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STABLE SOLUTIONS AND BIFURCATION PROBLEM FOR ASYMPTOTICALLY LINEAR HELMHOLTZ EQUATIONS

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Abstract. In this note, we investigate the existence of positive solutions for the Helmholtz equation $-\Delta u + cu = \lambda f(u)$ on a bounded smooth domain of $\mathbb{R}^n$ with Dirichlet boundary conditions. Here $\lambda > 0$, $c > 0$ are positive constants and f is a positive nondecreasing convex function, asymptotically linear that is $\lim_{t\to\infty} \frac{f(t)}{t} = a < \infty$. We show that there exists an extremal parameter $\lambda^* > 0$ but the extremal solution exists and it is regular provided that $\lim_{t\to\infty} f(t) - at = l < 0$.

1. Introduction

Consider the problem

$$(P_\lambda) \quad \begin{cases} -\Delta\, u + c\, u &= \lambda f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$, $c > 0$ and $\lambda > 0$. The function f defined in $[0, \infty)$ and satisfies

$$f \text{ is } C^1, \text{ positive, nondecreasing and convex} \tag{1.1}$$

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and

$$\lim_{t\longrightarrow\infty}\frac{f(t)}{t}=a\in(0,+\infty). \tag{1.2}$$

By a solution of (P_λ) we mean a function $u\in C^2(\overline{\Omega})$ satisfying (P_λ). In the sequel we are interested only in nonnegative solutions and for which we have considered only $\lambda>0$. From maximum principle that if u is a nonnegative solution, then $u(x)>0$ for $x\in\Omega$.

Problems of the form (P_λ) occur in a variety of situations. For $c(x)=$ *cte.*, (P_λ) is known as the Helmholtz problems. They arise in models of combustion [4, 5], thermal explosions [4], nonlinear heat generation [8], and the gravitational equilibrium of polytropic stars [3, 7]. In particular, the Helmholtz problem occur in the study of electromagnetic radiation, seismology, acoustics.

For $c=0$, various authors have studied the bifurcation problem

$$(E_\lambda)\quad\left\{\begin{array}{rcll}-\Delta u&=&\lambda f(u)&\text{in}\quad\Omega,\\ u&=&0&\text{on}\quad\partial\Omega,\end{array}\right.$$

Brezis *et al.* have proved in [2] that there exists $0<\lambda^*<\infty$, a critical value of the parameter λ, such that (E_λ) has a minimal, positive, classical solution u_λ for $0<\lambda<\lambda^*$ and does not have solutions for $\lambda>\lambda^*$.

The value of a was crucial in the study of (E_{λ^*}) and of the behavior of u_λ when λ approaches λ^*. In the case when $a=+\infty$, it is proved in [2] that a minimal weak solution u^* exists for $\lambda=\lambda^*$. In [9], Martel proves that in this case u^* is the unique weak solution of (E_{λ^*}). Recently, Sanchon in [11] generalizes these results for the p-Laplacian. When a is finite, Mironescu and Rădulescu proved in [10] that there exists a unique classical solution u^* of (E_{λ^*}) if and only if $\lim\limits_{t\to\infty}(f(t)-at)<0$.

In this paper,we deal with weak solution in the following sense.

Definition 1.1. A weak solution of (P_λ) is a function $u\in L^1(\Omega), u\geq 0$ such that $f(u)\in L^1(\Omega)$, and

$$-\int_\Omega u\Delta\zeta+\int_\Omega cu\,\zeta=\lambda\int_\Omega f(u)\zeta \tag{1.3}$$

for all $\zeta\in C^2(\overline{\Omega})$ with $\zeta=0$ on $\partial\Omega$.

We say that u is a weak super-solution of (P_λ) if "=" is replaced by "$\geq$" for all $\zeta\in C^2(\overline{\Omega})$, $\zeta\geq 0$ and $\zeta=0$ on $\partial\Omega$.

Remark 1.2. If $u\in L^1(\Omega)$ is a weak solution of (P_λ) and $u\in L^\infty(\Omega)$, we say that u is regular. By elliptic regularity, we know that regular solutions are smooth and solve (P_λ) in the classical sense.

For regular solutions, we introduce a notion of stability.

Definition 1.3. A regular solution u of (P_λ) is said to be stable if the first eigenvalue $\eta_1(c,\lambda,u)$ of the linearized operator $L_{c,\lambda,u} = -\Delta + c - \lambda f'(u)$ given by

$$\eta_1(c,\lambda,u) := \inf_{\varphi\in H_0^1(\Omega)-\{\,0\}} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2 + \int_\Omega c\varphi^2 - \lambda\int_\Omega f'(u)\varphi^2}{\|\varphi\|_2^2},$$

is positive in $H_0^1(\Omega)$. In other words,

$$\lambda\int_\Omega f'(u)\varphi^2 \le \int_\Omega |\nabla\varphi|^2 + c\int_\Omega \varphi^2 \qquad \text{for any} \quad \varphi\in H_0^1(\Omega). \tag{1.4}$$

If $\eta_1(c,\lambda,u) < 0$, the solution u is said to be *unstable*.

