (P), then there are $\bar{y} \in M', \varepsilon \in R^k$ such that $\bar{x} = E(\bar{y})$ and $\bar{y}$ is an optimal solution of problem $\mathrm{P}_q(\varepsilon)$ for every q.

Proof. Since $\bar{x} \in M$ is an efficient solution for problem (P), from Remark 2.1, there is $\bar{y} \in M'$ which is an efficient solution for problem (PE), i.e. $\bar{y} \in X'$ and $\bar{x} = E(\bar{y})$.

Let $\bar{y}$ be a non optimal solution for $\mathrm{P}_q(\varepsilon)$ for some q. Then there is $\tilde{y} \in M'$ such that $(f_q \circ E)(\tilde{y}) < (f_q \circ E)(\bar{y})$ and $(f_j \circ E)(\tilde{y}) \leq \bar{\varepsilon}_j,\ j = 1, 2, ..., k,\ j \neq q$.

By putting $\bar{\varepsilon}_j = (f_j \circ E)(\bar{y}), j = 1, 2, ..., k, \ j \neq q$, then we get

$$(f_j \circ E)(\tilde{y}) \ \leq (f_j \circ E)(\bar{y})$$

with strict inequality for at least one j. Thus, $\bar{y} \notin X'$ and $E(\bar{y}) = \bar{x} \notin X$ which is a contradiction. Hence $\bar{y}$ is an optimal solution for problem $P_q(\varepsilon)$. □

Theorem 4.2. Let $\bar{y} \in M'$ be an optimal solution of problem $P_q(\varepsilon)$, for each q. Then $E(\bar{y})$ is an efficient for problem (P).

Proof. Let $h : R^n \to [\frac{-1}{2}, \infty)$ be a map defined by
$h(y) = \inf_{(\tilde{y},\beta)} \{\frac{1}{2}\beta^2 + \beta : (f_j \circ E)(\tilde{y}) - (f_j \circ E)(y) \leq \ \beta e, \ j = 1, 2, ..., k, \ \ \tilde{y} \in M'\}$
where $e = (1, 1, ..., 1) \in R^k$. From [2] we obtain that
(i)$X' = \{y \in M' : h(y) = 0\}$,
(ii)$h(y) \ \leq \ 0, \ \ for \ each \ \ y \in M'$.
are equivalent, where X' is a set of efficient solution for problem (P_E).

Let $E(\bar{y}) \notin X$. Then, from Remark 2.1, $\bar{y} \notin X'$, that is $h(\bar{y}) < 0$, and hence, for some $\tilde{y} \in M'$, we have

$$(f_j \circ E)(\tilde{y}) - (f_j \circ E)(\bar{y}) < 0, \ j = 1, 2, ..., k.$$

Therefore $\bar{y}$ is not optimal solution for problem $P_q(\varepsilon)$, which is a contradiction, then $E(\bar{y})$ is an efficient for problem (P). □

5. Optimality Criteria

In this section, we will drive the sufficient and necessary conditions for a feasible solution $E(x^*)$, $x^* \in M^{'}$, to be efficient or properly efficient solution for problem (P) as in the following theorems.

Theorem 5.1. Suppose that $M \subseteq E(R^n)$, there exists a feasible solution x^* for (PE), and scalars $\ \lambda_i > 0, \ i = 1, 2, ..., k, \ u_i \geq 0, \ i \in I(x^*)$ such that

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(Ex^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(Ex^*) = 0. \tag{5.1}$$

If $f_i, \ i = 1, 2, ..., k, \ and \ g_i \ , i \in I(x^*)$ are strongly E-convex functions at $x^* \in M^{'}$ with respect to the same E. Then $x^*$is a properly efficient solution for problem (PE) and $E(x^*)$ is a properly efficient solution for problem (P).

