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FIXED POINT THEOREMS FOR CONTRACTIVE SET-VALUED MAPPINGS

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Abstract. In this paper we prove several fixed point theorems for set-valued mappings on complete partial ordered metric spaces which generalize some results in [2] and extend some results of [6] in T_0-quasipseudometric spaces. We also generalize some results of [11] for set-valued mappings on topological space. Some example are presented to illustrate the results.

1. Introduction

It is well known that fixed point theory plays an important role in various fields of applied mathematical analysis and scientific applications. In 2013, Beg and Butt [2] extended the fixed point theorems for set valued mappings in partially ordered metric space. In 2014, Tirado et al. [6] generalize some fixed point theorems in terms of Q-function on complete quasimetric space. Recently, Shaha et al. [11] give the existence of fixed points of mappings on general topological spaces. In this paper, we also extend their results. For convenience, we recall some definitions below.

2. Preliminaries

Definition 2.1. Let (X, d) be a complete metric space, and $CB(X)$ the class of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, let

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$$D(A,B)=\max\left\{\sup_{b\in B}d(A,b),\sup_{a\in A}d(a,B)\right\},$$

where, D is said to be a Hausdorff metric induced by d.

Definition 2.2. A partial order is a binary relation $\leq$ over a set X which satisfies the following conditions:

(1) $x\leq x$;
(2) if $x\leq y$ and $y\leq x$, then $x=y$;
(3) if $x\leq y$ and $y\leq z$, then $x\leq z$.

Definition 2.3. Let X be a nonempty set and $d:X\times X\to[0,\infty)$ a function such that:

(a) $d(x,y)=d(y,x)=0\Leftrightarrow x=y$;
(b) $d(x,z)\leq d(x,y)+d(y,z),\quad\forall\, x,y,z\in X$.

Then d is called a T_0-quasipseudometric on a set X. The pair (X,d) is said to be a T_0-quasipseudometric space. If one replaces the condition (a) with the stronger condition

(a*) $d(x,y)=0\Leftrightarrow x=y$.

Then d is called a quasimetric on X. In this case the pair (X,d) is said to be a T_0-quasimetric space. In the sequel we will use the abbreviation T_0-qpm (respectively, T_0-qpm space) instead of T_0-quasipsrudometric (respectively, T_0-quasipsrudometric space).

Definition 2.4. A sequence $\{x_n\}_{n=1}^{\infty}$ in a T_0-qpm space (X,d) is τ_d-convergent to $x\in X$, if only if $\lim\limits_{n\to\infty}d(x,x_n)=0$. Analogously, a sequence $\{x_n\}_{n=1}^{\infty}$ in a T_0-qpm space (X,d) is $\tau_{d^{-1}}$-convergent to $x\in X$, if only if $\lim\limits_{n\to\infty}d(x_n,x)=0$.

Definition 2.5. A T_0-qpm space (X,d) is said to be complete if every Cauchy sequence is $\tau_{d^{-1}}$-convergent in the metric space.

Definition 2.6. Let (X,d) be a T_0-qpm space and $q:X\times X\to[0,\infty)$ be a function which satisfies:

(Q1) $q(x,z)\leq q(x,y)+q(y,z),\ \forall\, x,y,z\in X$;
(Q2) if $x\in X, M>0$, $\{y_n\}_{n=1}^{\infty}$ is $\tau_{d^{-1}}$-converges to $y\in X$ and satisfies $q(x,y_n)\leq M$, for all $n\in N*$, then $q(x,y)\leq M$;
(Q3) for $\forall\,\varepsilon>0,\exists\,\delta>0$, such that $q(x,y)\leq\delta$ and $q(y,z)\leq\delta$, imply $d(y,z)\leq\varepsilon$;

then q is call a Q-function on (X,d).

Definition 2.7. Let q be a Q-function on (X,d). $T : X \to 2^X$ is said to be q-l.s.c operator. If the function $x \mapsto q(x,Tx)$ is $\tau_{d^{-1}}$-lower semicontinuous on (X,d), where $q(x,Tx) = \inf\{q(x,y) : y \in Tx\}$.

Definition 2.8. Let X be a topological space.