We denote by λ_1 the first eigenvalue of $L = -\Delta + c$ in Ω with Dirichlet boundary condition and φ_1 a positive normalized eigenfunction associated, that is, such that

$$\begin{cases} -\Delta\,\varphi_1 + c\varphi_1 &= \lambda_1\,\varphi_1 \quad \text{in} \quad \Omega,\\ \varphi_1 &> 0 \qquad\ \text{in} \quad \Omega,\\ \varphi_1 &= 0 \qquad\ \text{on} \quad \partial\Omega,\\ \|\varphi_1\|_2^2 &= 1. \end{cases} \tag{1.5}$$

Next, we let

$$\Lambda := \{\lambda > 0 \ \text{ such that } (P_\lambda) \text{ admits a solution}\} \text{ and } \lambda^* := \sup\Lambda \le +\infty.$$

We denote

$$r_0 := \inf_{t>0} \frac{f(t)}{t}. \tag{1.6}$$

Our first main statement asserts the existence of the critical value λ^*.

Theorem 1.4. *There exists a critical value $\lambda^* \in (0,\infty)$ such that the following properties hold true.*

(i) *For any $\lambda\in(0,\lambda^*)$, problem (P_λ) has a minimal solution u_λ, which is the unique stable solution of (P_λ) and the mapping $\lambda\mapsto u_\lambda$ is increasing.*

(ii) *For any $\lambda\in(0,\frac{\lambda_1}{a})$, u_λ is the unique solution of problem (P_λ).*

(iii) *If problem (P_{λ^*}) has a solution u, then*

$$u = u^* = \lim_{\lambda\to\lambda^*} u_\lambda,$$

and $\eta_1(c,\lambda^*,u^*) = 0$.

(iv) *For $\lambda > \lambda^*$, the problem (P_λ) has no weak solution.*

For the next results, let

$$l := \lim_{t\to\infty}\Big(f(t) - at\Big). \tag{1.7}$$

We distinguish two different situations strongly depending on the sign of l.

Theorem 1.5. *Assume that $l \geq 0$. The following results hold.*

(i) $\lambda^* = \lambda_1/a$.
(ii) *Problem (P_{λ^*}) has no solution.*
(iii) $\lim\limits_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* Ω.

Theorem 1.6. *Assume that $l < 0$. Then we have.*

(i) *The critical value λ^* belongs to $(\frac{\lambda_1}{a}, \frac{\lambda_1}{r_0})$.*
(ii) *(P_{λ^*}) has a unique solution u^*.*
(iii) *The problem (P_λ) has an unstable solution v_λ for any $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$ and the sequence $(v_\lambda)_\lambda$ satisfies*
  (a) $\lim\limits_{\lambda\to\frac{\lambda_1}{a}} v_\lambda = \infty$ *uniformly on compact subsets of* Ω,
  (b) $\lim\limits_{\lambda\to\lambda^*} v_\lambda = u^*$ *uniformly in* Ω.

2. Proof of Theorem 1.4

In the proof of this Theorem we shall make use of the following auxiliary results.

Lemma 2.1. ([2]) *Given $g \in L^1(\Omega)$, there exists an unique $v \in L^1(\Omega)$ which is a weak solution of*

$$\begin{cases} -\Delta\, v + c\, v &= g \quad in \quad \Omega, \\ v &= 0 \quad on \quad \partial\Omega, \end{cases}$$

in the sense that

$$\int_\Omega -v\Delta\,\zeta + \int_\Omega c\; v\,\zeta = \int_\Omega g\,\zeta$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Moreover, there exists a constant c_0 independents of g such that

$$\|v\|_{L^1(\Omega)} \leq c_0\,\|g\|_{L^1(\Omega)}.$$

In addition, if $g \geq 0$ a.e in Ω, then $v \geq 0$ a.e in Ω.

Lemma 2.2. *If (P_λ) has a weak super solution $\overline{u}$, then there exists a weak solution u of (P_λ) such that $0 \leq u \leq \overline{u}$ and u does not depend on $\overline{u}$.*

Proof. We use a standard monotone iteration argument and maximum principle for the operator $-\Delta + c$. Let $u_0 = 0$ and u_{n+1} the solution of

$$\begin{cases} -\Delta\, u_{n+1} + c\, u_{n+1} &= \lambda f(u_n) \quad \text{in} \quad \Omega, \\ u_{n+1} &= 0 \qquad\quad \text{on} \quad \partial\Omega, \end{cases}$$

which exists by Lemma 2.1. We prove that $0 = u_0 \leq u_1 \leq ... \leq u_n \leq ... \leq \overline{u}$ and $(u_n)_n$ converge to $u \in L^1(\Omega)$ which is a weak solution of (P_λ). Moreover u is independent of $\overline{u}$ by construction. □

The existence of the critical value λ^* is a consequence of the following auxiliary result.

Lemma 2.3. *The problem (P_λ) has no solution for any $\lambda > \lambda_1/r_0$, but has at least one solution provided λ is positive and small enough.*

Proof. To show that (P_λ) has a solution, we use the barrier method and so the Lemma 2.2. To this aim, let $\xi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$ which satisfies $-\Delta\xi + c\,\xi = 1$ in Ω. The choice of ξ implies that ξ is a super solution of (P_λ) for $\lambda \leq 1/f(\|\xi\|_\infty)$. By Lemma 2.2, there exist a weak solution u of (P_λ) such that $0 \leq u \leq \xi$. Because $\xi \in C^2(\overline{\Omega})$, $u \in L^\infty(\Omega)$ (u is a regular solution) and then $u \in C^2(\overline{\Omega})$. It follows that problem (P_λ) has a solution for $\lambda \leq 1/f(\|\xi\|_\infty)$.