Proof. For all $x \in M^{'}$, we have from [11]

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i f_i(Ex) - \sum_{i=1}^{k} \lambda_i f_i(Ex^*) &\geq (Ex - Ex^*) \sum_{i=1}^{k} \lambda_i [\nabla f_i(Ex^*)]^t \\ &= -(Ex - Ex^*) \sum_{i \in I(x^*)} u_i [\nabla g_i(Ex^*)]^t\end{aligned}$$

by (5.1), it can be rewritten as

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i f_i(Ex)-\sum_{i=1}^{k}\lambda_i f_i(Ex^*) &\geq \sum_{i=1}^{k}u_i g_i(Ex^*)-\sum_{i=1}^{k}u_i g_i(Ex)\\ &= -\sum_{i\in I(x^*)}u_i g_i(Ex)\geq 0,\end{aligned}$$

thus $\sum_{i=1}^{k}\lambda_i f_i(Ex)\geq\sum_{i=1}^{k}\lambda_i f_i(Ex^*)$, for all $x\in M'$, which implies that x^* is the minimizer of $\sum_{i=1}^{k}\lambda_i(f_i\circ E)(x)$ under the constraint $(g\circ E)(x)\leq 0$. Hence, x^* is a properly efficient solution for problem (PE) due to Theorem 4.1 of [3], and $E(x^*)$ is a properly efficient solution for problem (P) due to Lemma 2.5. □

Theorem 5.2. Let $M\subseteq E(R^n)$ and x^* be a feasible solution for (PE). If there exist scalars $\lambda_i\geq 0, i=1,2,...,k, \sum_{i=1}^{k}\lambda_i=1, u_i\geq 0, i\in I(x^*)$, such that the triplet (x^*,λ_i,u_i) satisfies (5.1) in Theorem 5.1, $\sum_{i=1}^{k}\lambda_i f_i$ is sharp strongly E-convex, and g_I is strongly E-convex at x^* with respect to E. Then x^* is an efficient solution for problem (PE), and $E(x^*)$ is an efficient solution for problem (P).

Proof. Suppose that x^* is not an efficient solution for (PE). There exists a feasible x for (PE) and index r such that

$$\begin{aligned}&(f_r\circ E)(x)<(f_r\circ E)(x^*),\\ &(f_i\circ E)(x)\leq(f_i\circ E)(x^*),\ \ for\ all\ i\neq r.\end{aligned}$$

These two inequalities lead to

$$0\geq\sum_{i=1}^{k}\lambda_i f_i(Ex)-\sum_{i=1}^{k}\lambda_i f_i(Ex^*)\quad or\quad 0>(Ex-Ex^*)\sum_{i=1}^{k}\lambda_i[\nabla f_i(Ex^*)]^t, \tag{5.2}$$

since $\sum_{i=1}^{k}\lambda_i f_i$ is sharp strongly E-convex at x^*, and

$$(Ex-Ex^*)\nabla g_i(Ex^*)\leq 0,$$

since $g_i,\ i\in I(x^*)$, is strongly E-convex at x^* and $(g\circ E)(x)\leq 0$. The above inequality yields

$$(Ex-Ex^*)\sum_{i\in I(x^*)}u_i[\nabla g_i(Ex^*)]^t\leq 0. \tag{5.3}$$

Adding (5.2) and (5.3), we obtain a contradiction to (5.1). Hence, x^* is an efficient solution for problem (PE), and $E(x^*)$ is an efficient solution for problem (P) due to Lemma 2.3. □

Remark 5.1. Proceeding along the same lines as in Theorem 5.1, it can be easily seen that x^* and $E(x^*)$ become properly efficient solutions for (PE), and (P), respectively in the above theorem, if $\lambda_i > 0, \ \ for\ all \ \ i = 1, 2, ..., k.$

Theorem 5.3. Suppose that $M \subseteq E(R^n)$, there exists a feasible solution x^* for (PE), and scalars $\lambda_i > 0, i = 1, 2, ..., k, u_i \geq 0, i \in I(x^*)$, such that (5.1) of Theorem 5.1 holds. If $\sum_{i=1}^{k} \lambda_i f_i$ is pseudo-strongly E-convex, and g_I are quasi-strongly E-convex at $x^* \in M'$ with respect to E. Then x^* is a properly efficient solution for problem (PE), and $E(x^*)$ is a properly efficient solution for problem (P).