1. A function $f : X \to R$ is said to be lower-continuous from above (lsca) at a point $x_0 \in X$, if for any net $\{x_\lambda\}_{\lambda\in\Lambda}$ converging to x_0 such that $f(x_{\lambda_1}) \le f(x_{\lambda_2})$ for any $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_2 \le \lambda_1$, imply $f(x_0) \le \lim\limits_{\lambda\in\Lambda} f(x_\lambda)$.
2. A function $f : X \times X \to R$ is said to be lower-continuous from above (lsca) at a point $(x_0,y_0) \in X \times X$, if for any net $\{(x_\lambda,y_\lambda)\}_{\lambda\in\Lambda}$ converging to (x_0,y_0) such that $f(x_{\lambda_1},y_{\lambda_1}) \le f(x_{\lambda_2},y_{\lambda_2})$ for any $\lambda_1,\lambda_2 \in \Lambda$ with $\lambda_2 \le \lambda_1$, imply $f(x_0,y_0) \le \lim\limits_{\lambda\in\Lambda} f(x_\lambda,y_\lambda)$.
3. A function $f : X \times X \to R$ is said to be lsca on $X \times X$, if it is lsca at all points $(x,y) \in X \times X$.

Definition 2.9. Ψ=$\{F : X \times X \to R \,|\, F$ is lsca and $F(x,y) = 0$ if $x = y\}$.

Lemma 2.10. ([2]) *Let $\{A_n\} \subset CB(X)$ and $\lim\limits_{n\to\infty} D(A_n,A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim\limits_{n\to\infty} d(x_n,x) = 0$, then $x \in A$.*

Lemma 2.11. ([7]) *Let q be a Q-function on a T_0-qpm space (X,d), let $\varepsilon > 0$, and $\delta = \delta(x) > 0$ for which condition* (Q3) *holds. If $q(x,y) \le \delta$ and $q(x,z) \le \delta$, then $d^s(y,z) \le \varepsilon$.*

Lemma 2.12. ([3]) *Let X be a compact topological space and $f : X \times X \to R$ be a lsca function. Then there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) : x \in X\}$.*

3. Main Results

Theorem 3.1. *Let $(X,\le)$ be a partially order set and d be a metric on X such that (X,d) is a complete metric space. Assume that X satisfies: if a non-decreasing sequence $x_n \to x \in X$, then $x_n \le x$, for $\forall\, n \in N^*$. Let $F : X \to CB(X)$ satisfying the following three conditions:*

(1) *$D(F(x),F(y)) < \varphi[d(x,y)]$, $\forall\, x \le y$ ($\varphi : R^+ \to R^+$ be a non-decreasing, satisfies $\varphi^n(t) \to 0$, $\lim\limits_{t\to 0^+} \varphi(t) = 0$, $\forall\, t > 0$.);*

(2) *if $d(x,y) < \varepsilon < 1$, for some $y \in F(x)$, then $x \le y$;*

(3) *there exists* $x_0 \in X,\ x_1 \in F(x_0)$ *with* $x_0 \leq x_1$ *such that* $d(x_0, x_1) < 1$.
Then F *has a fixed point.*

Proof. For assumption (3), there exists $x_0 \in X, x_1 \in F(x_0)$ with $x_0 \leq x_1$ such that

$$d(x_0, x_1) < 1. \tag{3.1}$$

Then, from assumption (1) and inequality (3.1), we deduce,

$$d(x_1, F(x_1)) \leq D(F(x_0), F(x_1)) < \varphi[d(x_0, x_1)] < 1.$$

So there exists $x_2 \in F(x_1)$ such that $x_1 \leq x_2$, and

$$d(x_1, x_2) \leq \varphi[d(x_0, x_1)].$$

Since

$$d(x_2, F(x_2)) \leq D(F(x_1), F(x_2)) < \varphi[d(x_1, x_2)] \leq \varphi^2[d(x_0, x_1)],$$

there exists $x_n \in F(x_{n-1})$ such that $x_{n-1} \leq x_n$ and

$$d(x_{n-1}, x_n) \leq \varphi[d(x_{n-2}, x_{n-1})].$$

Again, since

$$d(x_n, F(x_n)) \leq \varphi^n[d(x_0, x_1)],$$

there exists $x_{n+1} \in F(x_n)$ such that $x_n \leq x_{n+1}$, and

$$d(x_n, x_{n+1}) \leq \varphi^n[d(x_0, x_1)] \to 0, \text{ as } n \to \infty. \tag{3.2}$$