Assume now that u is a solution of (P_λ) for some $\lambda > 0$. Using φ_1 given by (1.5) as a test function, we get

$$-\int_\Omega u\Delta\varphi_1 + \int_\Omega cu\varphi_1 = \lambda \int_\Omega f(u)\varphi_1.$$

This yields

$$(\lambda_1 - \lambda r_0) \int_\Omega \varphi_1 u \geq 0.$$

Since $\varphi_1 > 0$ and $u > 0$, we conclude that the parameter λ should belong to $(0, \lambda_1/r_0)$. This completes our proof. □

As a consequence we have that λ^* is a real. Another useful result is stated in what follows.

Lemma 2.4. *Assume that the problem (P_λ) has a solution for some $\lambda \in (0, \lambda^*)$. Then there exists a minimal solution denoted by u_λ for the problem (P_λ). Moreover, for any $\lambda' \in (0, \lambda)$, the problem $(P_{\lambda'})$ has a solution.*

Proof. Fix $\lambda \in (0, \lambda^*)$ and let u be a solution of (P_λ). As above, we use the Lemma 2.2 to obtain a solution of (P_λ), u_λ which is independent of u used as super solution (as mentioned in the proof of Lemma 2.2). Since u_λ is independent of the choice of u, then it is a minimal solution.

Now, if u is a solution of (P_λ), then u is a super solution of the problem $(P_{\lambda'})$ for any λ' in $(0,\lambda)$ and Lemma 2.2 completes the proof. □

Proof of Theorem 1.4 **(i)**. First, we claim that u_λ is stable. Indeed, arguing by contradiction, we deduce that the first eigenvalue $\eta_1 = \eta_1(c,\lambda,u_\lambda)$ is non positive. Then, there exists an eigenfunction

$$\psi \in C^2(\overline{\Omega}) \text{ and } \psi = 0 \text{ on } \partial\Omega,$$

such that

$$-\Delta\psi + c\psi - \lambda f'(u_\lambda)\psi = \eta_1\psi \text{ in } \Omega \text{ and } \psi > 0 \text{ in } \Omega.$$

Consider $u^\varepsilon := u_\lambda - \varepsilon\psi$. Hence

$$\begin{aligned}-\Delta u^\varepsilon + c\,u^\varepsilon - \lambda f(u^\varepsilon) &= -\eta_1\varepsilon\psi + \lambda\Big[f(u_\lambda) - f(u_\lambda - \varepsilon\psi) - \varepsilon f'(u_\lambda)\psi\Big]\\ &= \varepsilon\psi(-\eta_1 + o_\varepsilon(1)).\end{aligned}$$

Since $\eta_1(c,\lambda,u_\lambda) \leq 0$ for $\varepsilon > 0$ small enough, we have

$$-\Delta u^\varepsilon + c\,u^\varepsilon - \lambda f(u^\varepsilon) \geq 0 \text{ in } \Omega.$$

Then, for $\varepsilon > 0$ small enough, we use the strong maximum principle (Hopf's Lemma) to deduce that $u^\varepsilon \geq 0$. u^ε is a super solution of (P_λ), so by Lemma 2.2 we obtain a solution u such that $u \leq u^\varepsilon$ and since $u^\varepsilon < u_\lambda$, then we contradict the minimality of u_λ.

Now, we show that (P_λ) has at most one stable solution. Assume the existence of another stable solution $v \neq u_\lambda$ of problem (P_λ). Let $w := v - u_\lambda$, then by maximum principle $w > 0$ and from (1.4) taking w as a test function, we have

$$\begin{aligned}\lambda\int_\Omega f'(v)\,w^2 &\leq \int_\Omega |\nabla w|^2 + \int_\Omega cw^2\\ &= -\int_\Omega w\Delta w + \int_\Omega cw^2 = \lambda\int_\Omega \Big[f(v) - f(u_\lambda)\Big]w.\end{aligned}$$

Therefore

$$\int_\Omega \Big[f(v) - f(u_\lambda) - f'(v)(v - u_\lambda)\Big]w \geq 0.$$

Thanks to the convexity of f, the term in the brackets is non positive, hence

$$f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) = 0 \text{ in } \Omega,$$

which implies that f is affine over $[u_\lambda, v]$ in Ω. So, there exists two real numbers $\bar{a}$ and b such that

$$f(x) = \bar{a}x + b \quad \text{ in } [0, \max_\Omega v].$$

Finally, since u_λ and v are two solutions to $-\Delta w + cw = \bar{a}w + b$, we obtain that
$$0 = \int_\Omega (u_\lambda \Delta v - v\Delta u_\lambda) = b\int_\Omega (v - u_\lambda) = b\int_\Omega w.$$
This is impossible since $b = f(0) > 0$ and w is positive in Ω.