Proof. Since $g_I(Ex) \leq g_I(Ex^*) = 0, \ u_i \geq 0$, and g_I are quasi-strongly E-convex at x^*, we have

$$(Ex - Ex^*) \sum_{i \in I(x^*)} u_i [\nabla g_i(Ex^*)]^t \leq 0, \ forall\ x \in M'.$$

By using (5.1), we have

$$(Ex - Ex^*) \sum_{i=1}^{k} \lambda_i [\nabla f_i(Ex^*)]^t \geq 0,$$

which implies that

$$\sum_{i=1}^{k} \lambda_i f_i(Ex) \geq \sum_{i=1}^{k} \lambda_i f_i(Ex^*),$$

since $\sum_{i=1}^{k} \lambda_i f_i$ is pseudo-strongly E-convex at x^*. Hence, x^* is the minimizer of $\sum_{i=1}^{k} \lambda_i (f_i \circ E)(x)$ under the constraint $(g \circ E)(x) \leq 0$. Therefore, x^* is a properly efficient solution for problem (PE), and $E(x^*)$ is a properly efficient solution for problem (P). □

Theorem 5.4. Suppose that $M \subseteq E(R^n)$, there exist a feasible solution x^* for (PE) and scalars $\lambda_i \geq 0, i = 1, 2, ..., k, \sum_{i=1}^{k} \lambda_i = 1, \ u_i \geq 0, i \in I(x^*)$, such that (5.1) of Theorem 5.1 holds. Let $\sum_{i=1}^{k} \lambda_i f_i$ be strictly pseudo-strongly E-convex and g_I be quasi-strongly E-convex at x^* with respect to E. Then x^* is an efficient solution for problem (PE), and $E(x^*)$ is an efficient solution for problem (P).

Proof. Suppose that x^* is not an efficient solution for (PE). Then, there exists a feasible x for (PE), and index r such that

$$(f_r \circ E)(x) < (f_r \circ E)(x^*),$$
$$(f_i \circ E)(x) \leq (f_i \circ E)(x^*), \ \ for\ all\ i \neq r.$$

Therefore,

$$\sum_{i=1}^{k}\lambda_i f_i(Ex) \le \sum_{i=1}^{k}\lambda_i f_i(Ex^*) \ \ or \quad (Ex-Ex^*)\sum_{i=1}^{k}\lambda_i[\nabla f_i(Ex^*)]^t < 0,$$

since $\sum_{i=1}^{k}\lambda_i f_i$ is strictly pseudo-strongly E-convex at x^*. Since g_I is quasi strongly E-convex at x^*, $g_I(Ex) \le g_I(Ex^*) = 0$ which implies

$$(Ex - Ex^*)\nabla g_I(Ex^*) \le 0.$$

The proof now is similar to the proof of Theorem 5.2. □

Remark 5.2. Proceeding along similar lines as in Theorem 5.3, it can be easily seen that x^* and $E(x^*)$ become properly efficient solutions for (PE), and (P), respectively in the above Theorem, if $\lambda_i > 0$, $for\ all$ $i = 1, 2, ..., k$.

Theorem 5.5. Suppose that $M \subseteq E(R^n)$, there exists a feasible solution x^* for (PE), and scalars $\lambda_i > 0, i = 1, 2, ..., k, u_i \ge 0, i \in I(x^*)$, such that (5.1) in Theorem 5.1 holds. Let $\sum_{i=1}^{k}\lambda_i f_i$ be pseudo-strongly E-convex, and $u_I\, g_I$ be quasi-strongly E-convex at x^* with respect to E. Then $x^*$is a properly efficient solution for problem (PE), and $E(x^*)$ is a properly efficient solution for problem (P).