Next we will show that $\{x_n\}$ is a Cauchy sequence in X. From inequality (3.2), for all $\varepsilon > 0$, there exists N such that

$$d(x_n, x_{n+1}) < \varepsilon - \varphi(\varepsilon), \quad \forall\, n \geq N.$$

By using assumption (1), we have,

$$d(x_{n+1}, x_{n+2}) \leq \varphi[d(x_n, x_{n+1})].$$

Therefore,

$$\begin{aligned} d(x_n, x_{n+2}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\leq \varepsilon - \varphi(\varepsilon) + \varphi[d(x_n, x_{n+1})] \\ &\leq \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon. \end{aligned}$$

Using the mathematical induction, we have,

$$d(x_n, x_{n+k}) \leq \varepsilon, \quad \forall\, k \in N^*,\ n \geq N.$$

Therefore $\{x_n\}$ is a Cauchy sequence in X and hence converges to some point (say) x in the complete metric space X.

Now we will show that x is a fixed point of the mapping F. For $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, we deduce

$$x_n \leq x, \quad \forall\, n \in N^*.$$

From assumption (1), it follows that

$$D(F(x_n), F(x)) < \varphi[d(x_n, x)] \to 0 \quad \text{as} \quad n \to \infty.$$

Now since $x_{n+1} \in F(x_n)$, it follows by using Lemma 2.11 that $x \in F(x)$, i.e., x is a fixed point of the mapping F. □

Theorem 3.2. *Let (X, d) be a complete metric space and satisfying:*

$$D(Fx, Fy) < \varphi[d(x, y)], \quad \forall\, x, y \in X, \tag{3.3}$$

where $\varphi : R^+ \to R^+$ is any monotone non-decreasing (not necessarily continuous) function such that

$$\lim_{n\to\infty} \varphi^n(t) = 0 \quad \textit{and} \quad \lim_{t\to 0^+} \varphi(t) = 0, \quad \forall\, t > 0.$$

Then F has a fixed point in X.

Proof. For all $x \in X$, there exists $x_1 \in Fx$ such that $Fx_1 \subseteq F^2x$. Since

$$d(Fx_1, x_1) \le D(Fx_1, Fx) < \varphi[d(x_1, x)],$$

there exists $x_2 \in Fx_1 \subseteq F^2x = F(Fx)$ such that

$$d(x_2, x_1) \le \varphi[d(x_1, x)].$$

Then, we have $Fx_2 \subseteq F^3x$. Similarly, we have, $x_{n+1} \in Fx_n \subseteq F^{n+1}x$ such that

$$d(x_{n+1}, x_n) \le \varphi[d(x_n, x_{n-1})].$$

Next we will show that $\{x_n\}$ is a Cauchy sequence in X. By the structuring of $\{x_n\}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\le \varphi[d(x_n, x_{n-1})] \\ &\le \varphi^2[d(x_{n-1}, x_{n-2})] \\ &\;\;\vdots \\ &\le \varphi^n[d(x_1, x)] \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

For all $\varepsilon > 0$, choose N so large that

$$d(x_{n+1}, x_n) \le \varepsilon - \varphi(\varepsilon), \quad \forall\, n \ge N.$$

Then, we have

$$\begin{aligned} d(x_n, x_{n+2}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\le \varepsilon - \varphi(\varepsilon) + \varphi[d(x_n, x_{n+1})] \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon. \end{aligned}$$

Using the mathematical induction, we have,

$$d(x_n, x_{n+k}) \le \varepsilon, \quad \forall\, k \in N^*,\, n \ge N.$$

Therefore $\{x_n\}$ is a Cauchy sequence in X and hence converges to some point (say) x_0 in the complete metric space X.