Finally, by Lemma 2.4 and the definition of u_λ, we have that the function $\lambda \to u_\lambda$ is an increasing mapping. □

Proof of Theorem 1.4 **(ii)**. In this stage, we need the following results.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Assume that f is a function satisfying* (1.1) *and* (1.2). *If (P_λ) has a weak solution u, then u is a regular solution and hence a classical solution.*

Proof. By convexity of f, we have $a = \sup_{t\geq 0} f'(t)$ and
$$f(t) \leq at + f(0) \;\text{ for all }\; t \geq 0. \tag{2.1}$$
Let u a weak solution of (P_λ), $f(u) \in L^1(\Omega)$. By elliptic regularity, $u \in L^p(\Omega)$, for all $p \geq 1$ such that
$$p < \frac{n}{n-2} \qquad (p < \infty \;\text{ if }\; n = 2). \tag{2.2}$$
Again by (2.1), $f(u) \in L^p$ for all p satisfying (2.2) so $u \in L^r(\Omega)$ for all $r \geq 1$ such that
$$r < \frac{n}{n-4} \qquad (p \leq \infty \;\text{ if }\; n = 2, 3 \;\text{ and }\; r < \infty \;\text{ if }\; n = 4). \tag{2.3}$$
By iteration and after $k(n) = [\frac{n}{2}] + 1$ operation, the solution u belongs to $L^\infty(\Omega)$. By elliptic regularity and standard bootstrap argument, $u \in C^2(\overline{\Omega})$. □

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Assume that $f(t) = at + b$, where $a, b > 0$. Then*

(i) $\lambda^* = \frac{\lambda_1}{a}$,

(ii) *The problem (P_λ) has no weak solution for $\lambda = \lambda^*$.*

Proof. Let $0 < \lambda < \dfrac{\lambda_1}{a}$, the problem (P_λ), given by
$$\begin{cases} -\Delta u + (c - \lambda a)u &= \lambda b \quad \text{in} \quad \Omega \\ \qquad\qquad\quad u &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.4}$$
has a unique solution in $C^2(\overline{\Omega})$. Since $\lambda a < \lambda_1$, by Maximum principle $u > 0$. Now let $\lambda = \dfrac{\lambda_1}{a}$. If the problem (2.4) has a solution u, then by multiplication

(2.4) by φ_1 the positive eigenfunction associated to λ_1 and introduced by (1.5) and integration by parts, it follows that $\int_\Omega \varphi_1 = 0$ which is impossible since $\varphi_1 > 0$ in Ω. So for $f(t) = at + b$, a and $b > 0$, we have $\lambda^* = \dfrac{\lambda_1}{a}$ and the equation (P_{λ^*}) has no solution. □

For the proof of Theorem 1.4 **(ii)**, let $\lambda \in (0, \frac{\lambda_1}{a})$, $b = f(0)$ and w a solution for the problem (2.4) when $f(t) = at + b$. Since we have for the general function f in Theorem 1.4, that $f(w) \leq aw + f(0)$, then w is a super solution of (P_λ) and hence by Lemma 2.2 and Proposition 2.5, the equation (P_λ) has a solution. For the uniqueness, let u a solution of (P_λ) for a reel $\lambda \in (0, \frac{\lambda_1}{a})$. We denote $\lambda_1(L)$ the first eigenvalue of an operator L, that is $\lambda_1(-\Delta + c) = \lambda_1$. Because $a = \sup\limits_{t\geq 0} f'(t)$, we have $-\Delta + c - \lambda f'(u) \geq -\Delta + c - \lambda a$ and so

$$\lambda_1(-\Delta + c - \lambda f'(u)) \geq \lambda_1(-\Delta + c - \lambda a)$$

that is

$$\eta_1(c, \lambda, u) \geq \lambda_1 - \lambda a > 0.$$

The solution u is stable then, by Theorem 1.4 **(i)**, we obtain $u = u_\lambda$. □

Proof of Theorem 1.4 **(iii)**. Suppose that (P_{λ^*}) has a solution u. then, for every $\lambda \in (0, \lambda^*)$ we have $u_\lambda \leq u$ and so $u^* = \lim\limits_{\lambda \to \lambda^*} u_\lambda$ is well defined in $L^1(\Omega)$ and furthermore u^* is a weak then classical solution for (P_{λ^*}). Since $0 \leq u^* \leq u$, u^* is a minimal solution and also satisfies (1.4) for $\lambda = \lambda^*$ so $\eta_1(c, \lambda^*, u^*) \geq 0$. Furthermore, u^* is a the unique solution for (P_{λ^*}) and we can proceed as in [9].

Now, consider the nonlinear operator

$$\begin{array}{llll} G : (0, +\infty) \times C^{2,\alpha}(\bar{\Omega}) \cap H_0^1(\Omega) & \longrightarrow & C^{0,\alpha}(\bar{\Omega}) \\ (\lambda, u) & \longmapsto & -\Delta u + cu - \lambda f(u), \end{array}$$

where $\alpha \in (0, 1)$. Assuming that the first eigenvalue $\eta_1(c, \lambda^*, u^*)$ is positive. By the implicit function theorem applied to the operator G, it follows that problem (P_λ) has a solution for λ in a neighborhood of λ^*. But this contradicts the definition of λ^* so $\eta_1(c, \lambda^*, u^*) = 0$ and this completes the proof of Theorem 1.4 **(iii)**. □

Proof of Theorem 1.4 **(iv)**. If the problem (P_λ) has a weak solution u for $\lambda > \lambda^*$, then by Proposition 2.5, u is a classical solution for (P_λ) and this contradicts the definition of λ^*. □

3. Proof of Theorem 1.5

In the proof of Theorem 1.5, we shall use the following auxiliary result which is a reformulation of Theorem due to Hörmander [6] and maximum principle and it is also true for the operator $-\Delta + c$.