Proof. The proof is similar to the proof of Theorem 5.3. □

Theorem 5.6. Suppose that $M \subseteq E(R^n)$, there exists a feasible solution x^* for (PE), and scalars $\lambda_i \ge 0, i = 1, 2, ..., k, \sum_{i=1}^{k}\lambda_i = 1, u_i \ge 0, i \in I(x^*)$, such that (5.1) of Theorem 5.1 holds. If $I(x^*) \ne \phi, \sum_{i=1}^{k}\lambda_i f_i$ is quasi-strongly E-convex and $u_I\, g_I$ is strictly pseudo-strongly E-convex at x^* with respect to E. Then x^* is an efficient solution for problem (PE), and $E(x^*)$ is an efficient solution for problem (P).

Proof. The proof is similar to the proof of Theorem 5.4. □

Remark 5.3. Proceeding along similar lines as in Theorem 5.3, it can be easily seen that x^* and $E(x^*)$ become properly efficient solutions for (PE), and (P), respectively in the above Theorem, if $\lambda_i > 0$, $for\ all$ $i = 1, 2, ..., k$.

Theorem 5.7. (Necessary Optimality Criteria) Assume that $y^* = Ex^*$ is a properly efficient solution for problem (P). Assume also that there exists a feasible point $y' = Ex'$ for such that $g_i(y') < 0$, $i = 1, 2, ..., m$, and each g_i, $i \in I(x^*)$ is strongly E-convex at x^* with respect to the same map $E : R^n \to R^n$. Then, there exists scalars $\lambda_i > 0, i = 1, 2, ..., k$ and $u_i \ge 0, i \in I(x^*)$, such that the triplet (x^*, λ_i, u_i) satisfies

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(Ex^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(Ex^*) = 0. \tag{5.4}$$

Proof. we show that the system

$$\begin{array}{ll} (Ex - Ex^*)^t \nabla f_q(Ex^*) < 0, & \\ (Ex - Ex^*)^t \nabla f_i(Ex^*) \leq 0, & for\ all\ i \neq q\ . \\ (Ex - Ex^*)^t \nabla g_i(Ex^*) \leq 0, & i \in I(x^*), \end{array} \tag{5.5}$$

has no solution for every $q = 1, 2, ..., k$. Since by the assumed Slater-type condition,

$$g_i(y^{'}) - g_i(Ex^*) = g_i(Ex^{'}) - g_i(Ex^*) < 0,\ i \in I(x^*),$$

and then from strong E-convexity of $g_i\ at\ x^*$, we get

$$(Ex^{'} - Ex^*)^t \nabla g_i(Ex^*) < 0,\ i \in I(x^*). \tag{5.6}$$

Therefore from (5.5) and (5.6)

$$[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)]^t \nabla g_i(Ex^*) < 0,\ i \in I(x^*),$$

for all $\rho > 0$. Hence for some positive λ small enough

$$g_i(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)]) < g_i(Ex^*) = 0,\ i \in I(x^*).$$

Similarly, for $i \notin I(x^*)$, $g_i(Ex^*) < 0$ and for $\lambda > 0$ small enough

$$g_i(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)]) \leq 0,\ i \notin I(x^*).$$

Thus, for λ sufficiently small and all $\rho > 0$, $Ex^* + \lambda[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)]$ is feasible for problem (P). For sufficiently small $\rho > 0$, (5.5) gives

$$f_q(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)]) < f_q(Ex^*), \tag{5.7}$$

now for all $j \neq q$ such that

$$f_j(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)]) > f_j(Ex^*). \tag{5.8}$$

Consider the ratio

$$\frac{N(\lambda, \rho)}{D(\lambda, \rho)} = \frac{[f_q(Ex^*) - f_q(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)])]/\lambda}{[f_j(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ex^{'} - Ex^*)]) - f_j(Ex^*)]/\lambda}. \tag{5.9}$$

From (5.5), $N(\lambda, \rho) \to -(Ex - Ex^*)^t \nabla f_q(Ex^*) > 0$. Similarly, $D(\lambda, \rho) \to (Ex - Ex^*)^t \nabla f_j(Ex^*) \leq 0$; but, by (5.8) $D(\lambda, \rho) > 0$, so $D(\lambda, \rho) \to 0$. Thus, the ratio in (5.9) becomes unbounded, contradicting the proper efficiency of