Next we have to show that x_0 is a fixed point of the mapping F.
From assumption (3), it follows that

$$D(Fx_n, Fx_0) < \varphi[d(x_n, x_0)] \to 0, \quad \text{as} \quad n \to \infty.$$

Now because $x_{n+1} \in F(x_n)$, we have

$$\lim_{n\to\infty} d(x_{n+1}, Fx_0) = 0.$$

It follows by using Lemma 2.11 that $x_0 \in F(x)$, i.e., x_0 is a fixed point of the mapping F. □

Theorem 3.3. *Let (X, d) be a complete metric space and $F : X \to CB(X)$ satisfying: for all $\varepsilon > 0$, there exists $\delta(x) > 0$ such that $d(x, Fx) < \delta$ imply $F[B(x, \varepsilon)] \subset B(x, \varepsilon)$. If there exists a point $u \in X$ with*

$$\lim_{n\to\infty} D(F^n(u), F^{n+1}(u)) = 0.$$

Then exists a sequence $\{u_n\} = \{u_i |\, u_i \in F^i(u),\, i = 1, 2, \cdots\}$ such that $u_n \to u_0$ and u_0 is a fixed point of F in X.

Proof. Take $u_1 \in Fu$, then $Fu_1 \subset F^2u = F(Fu)$ and take $u_2 \in Fu_1 \subseteq F^2u$, then $Fu_2 \subseteq F^3u$. Similarly, we get,

$$u_{n+1} \in Fu_n \subseteq F^{n+1}u.$$

Next we will show that $\{x_n\}$ is a Cauchy sequence in X.
Since $\lim_{n\to\infty} D(F^nu, F^{n+1}u) = 0$, for all $\varepsilon > 0$, choose N so large that

$$D(F^nu, F^{n+1}u) < \delta(\varepsilon), \quad \forall\, n \geq N,$$

then

$$d(u_n, Fu_n) \leq D(F^nu, F^{n+1}u) < \delta(\varepsilon), \quad \forall\, n \geq N.$$

Specifically, we have

$$d(u_N, Fu_N) < \delta(\varepsilon).$$

So, we get

$$F[B(u_N, \varepsilon)] \subset B(u_N, \varepsilon).$$

On the other hand, for $u_{n+1} \in Fu_n$, we have $u_{N+1} \in B(u_N, \varepsilon)$. Then

$$Fu_{N+1} \subseteq F[B(u_N, \varepsilon)] \subset B(u_N, \varepsilon).$$

Therefore, $u_{N+2} \in B(u_N, \varepsilon)$. Following the same way, we have

$$Fu_{N+k} \subseteq F[B(u_N, \varepsilon)] \subset B(u_N, \varepsilon),$$

then $u_{N+k} \in B(u_N, \varepsilon)$, $\forall\, k \in N^*$. So, we have

$$d(u_N, u_{N+k}) < \varepsilon, \quad \forall\, k \in N^*.$$

Therefore $\{u_n\}$ is a Cauchy sequence in X and we get $u_n \to u_0 \in X$ in the complete metric space X.

Next we have to show that u_0 is a fixed point of the mapping F(proof by contradiction).

Suppose, to the contrary, that $d(Fu_0, u_0) = a > 0$. For $u_n \to u_0$ we have

$$d(Fu_0, u_n) \to a, \quad \text{as} \quad n \to \infty.$$

So we could take $u_n \in B(u_0, \frac{a}{3})$ such that

$$d(Fu_0, u_n) > \frac{2a}{3} \ \text{ and } \ d(u_n, u_{n+1}) < \delta(\frac{a}{3}).$$

By the assumption we have

$$F[B(u_n, \frac{a}{3})] \subset B(u_n, \frac{a}{3}).$$

On the other hand, for $u_0 \in B(u_n, \frac{a}{3})$ we have

$$Fu_0 \subseteq B(u_n, \frac{a}{3}),$$

a contradiction to $d(Fu_0, u_n) > \frac{2a}{3}$. Thus $d(Fu_0, u_0) = a = 0$, i.e., u_0 is a fixed point of the mapping F. $\square$

Theorem 3.4. *Let (X, d) be a T_0-qpm space, q a Q-function on (X, d). $T : X \to 2^X$ a multivalued map such that for all $x, y \in X, u \in Tx$, there exists $v \in Ty$ satisfying:*

$$q(u, v) \leq \varphi(\max\{q(x, y), q(x, u), q(y, v)\})$$

where $\varphi : R^+ \to R^+$ is any monotone non-decreasing (not necessarily continuous) function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ and $\lim_{t\to 0^+} \varphi(t) = 0$, $\forall\, t > 0$. Then for all $x_0 \in X$, there exists $\{x_n\}$ satisfying the following three conditions:

(a) *$x_{n+1} \in Tx_n$, for all $n \in N^*$;*
(b) *for $\delta > 0$, there exists $n_\delta \in N$ such that $q(x_n, x_m) < \delta$, $\forall\, m > n \geq n_\delta$;*
(c) *$\{x_n\}$ is a Cauchy sequence in the metric space (X, d^s).*

Proof. Following the same process with Lemma 2.12 of literature [6]. We get a sequence $\{x_n\} \subset X$ with $x_{n+1} \in Tx_n$ and for all $n \in N$

$$q(x_{n+1}, x_{n+2}) \leq \varphi(\max\{q(x_n, x_{n+1}), q(x_{n+1}, x_{n+2})\}). \tag{3.4}$$

Now we distinguish two cases.

Case 1. The same discussion with Lemma 2.12 of literature [6].

Case 2. For all $n \in N^*$, $q(x_n, x_{n+1}) > 0$. Then, for the same way with Lemma 2.12 of literature [6], we have

$$q(x_n, x_{n+1}) \leq \varphi^n(q(x_0, x_1)), \quad \forall\, n \in N^*.$$

Since $\lim\limits_{n\to\infty} \varphi^n(t) = 0$, $\forall\, t > 0$, then we have

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0.$$

So, for $\delta > 0$, there exists N_δ so large that

$$q(x_n, x_{n+1}) < \delta - \varphi(\delta),$$

then, for all $n > N_\delta$,

$$\begin{aligned} q(x_n, x_{n+2}) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) \\ &\leq \delta - \varphi(\delta) + \varphi(q(x_n, x_{n+1})) \\ &< \delta - \varphi(\delta) + \varphi(\delta) = \delta. \end{aligned}$$

Using the mathematical induction we have

$$q(x_n, x_{n+k}) < \delta, \quad \forall\, n > N_\delta,\ k \in N^*.$$

In particular, $q(x_{N_\delta}, x_n) \leq \delta$ and $q(x_{N_\delta}, x_m) \leq \delta$, for all $n, m > N_\delta$. Thus by Lemma 2.12 of this article, for $\forall\, \varepsilon > 0$, we have $d^s(x_n, x_m) \leq \varepsilon$, for all $n, m > N_\delta$. Hence $\{x_n\}$ is a Cauchy sequence in (X, d^s). This concludes the proof. □

Theorem 3.5. *Let (X, d) be a T_0-qpm space and q is a Q-function on (X, d), $T : X \to CB(X)$, a multivalued map, is q-l.s.c operator and for all $x, y \in X, u \in Tx$, there exists $v \in Ty$ satisfying:*

$$q(u, v) \leq \varphi(\max\{q(x, y), q(x, u), q(y, v)\})$$

where $\varphi : R^+ \to R^+$ is any monotone non-decreasing (not necessarily continuous) function such that $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ and $\lim\limits_{t\to 0^+} \varphi(t) = 0$, $\forall\, t > 0$. Then, there exists $z \in X$ such that $z \in Tz$ and $q(z, z) = 0$.

Proof. Using the same proving way with Theorem 1 of article [6]. □

Theorem 3.6. *Let (X, d) be a T_0-qpm space and q is a Q-function on (X, d), $T : X \to CB(X)$, a multivalued map, such that for all $x, y \in X, u \in Tx$, there exists $v \in Ty$ satisfying:*

$$q(u, v) \leq \varphi(\max\{q(x, y), q(x, u)\}),$$

where $\varphi : R^+ \to R^+$ is any monotone non-decreasing (not necessarily continuous) function such that $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ and $\lim\limits_{t\to 0^+} \varphi(t) = 0$, $\forall\, t > 0$. Then, there exists $z \in X$ such that $z \in Tz$ and $q(z, z) = 0$.

Proof. Using the same proving way with Theorem 2 of article [6], we get a $z \in X$ such that $z \in Tz$.