Lemma 3.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$ with smooth boundary. Let (u_n) be a sequence of super harmonic nonnegative functions defined on Ω. Then the following alternative holds.*

(i) $\lim\limits_{n\to\infty} u_n = \infty$ *uniformly on compact subsets of* Ω,
or
(ii) (u_n) *contains a subsequence which converges in* $L^1_{loc}(\Omega)$ *to some function* u.

We first prove the following result.

Proposition 3.2. *Let f be a positive function satisfying* (1.1) *and* (1.2). *Then the following assertions are equivalent:*

(i) $\lambda^* = \dfrac{\lambda_1}{a}$.
(ii) (P_{λ^*}) *has no solution.*
(iii) $\lim\limits_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* Ω.

Proof. (i)⇒(ii). By contradiction. Assume that (P_{λ^*}) has a solution u. By (ii) of Theorem 1.4, $u = u^* = \lim\limits_{\lambda\to\lambda^*} u_\lambda$ and $\eta_1(c, \lambda^*, u^*) = 0$. Thus there exists $\psi \in C^2(\overline{\Omega}) \cap H^1_0(\Omega)$ satisfying

$$-\Delta\psi + c\psi - \lambda^* f'(u^*)\psi = 0 \text{ and } \psi > 0 \text{ in } \Omega. \tag{3.1}$$

Using φ_1 given by (1.5) as a test function, we obtain

$$\int_\Omega (\lambda_1 - \lambda^* f'(u^*))\varphi_1\psi = 0. \tag{3.2}$$

Since $\varphi_1 > 0$, $\psi > 0$, $\lambda^* = \dfrac{\lambda_1}{a}$ and $a = \sup\limits_{t>0} f'(t)$, we have $\lambda_1 - \lambda^* f'(u^*) \geq 0$. Then equality (3.2) gives $f'(u^*) = a$ in Ω. This implies that $f(t) = at + b$ in $[0, max_\Omega u^*]$ for some scalar $b > 0$ and this impossible by Proposition 2.6. Hence (P_{λ^*}) has no solution.

(ii)⇒(iii). By contradiction, suppose that (iii) doesn't hold. By Lemma 3.1 and up to subsequence, u_λ converges locally in $L^1(\Omega)$ to a function u as $\lambda \to \lambda^*$.
Claim: u_λ is bounded in $L^2(\Omega)$.
Indeed, if not, we may assume that

$$u_\lambda = k_\lambda w_\lambda$$

with

$$\int_\Omega {w_\lambda}^2 dx = 1 \quad \text{and} \quad \lim_{\lambda\to\lambda^*} k_\lambda = \infty. \tag{3.3}$$

We have

$$\frac{\lambda}{k_\lambda} f(u_\lambda) \to 0 \text{ in } L^1_{loc}(\Omega) \text{ as } \lambda \to \lambda^*$$

and then

$$-\Delta w_\lambda + c w_\lambda \to 0 \text{ in } L^1_{loc}(\Omega). \tag{3.4}$$

We have

$$\int_\Omega |\nabla w_\lambda|^2 = -\int_\Omega \Delta w_\lambda \, w_\lambda = \int_\Omega (\frac{\lambda f(u_\lambda)}{k_\lambda} - c w_\lambda) w_\lambda,$$

then

$$\int_\Omega |\nabla w_\lambda|^2 \le \int_\Omega \frac{\lambda f(u_\lambda)}{k_\lambda} w_\lambda \le \lambda^* \int_\Omega a\, w_\lambda^2 + \frac{f(0)}{k_\lambda} w_\lambda$$
$$\le a\lambda^* + c_0 \int_\Omega w_\lambda \le a\lambda^* + c_0\sqrt{|\Omega|},$$

for some $c_0 > 0$ independent of λ. Then (w_λ) is bounded in $H^1_0(\Omega)$ and up to a subsequence, we obtain

$$w_\lambda \rightharpoonup w \text{ weakly in } H^1_0(\Omega) \text{ and}$$
$$w_\lambda \to w \text{ in } L^2(\Omega) \text{ as } \lambda \to \lambda^*. \tag{3.5}$$

It follows by (3.4) and (3.5) that $w = 0$ in Ω and this contradicts (3.3). This complete the proof of the claim.
Thus u_λ is bounded in $L^2(\Omega)$ and with the same argument above, u_λ is bounded in $H^1_0(\Omega)$ and up to a subsequence, we have

$$u_\lambda \rightharpoonup u \text{ weakly in } H^1_0(\Omega) \text{ and}$$
$$u_\lambda \to u \text{ in } L^2(\Omega) \text{ as } \lambda \to \lambda^*,$$
$$\begin{cases} -\Delta u + cu = \lambda^* f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and this impossible by the hypothesis (ii). We remark clearly that (iii)⇒(ii) and hence (ii)⇔(iii).