$y* = Ex^*$ for (P). Hence, for each $q = 1, 2, ..., k$, the system (5.5) has no solution. The result then follows from an application of the Farkas Lemma as in [1], namely

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(Ex^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(Ex^*) = 0, \quad u \geq 0.$$

□

Remark 5.4. The above Theorem continues to hold when the functions $g_i,\ i \in I(x^*)$, are strictly pseudo-strongly E-convex at x^* with respect to the map $E : R^n \to R^n$.

Theorem 5.8. Assume that $y^* = Ex^*$ is an efficient solution for problem (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then, there exist scalars $\lambda_i \geq 0, i = 1, 2, ..., k, \sum_{i=1}^{k} \lambda_i = 1, u_j \geq 0, j = 1, 2, ..., m$, such that

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(Ex^*) + \sum_{j=1}^{m} u_j \nabla g_j(Ex^*) = 0,$$

$$\sum_{j=1}^{m} u_j g_j(Ex^*) = 0.$$

Proof. Since every efficient solution is a weak minimum, by applying Theorem 2.2 of Weir and Mond [6] for $y^* = Ex^*$, we get $\exists \lambda \in R^k,\ y \in R^m$ such that

$$\lambda^t \nabla f(Ex^*) + u^t \nabla g(Ex^*) = 0,$$

$$u^t g(Ex^*) = 0,$$

$$u \geq 0, \quad \lambda \geq 0, \quad \lambda^t e = 0,$$

where $e = (1, 1, ..., 1) \in R^k$. □

Example 1. Let $E_{\bar{\alpha}} : R^2 \to R^2$ be defined as $E_{\bar{\alpha}}(x, y) = ((1 - \bar{\alpha})\frac{x}{2}, (1 - \bar{\alpha})y)$ for fixed $\bar{\alpha} \in [0, 1)$, and let $M'_{\bar{\alpha}}$ be given by

$$M'_{\bar{\alpha}} = \{(x, y) \in R^2 :\ x - 2y \leq 0,\ \ (1 - \bar{\alpha})(x + y) - 3 \leq 0,\ \ y \geq 0,\ \ x \geq 0\}.$$

Consider the following bi-objective programming

$$min f_1(x, y) = (1 - x)^3$$

$$min f_2(x, y) = (6 + x - 2y)^3$$

$$s.t.\ \ (x, y) \in M = \{(x, y) \in R^2 :\ x - y \leq 0,\ 2x + y - 3 \leq 0,\ y \geq 0,\ x \geq 0\}.$$

Where f_1, and f_2 are strongly $E_{\bar{\alpha}}$-convex functions on strongly $E_{\bar{\alpha}}$-convex set $M'_{\bar{\alpha}}$ with respect to $E_{\bar{\alpha}}$, for fixed $\bar{\alpha}$.

We shall, now formulate the problem $(P_{E_{\bar{\alpha}}})$ as follow:

$$\begin{aligned} &min(f_1 \circ E_{\bar{\alpha}})(x,y) \\ &min(f_2 \circ E_{\bar{\alpha}})(x,y) \\ &s.t. \quad (x,y) \in M'_{\bar{\alpha}}. \end{aligned}$$

Now, we characterize the efficient solution for considered example by weighting approach, in part (i) and by ε-constraint approach in part (ii). Finally, in part (iii), we obtain sufficient and necessary conditions for efficient solution for this example:

i) Formulate the weighting problem (P_w) as

$$\begin{aligned} &Min\ \{w_1(1-x)^3 + w_2(6+x-2y)^3\}, \\ &s.t. \quad x \in M \end{aligned}$$

where $w_1, w_2 \geq 0$, $w_1 + w_2 = 1$.