Now we prove that $q(z,z) = 0$. Since $z \in Tz$, hence, there exists $\{z_n\}$ with $z_1 \in Tz,\ z_{n+1} \in Tz_n$ and $\lim\limits_{n\to\infty} d(z_n, z) = 0$ such that

$$q(z, z_n) \le \varphi(\max\{q(z, z_{n-1}), q(z,z)\}), \quad \forall\, n \ge 2.$$

Then, there is N_0 such that for any $n \ge N_0$, we have $q(z, z_n) \le \varphi(q(z,z))$. Thus, we obtain

$$\overline{\lim_{n\to\infty}}\, q(z, z_n) \le \varphi(q(z,z)).$$

By hypotheses $q(z,z) > 0$, we have

$$\varphi(q(z,z)) < q(z,z) \le q(z, z_n) + q(z_n, z).$$

Since $\lim\limits_{n\to\infty} d(z_n, z) = 0$, then $\lim\limits_{n\to\infty} q(z_n, z) = 0$. Thus, we obtain

$$\underline{\lim_{n\to\infty}}\, q(z, z_n) \ge q(z,z).$$

So, we get a contradiction that $\underline{\lim\limits_{n\to\infty}}\, q(z, z_n) > \overline{\lim\limits_{n\to\infty}} q(z, z_n)$. Thus we have

$$q(z,z) = 0.$$

$\square$

Theorem 3.7. *Let X be a topological space, K be a nonempty compact subset of X and $T : K \to CB(K)$ be a set valued operator. If $F \in \Psi$ and for all $x, y \in K, x \neq y, u \in Tx$, there exists $v \in Ty$ satisfying*

$$\begin{aligned} F(u,v) < \max\{&F(x,y), \min\{F(x,u), F(y,v)\}\} \\ &+ \lambda \min\{F(x,v), F(u,y)\}, \end{aligned} \tag{3.5}$$

where λ is an arbitrary positive real number. Then T has at least one fixed point.

Proof. Define $\varphi : K \to R$ by $\varphi(x) = \inf\limits_{u \in Tx} F(x,u), x \in K$. We shall show that φ is also lsca on K. For any $x_0 \in K$, there exists $\{x_n\} \subset K$ such that $x_n \to x_0$. Since Tx_n is close and K is compact, there exists $y_n \in Tx_n$ that $y_n \to y_0 \in Tx_0$. For F is lsca, we have

$$F(x_0, y_0) \le \lim_{n\to\infty} F(x_n, y_n).$$

Then

$$\begin{aligned} \varphi(x_0) &= \inf_{y \in Tx_0} F(x_0, y) \le F(x_0, y_0) \\ &\le \lim_{n\to\infty} F(x_n, y_n) = \lim_{\substack{n\to\infty \\ y_n \in Tx_n}} F(x_n, y_n) = \lim_{n\to\infty} \varphi(x_n). \end{aligned}$$

So, φ is lsca on K for the arbitrary of x_0. Then, by using Lemma 2.12, there exists a point, (say, $w \in K$) such that

$$\varphi(w) = \inf_{x \in K} \varphi(x) = \inf_{x \in K} \{ \inf_{u \in Tx} F(x, u) \}.$$

Generally, We note that $\varphi(w) = \inf_{x \in K} \varphi(x) = F(x, u_0)$, where $u_0 \in Tw$.

Next we shall prove that $u_0 = w$. Suppose, to the contrary, that $u_0 \neq w$, then, by using condition (3.5), for $w, u_0 \in K$, $u_0 \in Tw$, $\exists\, v \in Ty$, for λ an arbitrary positive real number that satisfying

$$\begin{aligned} F(u_0, v) &< \max\{F(w, u_0), \min\{F(w, u_0), F(u_0, v)\}\} \\ &\quad + \lambda \min\{F(w, v), F(u_0, u_0)\} \\ &= \max\{F(w, u_0), \min\{F(w, u_0), F(u_0, v)\}\}. \end{aligned}$$

Since $\varphi(w) = \inf_{x \in K} \varphi(x)$, we have

$$F(w, u_0) = \varphi(w) \leq \varphi(u_0) = \inf_{y \in Tu_0} F(u_0, y) \leq F(u_0, v).$$

Then,

$$\min\{F(w, u_0), F(u_0, v)\} = F(w, u_0). \tag{3.6}$$

So, we deduce

$$F(u_0, v) < F(w, u_0),$$

a contradiction to (3.6). Thus $u_0 = w$ and w is a fixed point of T on K. $\square$

Corollary 3.8. *Let X be a compact topological space, K be a nonempty compact subset of X and $T : K \to CB(X)$ be a set valued operator. If $F \in \Psi$ and for $\forall\, x, y \in K, x \neq y, u \in Tx$, there exists $v \in Ty$ satisfying*

$$F(u, v) < \max\{F(x, y), \min\{F(x, u), F(y, v)\}\} + \lambda \min\{F(x, v), F(u, y)\},$$

where λ is an arbitrary positive real number. Then T has at least one fixed point.