(iii)⇒(i). If (iii) occurs, that (ii) also is true and we have $\lim_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Let

$$u_\lambda = k_\lambda w_\lambda \text{ with } \|w_\lambda\|_2 = 1. \tag{3.6}$$

Up to subsequence, we obtain

$$w_\lambda \rightharpoonup w \text{ weakly in } H^1_0(\Omega) \text{ and}$$
$$w_\lambda \to w \text{ in } L^2(\Omega) \text{ as } \lambda \to \lambda^*. \tag{3.7}$$

We have also

$$\frac{\lambda}{k_\lambda} f(u_\lambda) \to \lambda^* a w \quad \text{as} \quad \lambda \to \lambda^*, \tag{3.8}$$

$$-\Delta w_\lambda + c w_\lambda \to -\Delta w + c w \text{ in } L^2(\Omega)$$

and then

$$\begin{cases} -\Delta w + c w &= a\lambda^* w \quad \text{in} \quad \Omega, \\ w &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.9}$$

Taking φ_1 as a test function in (3.9), we obtain

$$\lambda_1 \int_\Omega w\varphi_1 = \int_\Omega w(-\Delta\varphi_1 + c\varphi_1)) = \int_\Omega a\lambda^* w\varphi_1.$$

Since $\varphi_1 > 0$ and $w > 0$ in Ω, we have $\lambda^* = \dfrac{\lambda_1}{a}$ and this complete the proof of Proposition 3.2.

To finish the proof of Theorem 1.5, we need only to show that $(P_{\frac{\lambda_1}{a}})$ has no solution. Assume that u is a solution of $(P_{\frac{\lambda_1}{a}})$. Since

$$l := \lim_{t\to\infty} \Big(f(t) - at \Big) \geq 0 \text{ and } a = \sup_{t\geq 0} f'(t),$$

we have $l \in (0, \infty)$ and $f(t) - at \geq 0$ and

$$-\Delta u + cu = \frac{\lambda_1}{a} f(u) \quad \text{in} \quad \Omega. \tag{3.10}$$

Taking φ_1 as a test function in (3.10), we get $f(u) = a\,u$ in Ω, which contradicts $f(0) > 0$. This concludes the proof of Theorem 1.5. □

4. Proof of Theorem 1.6

(i) We have shown that

$$\frac{\lambda_1}{a} \leq \lambda^* \leq \frac{\lambda_1}{r_0}.$$

Suppose that $\lambda^* = \frac{\lambda_1}{a}$. By Proposition 3.2, we have

$$\lim_{\lambda\to\lambda^*} u_\lambda = \infty \text{ uniformly on compact subsets of } \Omega.$$

Let u_λ be the minimal solution of (P_λ). Then, multiplying (P_λ) by φ_1 and integrating, we obtain

$$\int_\Omega \varphi_1 \Big(\lambda_1 u_\lambda - \lambda f(u_\lambda) \Big) = \int_\Omega \varphi_1 \Big((\lambda_1 - a\lambda) u_\lambda - \lambda (f(u_\lambda) - a u_\lambda) \Big) = 0 \tag{4.1}$$

and then

$$\lambda \int_\Omega \varphi_1 \Big(f(u_\lambda) - a u_\lambda \Big) \geq 0. \tag{4.2}$$

Passing to the limit in the inequality (4.2) as λ tends to λ^*, we find

$$0 \leq l\lambda^* \int_\Omega \varphi_1 < 0,$$

which is impossible and then $\lambda^* \neq \frac{\lambda_1}{a}$.

If $\lambda^* = \frac{\lambda_1}{r_0}$, let u be a solution of problem (P_{λ^*}) which exists by Proposition 3.2. Multiplying (P_{λ^*}) by φ_1 and integrating by parts, we obtain

$$\lambda_1 \int_\Omega u\varphi_1 = \frac{\lambda_1}{r_0} \int_\Omega f(u)\varphi_1$$

that is

$$\int_\Omega (f(u) - r_0 u)\varphi_1 = 0,$$

then $f(u) = r_0\, u$ in Ω, and this contradicts the fact that $f(0) > 0$.
(ii) Since $\lambda^* > \frac{\lambda_1}{a}$, the existence of a solution to (P_{λ^*}) is assured by Proposition 3.2 and the uniqueness is given by Theorem 1.4.
(iii) In this stage, we will use the mountain pass Theorem of Ambrosetti and Rabinowitz.

Theorem 4.1. ([1]) *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions.*

(1) *There exist positive constants R and ρ such that*

$$J(u) \geq J(u_0) + \rho, \text{ for all } u \in E \text{ with } \|u - u_0\| = R.$$

(2) *There exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$.*
Then the functional J possesses at least a critical point. The critical value is characterized by

$$c := \inf_{g\in\Gamma} \max_{u\in g([0,1])} J(u),$$

where

$$\Gamma := \Big\{ g \in C([0,1], E) \,|\, g(0) = u_0,\, g(1) = v_0 \Big\}$$

and satisfies

$$c \geq J(u_0) + \rho.$$

Let

$$\begin{array}{rccl} J: & H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 + \dfrac{1}{2}\int_\Omega cu^2 - \int_\Omega F(u), \end{array}$$

where

$$F(t) = \lambda \int_0^t f(s)ds, \text{ for all } t \geq 0.$$

We take u_0 as the stable solution u_λ for each $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$.

