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FIXED POINT RESULTS IN ORBITALLY 0-σ-COMPLETE METRIC-LIKE SPACES VIA RATIONAL CONTRACTIVE CONDITIONS

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Abstract. Inspired by the notion of metric-like space as a generalization of partial metric spaces, we present some results on fixed points in the framework of F-orbitally 0-σ-complete metric-like spaces. In this paper we bring into play two alternatives of rational contraction conditions. Some consequences are obtained and examples are presented, showing how the given results can be used for proving the existence of (common) fixed points.

1. Introduction

Nonlinear and convex analysis have as one of their goals solving equilibrium problems arising in applied sciences. In fact, a lot of these problems can be modelled in an abstract form of an equation (algebraic, functional,

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differential, integral, etc.), and this can be further transferred into a form of a fixed point problem of a certain operator. In this context, finding solutions of fixed point problems, or at least proving that such solutions exist and can be approximately computed, is a very interesting area of research.

The Banach contraction principle is one of the cornerstones in the development of nonlinear analysis, in general, and metric fixed point theory, in particular. This principle was extended and improved in many directions and various fixed point theorems were established. Two usual ways for extending and improving the Banach contraction principle are obtained by:

(1) replacing the underlying metric space by certain generalized metric space;
(2) changing the contraction condition to more general ones.

There exist many generalizations of the concept of metric spaces in the literature. In the development of fixed point theory, one more has been added as the notion of a partial metric space. Matthews [14] introduced this concept as a part of the study of denotational semantics of dataflow networks. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Further, Romaguera showed in [17] that in this context it is more natural to use the so-called 0-completeness.

In 2012, Amini-Harandi [8] reintroduced the notion of metric-like space (treated earlier by Hitzler and Seda in [9] under the name of dislocated metric space) as a further generalization of partial metric space. In metric-like spaces, the assumption of smallest self distance of partial metric spaces was removed and the triangular inequality of partial metric was replaced by a weaker one. Amini-Harandi defined the σ-completeness of metric-like spaces and obtained some fixed point results. Further, Shukla et al. [20] introduced the notion of 0-σ-complete metric-like spaces and generalized the results of [8]. Fixed point results in metric-like (dislocated) and some related spaces were obtained also in [1, 2, 3, 10, 13, 19].

Completeness of the underlying space was relaxed and replaced by the so-called orbital completeness in the papers by Browder and Petryshin [5] and Ćirić [6]. Karapınar and Erhan [12] have extended the notions of orbitally completeness and orbitally continuity to partial metric spaces and obtained the corresponding fixed point results. Nashine et al. [16] proved some fixed point theorems in orbitally 0-complete partial metric space.

On the other hand, Dass and Gupta [7] and Jaggi [11] were the first to prove fixed point theorems in metric spaces using contractive conditions involving rational expressions. Nashine and Erdal [15] generalized the results from [12] and obtained fixed point results in orbitally complete partial metric spaces by using conditions involving a rational expression.

In this paper, motivated by the results discussed above, we extend the results of [16] to the setting of orbitally 0-σ-orbitally complete metric-like spaces, using contraction conditions involving rational expressions. A common fixed point result of this kind for two self-mappings is also obtained. Some consequences are deduced and examples are given to support the usability of our results.

2. Preliminaries

First, we recall some definitions and facts which will be used throughout the paper.

Definition 2.1. ([14]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space.

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [4, 14]. Obviously, one of the main features of this generalization of metric spaces is the so-called "non-zero self-distance". It is also a property of the following further generalization.

Definition 2.2. ([8]) A metric-like on a nonempty set X is a function $\sigma : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$,

(σ_1) $\sigma(x,y) = 0 \Rightarrow x = y$;
(σ_2) $\sigma(x,y) = \sigma(y,x)$;
(σ_3) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$.

A metric-like space is a pair (X,σ) such that X is a nonempty set and σ is a metric-like on X.

Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}, \text{ for all } x \in X \text{ and } \varepsilon > 0.$$

It is obvious that each metric space is a partial metric space and each partial metric space is a metric-like space, but the converse may not be true.

Example 2.3. ([8]) Let $X = \{0,1\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2, & \text{if } \ x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X,σ) is a metric-like space, but it is neither a metric space nor a partial metric space, since $\sigma(0,0) > \sigma(0,1)$.

Definition 2.4. ([8, 20]) Let (X,σ) be a metric-like space. Then

(1) A sequence $\{x_n\}$ in X converges to a point $x \in X$ if $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x)$. The sequence $\{x_n\}$ is said to be σ-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite. The space (X,σ) is called complete if for each σ-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m).$$

(2) A sequence $\{x_n\}$ in (X,σ) is called a 0-σ-Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n,x_m) = 0$. The space (X,σ) is said to be 0-σ-complete if every 0-σ-Cauchy sequence in X converges (in τ_σ) to a point $x \in X$ such that $\sigma(x,x) = 0$.

Remark 2.5. ([8]) Let $X = \{0,1\}$, let $\sigma(x,y) = 1$ for each $x,y \in X$, and let $x_n = 1$ for each $n \in N$. Then it is easy to see that $x_n \to 0$ and $x_n \to 1$, and so in metric-like spaces the limit of a convergent sequence is not necessarily unique.

Lemma 2.6. ([13]) *Let (X,σ) be a metric-like space.*

(a) *If $x,y \in X$ then $\sigma(x,y) = 0$ implies that $\sigma(x,x) = \sigma(y,y) = 0$.*
(b) *If a sequence $\{x_n\}$ in X converges to some $x \in X$ with $\sigma(x,x) = 0$ then $\lim_{n\to\infty} \sigma(x_n,y) = \sigma(x,y)$ for all $y \in X$.*

Remark 2.7. ([20]) If a metric-like space is σ-complete, then it is 0-σ-complete.
The converse assertion does not hold as the example given below shows.