Corollary 3.9. *Let X be a topological space, K be a nonempty compact subset of X and $T : K \to CB(X)$ be a set valued operator. If $F \in \Psi$ is symmetric and for $\forall\, x, y \in K, x \neq y, u \in Tx$, there exists $v \in Ty$ satisfying*

$$F(u, v) < \max\{F(x, y), \min\{F(x, u), F(y, v)\}\} + \lambda \min\{F(x, v), F(u, y)\},$$

where λ is an arbitrary positive real number. Then T has at least one fixed point.

Corollary 3.10. *Let (X,d) be a metric space, K be a nonempty compact subset of X and $T: K \to CB(x)$ be a set valued operator. If for $\forall\, x, y \in K, x \neq y, u \in Tx$, there exists $v \in Ty$ satisfying*

$$d(u,v) < \max\{d(x,y), \min\{d(x,u), d(y,v)\}\} + \lambda \min\{d(x,v), d(u,y)\},$$

where λ is an arbitrary positive real number. Then T has at least one fixed point.

Proof. We can easily verify that $d \in \Psi$. Thus, by using Theorem 3.7 we get the conclusion. □

4. Example

Cross reference to [5], we could know that a (c)-comparison function must be a comparison function. However, the opposite is not true. Thus, Theorem 3.6 and Theorem 3.7 both are the generalize of [6]. Next we will give two examples to show that.

Example 4.1. Let $X = \{0, 1, 2, 3, \cdots\}$ and let d be the quasimetric on X defined as: $d(x,x) = 0$, for all $x \in X$; $d(x,y) = x$, if $x > y$; and $d(x,y) = x+y$, if $x < y$. Clearly (X,d) is a complete quasimetric space and d is a w-distance on (X,d).
Now let $T: X \to CB(X)$ given as:
$T0 = 0;\ T1 = \{x \in N : x > 1\}$ and $Tx = \{0\} \cup \{y \in N : y > x\}, x \in N \backslash \{1\}$;
Consider the function φ given by

$$\begin{cases} \varphi(t) = \dfrac{t}{1+t}, & 0 \leq t < 2, \\ n, & n+1 \leq t < n+2,\ n \in N. \end{cases} \tag{4.1}$$

An easy computation of all different cases show that the condition of Theorem 3.6 is satisfied (but $\varphi(t)$ isn't a (c)-comparison function and cannot apply Theorem 1 of [6]).
Case 1: If $x = 0, y = 0,$ then $u = 0, v = 0$, we deduce that $q(u,v) = 0$.
Case 2: If $x = 0, y = 1,$ then $u = 0$. Take $v = 2 \in Ty$, we deduce that

$$d(u,v) = 2 = \varphi(3) = \varphi(d(y,v)).$$

Case 3: If $x = 0, y \in N\backslash\{1\},$ then $u = 0$. Take $v = y + 1 \in Ty$, we deduce that

$$d(u,v) \leq 2y = \varphi(d(y,v)).$$

Case 4: If $x = 1, y = 0$, then for all $u \in Tx$. Take $v = 0 \in Ty$, we deduce that

$$d(u,v) = u = \varphi(u+1) = \varphi(d(x,u)).$$

Case 5: If $x = 1, y = 1$, then for all $u \in Tx$. Take $v = 2 \in Ty$, we deduce that

$$d(u, v) \leq u = \varphi(d(x, u)).$$

Case 6: If $x = 1, y = N\backslash\{1\}$, then for all $u \in Tx$. Take $v = 0 \in Ty$, we deduce that

$$d(u, v) = u = \varphi(d(x, u).$$

Case 7: If $x \in N\backslash\{1\}, y = 0$, then for all $u \in Tx$. Take $v = 0 \in Ty$, we deduce that

$$d(u, v) = u \leq \varphi(d(x, u)).$$

Case 8: If $x \in N\backslash\{1\}, y = 1$, then for all $u \in Tx$. Take $v = 0 \in Ty$, we deduce that

$$d(u, v) = \max\{u, v\} \leq \max\{u + x - 1, v\} = \varphi(\max\{d(x, u), d(y, v)\}).$$

Case 9: If $x, y \in N\backslash\{1\}$, then for all $u \in Tx$. Take $v = 0 \in Ty$, we deduce that

$$d(u, v) = u < u + x - 1 = \varphi(u + x) = \varphi(d(x, u)).$$

Hence, all conditions of Theorem 3.6 are satisfied. So T has fixed point.