The energy functional J belongs to $C^1(H_0^1(\Omega), \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega cuv - \lambda \int_\Omega f(u)v,$$

for all $u, v \in H_0^1(\Omega)$.

Since $\eta_1(\lambda, u_\lambda) \geq 0$, the function u_λ is a local minimum for J, and in order to transform it into a local strict minimum, we apply the mountain pass theorem not for J, but to the perturbed functional J_ε defined by

$$\begin{array}{llll} J_\varepsilon : & H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & J(u) + \frac{\varepsilon}{2}\Big(\int_\Omega |\nabla(u-u_\lambda)|^2 + \int_\Omega c(u-u_\lambda)\Big), \end{array} \tag{4.3}$$

for all $\varepsilon \in [0,\ \varepsilon_0]$, where

$$\varepsilon_0 := \frac{3}{4}\ \frac{\lambda a - \lambda_1}{\lambda_1}.$$

We observe that J_ε is also in $C^1(H_0^1(\Omega), \mathbb{R})$ and

$$\begin{aligned} J'_\varepsilon(u)v = & \int_\Omega \nabla u \nabla v + \int_\Omega cuv - \lambda \int_\Omega f(u)v \\ & + \varepsilon\Big(\int_\Omega \nabla(u-u_\lambda)\nabla v + c\int_\Omega (u-u_\lambda)v\Big), \end{aligned}$$

for all $u, v \in H_0^1(\Omega)$.

Using the same arguments of Mironescu and Rădulescu in [10, Lemma 9], we show in the next lemma that J_ε satisfies the Palais-Smale compactness condition.

Lemma 4.2. *Let $(u_n) \subset E$ be a Palais-Smale sequence, that is,*

$$\sup_{n\in\mathbb{N}} |J_\varepsilon(u_n)| < +\infty, \tag{4.3}$$

$$\|J'_\varepsilon(u_n)\|_{E^*} \to 0 \text{ as } n \to \infty. \tag{4.4}$$

Then (u_n) is relatively compact in E.

Now, we need only to check that the two geometric assumptions are fulfilled. First, since u_λ is a local minimum of J, there exists $R > 0$ such that for all

$u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $J(u) \geq J(u_\lambda)$. Then

$$J_\varepsilon(u) \geq J_\varepsilon(u_\lambda) + \frac{\varepsilon}{2} \int_\Omega |\nabla(u - u_\lambda)|^2.$$

Since $u - u_\lambda$ is not harmonic, we can choose

$$\rho := \frac{2\, R^2}{\varepsilon} > 0$$

and u_λ becomes a strict local minimal for J_ε, which proves $(*)$.

Next, using the definition of φ_1 given in (1.5), we have

$$\begin{aligned} J_\varepsilon(t\varphi_1) = \frac{\lambda_1}{2} t^2 + \frac{\varepsilon}{2}\lambda_1 t^2 - \varepsilon\lambda_1 t \int_\Omega \varphi_1 u_\lambda \\ + \frac{\varepsilon}{2}\int_\Omega |\nabla u_\lambda|^2 - \int_\Omega F(t\varphi_1),\ \forall\, t \in \mathbb{R}. \end{aligned} \tag{4.5}$$

Recall that $\lim_{t \mapsto +\infty} (f(t) - a\, t)$ is finite, then there exists $\beta \in \mathbb{R}$ such that

$$f(t) \geq a\, t + \beta,\ \ \forall\, t > 0.$$

Hence

$$F(t) \geq \frac{a\,\lambda}{2} t^2 + \beta\,\lambda\, t,\ \ \forall\, t > 0.$$

This yields

$$\begin{aligned} \frac{J_\varepsilon(t\varphi_1)}{t^2} \leq \Big(\frac{\lambda_1}{2} + \frac{\varepsilon}{2}\lambda_1 - \frac{a\,\lambda}{2}\Big) - \frac{\varepsilon\lambda_1}{t}\int_\Omega \varphi_1 u_\lambda \\ - \frac{\beta\lambda}{t}\int_\Omega \varphi_1 + \frac{\varepsilon}{2t^2}\int_\Omega |\nabla u_\lambda|^2, \end{aligned}$$

which implies that

$$\limsup_{t \to +\infty} \frac{1}{t^2}\, J_\varepsilon(t\varphi_1) \leq \frac{\lambda_1 + \varepsilon_0\lambda_1 - a\,\lambda}{2} < 0,\ \ \forall\, \varepsilon \in [0,\ \varepsilon_0].$$

Therefore

$$\lim_{t \to +\infty} J_\varepsilon(t\varphi_1) = -\infty.$$

So, there exists $v_0 \in E$ such that

$$J_\varepsilon(v_0) \leq J_\varepsilon(u_\lambda),\ \ \forall\, \varepsilon \in [0,\ \varepsilon_0],$$

and $(**)$ is proved. Finally, for all $\varepsilon \in [0,\ \varepsilon_0]$, let v_ε (respectively. c_ε) be the critical point (respectively. critical value) of J_ε.

Remark 4.3. The fact that J_ε increases with ε implies that for all $\varepsilon \in [0, \varepsilon_0]$, $c_\varepsilon \in [c_0, c_{\varepsilon_0}[$. Then, c_ε is uniformly bounded. Thus, for all $\varepsilon \in [0,\ \varepsilon_0]$, the critical point v_ε satisfies $\|v_\varepsilon - u_\lambda\| \geq R$.