It is clear that a point $(1,1)$ is optimal solution for (P_w) corresponding $w = (w_1, 0)$, $0 < w_1 \leq 1$, and a point $(0,3)$ is optimal solution for (P_w) corresponding to $w = (0, w_2)$, $0 < w_2 \leq 1$. Hence the set of efficient solutions of problem (P) can be described as

$$X = \{(x,y) \in M :\ 2x + y = 3\}.$$

ii) Formulate the problem $\mathrm{P}_q(\varepsilon)$ as

$$\begin{aligned} &Min\ ((1-\bar{\alpha})\tfrac{x}{2})^3 \\ &s.t.\ (x,y) \in M', \\ &(6 + (1-\bar{\alpha})\tfrac{x}{2} - 2(1-\bar{\alpha})y)^3 \leq \varepsilon_1, \end{aligned}$$

and

$$\begin{aligned} &Min\ (6 + (1-\bar{\alpha})\tfrac{x}{2} - 2(1-\bar{\alpha})y)^3 \\ &s.t.\ (x,y) \in M', \\ &((1-\bar{\alpha})\tfrac{x}{2})^3 \leq \varepsilon_2. \end{aligned}$$

It is easy to see that the points $\{(x^*, y^*) \in M'_{\bar{\alpha}} :\ (1-\bar{\alpha})(x^* + y^*) = 3\}$ are optimal solutions corresponding to

$$(\varepsilon_1, \varepsilon_2) = \{(6 + (1-\bar{\alpha})\frac{x^*}{2} - 2(1-\bar{\alpha})y^*)^3, ((1-\bar{\alpha})\frac{x^*}{2})^3\};$$

the images of these optimal solutions by mapping $E_{\bar{\alpha}}$ are efficient solutions for problem (P) which can be described as

$$X = \{(x,y) \in M :\ 2x + y = 3\}.$$

iii) Applying the Kuhn-Tucker conditions (1) to $(P_{E_{\bar{\alpha}}})$ yields
$\frac{-3\lambda_1}{2}(1-\bar{\alpha})(1-(1-\bar{\alpha})\frac{x^*}{2})^2 + \frac{3\lambda_2}{2}(1-\bar{\alpha})(6+(1-\bar{\alpha})\frac{x^*}{2} - 2(1-\bar{\alpha})y^*)^2$

$$+u_1 + u_2\,(1-\bar{\alpha}) - u_4 = 0,$$

$$-6\lambda_2(1-\bar{\alpha})(6+(1-\bar{\alpha})\frac{x^*}{2} - 2(1-\bar{\alpha})y^*)^2 - 2u_1 + u_2\,(1-\bar{\alpha}) - u_3 = 0,$$
$$-6\lambda_2(1-\bar{\alpha})(6+(1-\bar{\alpha})\frac{x^*}{2} - 2(1-\bar{\alpha})y^*)^2 - 2u_1 + u_2\,(1-\bar{\alpha}) - u_3 = 0,$$
$$u_2(\,(1-\bar{\alpha})(x^*+y^*) - 3) = 0,$$
$$u_1(x^* - 2y^*) = 0,$$
$$-u_3\,y^* = 0,$$
$$-u_4 x^* = 0,$$

and

$$x^* - 2y^* \le 0,\ (1-\bar{\alpha})(x^*+y^*) - 3 \le 0,\ \ y^* \ge 0,\ \ x^* \ge 0,$$

where $\lambda_i \ge 0,\ i = 1,2,\ \lambda_1 + \lambda_2 = 1$, and $u_i \ge 0,\ i = 1,2,3,4$.

From the above system we conclude that the full set of efficient solutions of problem $(P_{E_{\bar{\alpha}}})$ can be described as

$$X'_{\bar{\alpha}} = \{(x^*, y^*) \in M'_{\bar{\alpha}} :\ (1-\bar{\alpha})(x^*+y^*) = 3\}.$$

Hence the images of these efficient solutions, by mapping $E_{\bar{\alpha}}$ are efficient solutions for problem (P) which can be described as $X = \{(x,y) \in M : 2x + y = 3\}$.

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