Example 4.2. Let $X = \{0, \frac{1}{4}, 2\} \cup [\frac{5}{12}, \frac{3}{4}]$ and let d be the T_0-qpm on X defined as:

$$\begin{cases} d(x, x) = 0, d(x, 0) = 0, & \forall\, x \in X, \\ d(x, y) = 1, & \text{otherwise}. \end{cases}$$

It is clear that d is complete and the function $q : X \times X \to [0, \infty)$ defined as:
$q(x, x) = 0,\ \forall\, x \in X\backslash\{1\};\ \ q(1, 1) = 1;\ \ q(0, x) = q(x, 0) = \frac{1}{2},\ \forall\, x \in X\backslash\{0\};$
$q(1, x) = x,\ \forall\, x \in A;\ q(x, 1) = 1 - x,\ \forall\, x \in A;\ q(x, y) = |x - y|\,,\ \forall\, x, y \in A$
is also a Q-function. Now let $T : X \to CB(X)$ given as: $T0 = 0$; $T1 = \{0, 1\}$; $Tx = \{0, \frac{1+x}{2}\}, x \in A$ and let φ given as:

$$\begin{cases} \varphi(t) = \dfrac{t}{1 + t}, & 0 \leq t < 1, \\ \dfrac{2t}{3}, & t \geq 1. \end{cases} \tag{4.2}$$

Notice that T is not q-l.s.c. Hence we cannot apply Theorem 3.6 to this example. We shall show that, nevertheless the conditions of Theorem 3.7 are satisfied.

Case 1: For $x \in \{0, \frac{1}{4}\}, y \in \{0, \frac{1}{4}\} \cup [\frac{5}{12}, \frac{3}{4}]$, then $u = 0$. Take $v = 0 \in Ty$, we deduce that

$$q(u, v) = 0.$$

Case 2: For $x \in \{0, \frac{1}{4}\}, y = 2$, then $u = 0$. Take $v = \frac{5}{12} \in Ty$, we deduce that

$$q(u, v) = \frac{5}{12} < \varphi(q(x, y)).$$

Case 3: For $\frac{5}{12} \leq x \leq \frac{3}{4}, y \in \{0, \frac{1}{4}\} \cup [\frac{5}{12}, \frac{3}{4}], \forall\, u \in Tx$. Take $v = 0 \in Ty$, and we have if $u = 0,$ then $q(u,v) = 0;$ else that $u = \frac{1}{4}$, we deduce that

$$q(u,v) = \frac{1}{4} \leq \frac{x+u}{1+x+u} = \varphi(x+u) = \varphi(q(x,u)).$$

Case 4: For $\frac{5}{12} \leq x \leq \frac{3}{4}, y = 2, \forall\, u \in Tx$. Take $v = \frac{5}{12} \in Ty$, we deduce that

$$q(u,v) \leq \frac{1}{4} + \frac{5}{12} = \varphi(1) \leq \varphi(q(x,y)).$$

Case 5: For $x = 2, y \in \{0, \frac{1}{4}\} \cup [\frac{5}{12}, \frac{3}{4}], \forall\, u \in Tx$. Take $v = 0 \in Ty$, we deduce that

$$q(u,v) = u < \frac{2(x+u)}{3} = \varphi(q(x,u)).$$

Case 6: For $x = 2, y = 2, \forall\, u \in Tx$. Take $v = \frac{5}{12} \in Ty$, we deduce that

$$q(u,v) \leq \frac{3}{4} + \frac{5}{12} < \frac{8}{3} = \varphi(q(x,y)).$$

Hence, all conditions of Theorem 3.7 are satisfied. So T has fixed point.

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