Recall that for any $\varepsilon \in [0, \varepsilon_0]$, the function v_ε belongs to E and satisfies

$$-\Delta v_\varepsilon + cv_\varepsilon = \frac{\lambda}{1+\varepsilon} f(v_\varepsilon) + \frac{\lambda\varepsilon}{1+\varepsilon} f(u_\lambda) \text{ in } \Omega \tag{4.6}$$

and

$$J_\varepsilon(v_\varepsilon) = c_\varepsilon. \tag{4.7}$$

Thanks to Lemma 4.2, Remark 4.3, (4.6) and (4.7) , there exists $v \in E$ such that

$$v_\varepsilon \to v \text{ in } E, \text{ as } \varepsilon \to 0,$$

satisfying

$$-\Delta v + cv = \lambda f(v) \text{ in } \Omega.$$

From Remark 4.3, we see that $v \neq u_\lambda$, which can be also proved using the same arguments of Mironescu and Rădulescu in [10]. Indeed, note that v_ε is a solution to (4.6) which is different from the unique stable solution u_λ. Then, v_ε is unstable, that is,

$$\eta_1\left(\frac{\lambda}{1+\varepsilon}, v_\varepsilon\right) < 0,$$

since (4.7) can be written as

$$-\Delta v_\varepsilon + cv_\varepsilon = g_\varepsilon(v_\varepsilon) + h_\varepsilon(x), \tag{4.8}$$

where g_ε is convex, positive and h_ε is positive. Thus, if (4.8) has solutions satisfying $v_\varepsilon = 0$ on $\partial\Omega$, then it has a minimal one, say w_ε, which is stable. Now, thanks to Theorem 1.4, all other solutions v_ε of (4.8) are unstable.

The next lemma states that the limit of a sequence of unstable solutions is also unstable (the proof is similar to that of Lemma 11 in [10]).

Lemma 4.4. *Let $u_n \rightharpoonup u$ in $H^2(\Omega) \cap H_0^1(\Omega)$ and $\mu_n \to \mu$ be such that $\eta_1(\mu_n, u_n) < 0$. Then, $\eta_1(\mu, u) \leq 0$.*

Finally, the fact that the function v belongs to $C^4(\bar{\Omega}) \cap E$ follows from a bootstrap argument.

Proof of Theorem 1.6 (iii) **(a)**. Thanks to Lemma 3.1, if (i) does not occur, then there is a sequence of positives scalars (μ_n) and a sequence (v_n) of unstable solutions to (P_{μ_n}) such that $v_n \to v$ in $L^1_{loc}(\Omega)$ as $\mu_n \to \lambda_1/a$ for some function v.

We first claim that (v_n) cannot be bounded in E. Otherwise, let $w \in E$ be such that, up to a subsequence,

$$v_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad v_n \to w \text{ strongly in } L^2(\Omega).$$

Therefore,

$$-\Delta v_n + cv_n \to -\Delta w + cw \text{ in } \mathcal{D}'(\Omega) \quad \text{and} \quad f(v_n) \to f(w) \text{ in } L^2(\Omega),$$

which implies that $-\Delta w + cw = \frac{\lambda_1}{a} f(w)$ in Ω. It follows that $w \in E$ and solves $(P_{\lambda_1/a})$. From Lemma 4.2, we deduce that

$$\eta_1\left(\frac{\lambda_1}{a}, w\right) \le 0. \tag{4.9}$$

Relation (4.9) shows that $w \neq u_{\lambda_1/a}$ which contradicts the fact that $(P_{\lambda_1/a})$ has a unique solution. Now, since $-\Delta v_n + cv_n = \mu_n f(v_n)$, the unboundedness of (v_n) in E implies that this sequence is unbounded in $L^2(\Omega)$, too. To see this, let

$$v_n = k_n\, w_n, \quad \text{where} \quad k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \to \infty.$$

Then

$$-\Delta w_n + cw_n = \frac{\mu_n}{k_n} f(v_n) \to 0 \qquad \text{in} \quad L^1_{loc}(\Omega).$$

So, we have convergence also in the sense of distributions and (w_n) is seen to be bounded in E with standard arguments. We obtain

$$-\Delta w + cw = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

The desired contradiction is obtained since $w \in E$. □

Proof of Theorem 1.6 (iii) **(b)**. As before, it is enough to prove the $L^2(\Omega)$ boundedness of v_λ near λ^* and to use the uniqueness property of u^*. Assume that $\|v_n\|_2 \to \infty$ as $\mu_n \to \lambda^*$, where v_n is a solution to (P_{μ_n}). We write again $v_n = l_n\, w_n$. Then,

$$-\Delta w_n + cw_n = \frac{\mu_n}{l_n} f(v_n). \tag{4.10}$$

The fact that the right-hand side of (4.10) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in E. Let (w_n) be such that (up to a subsequence)

$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \to w \text{ strongly in } L^2(\Omega).$$

A computation already done shows that

$$-\Delta w + cw = \lambda^* a w, \quad w \ge 0 \quad \text{and } \|w\|_2 = 1,$$

which forces λ^* to be λ_1/a. This contradiction concludes the proof. □

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