Example 2.8. ([20]) Let $X = [0,1) \cap Q$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2x, & \text{if } \ x = y, \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

for all $x,y \in X$. Then (X,σ) is a metric-like space. Note that (X,σ) is not a partial metric space, as $\sigma(1,1) = 2 > 1 = \sigma(1,0)$. Now, it is easy to see that (X,σ) is a 0-σ-complete metric-like space, while it is not a σ-complete metric-like space.

Recall that the set $\mathcal{O}(x_0;F)=\{F^n x_0 : n=0,1,2,\dots\}$ is called the orbit of a self-map $F:X\to X$ at the point $x_0\in X$. Now we introduce the notions of 0-σ-orbital completeness and orbital continuity in metric-like spaces.

Definition 2.9. Let (X,σ) be a metric-like space and $F:X\to X$. The space (X,σ) is said to be F-orbitally 0-σ-complete if every 0-Cauchy sequence contained in $\mathcal{O}(x;F)$ (for some x in X) converges in X to a point z such that $\sigma(z,z)=0$.

Here, it can be pointed out that every 0-σ-complete metric-like space is F-orbitally 0-σ-complete for any F, but the converse does not hold.

Definition 2.10. A self-map F defined on a partial metric space (X,σ) is said to be orbitally continuous at a point z in X if for any sequence $\{x_n\}\subset\mathcal{O}(x;F)$ (for some $x\in X$), $x_n\to z$ as $n\to\infty$ (in τ_σ) implies that $Fx_n\to Fz$ as $n\to\infty$.

Clearly, every continuous self-mapping of a metric-like space is orbitally continuous, but not conversely.

3. Fixed point results for a single mapping

In what follows, we will denote by Ω the set of functions $\omega:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(Ω_1) ω is continuous;
(Ω_2) $\omega(t)<t$ for all $t>0$.

Obviously, if $\omega\in\Omega$, then $\omega(0)=0$ and $\omega(t)\le t$ for all $t\ge 0$.

Theorem 3.1. *Let (X,σ) be an F-orbitally 0-σ-complete metric-like space, where $F:X\to X$ is an orbitally continuous mapping. Suppose there exist $\alpha,\beta,\gamma\in[0,1)$ with $\alpha+\frac{5}{2}\beta+3\gamma<1$, such that*

$$\sigma(Fx,Fy)\le \alpha\,\Lambda(x,y)+\beta\,\Delta(x,y)+\gamma\,\Theta(x,y), \tag{3.1}$$

for all $x,y\in\overline{\mathcal{O}(x_0,F)}$ (for some $x_0\in X$), where

$$\Lambda(x,y)=\omega\left(\frac{\sigma(x,Fx)\sigma(y,Fy)}{1+\sigma(x,y)}\right), \tag{3.2}$$

$$\begin{aligned}\Delta(x,y)=\max\Big\{&\omega(\sigma(x,y)),\omega(\sigma(x,Fx)),\omega(\sigma(y,Fy)),\\ &\omega\left(\frac{\sigma(y,Fx)+\sigma(x,Fy)}{2}\right)\Big\},\end{aligned} \tag{3.3}$$

$$\Theta(x,y)=\min\{\omega(\sigma(x,Fx)),\omega(\sigma(y,Fy)),\omega(\sigma(x,Fy)),\omega(\sigma(y,Fx))\}, \tag{3.4}$$

and $\omega\in\Omega$. Then there exists $z\in X$ such that $Fz=z$ and $\sigma(z,z)=0$.

Proof. Starting with the given point $x_0 \in X$, define $x_n = F^n x_0$ and so $x_n = Fx_{n-1}$ for $n \in \mathbb{N}$. If there exists $n_0 \in \{1, 2, \dots\}$ such that $\sigma(x_{n_0}, x_{n_0-1}) = 0$, then by (σ_1) we have $x_{n_0-1} = x_{n_0} = Fx_{n_0-1}$. By Lemma 2.6 we get $\sigma(x_{n_0-1}, x_{n_0-1}) = \sigma(x_{n_0}, x_{n_0}) = 0$. We are done in this case.

Suppose now that $\sigma(x_n, x_{n+1}) > 0$ for all $n \geq 0$. We claim that for all $n \geq 0$, we have

$$\sigma(x_{n+1}, x_{n+2}) \leq h^{n+1}\sigma(x_0, x_1), \tag{3.5}$$

for some h, $0 \leq h < 1$. From (3.1) with $x = x_{n+1}$ and $y = x_n$ we get

$$\begin{aligned} \sigma(x_{n+2}, x_{n+1}) &= \sigma(Fx_{n+1}, Fx_n) \\ &\leq \alpha\, \Lambda(x_{n+1}, x_n) + \beta\, \Delta(x_{n+1}, x_n) + \gamma\, \Theta(x_{n+1}, x_n). \end{aligned} \tag{3.6}$$

By (3.2) and the properties of $\omega \in \Omega$, we have

$$\Lambda(x_{n+1}, x_n) = \omega\left(\sigma(x_n, x_{n+1})\frac{\sigma(x_{n+1}, x_{n+2})}{1 + \sigma(x_{n+1}, x_n)}\right) < \sigma(x_{n+1}, x_{n+2}).$$

By (3.3) and (3.4), we have

$$\begin{aligned} \Delta(x_{n+1}, x_n) &= \max\Big\{\omega(\sigma(x_n, x_{n+1})), \omega(\sigma(x_n, Fx_n)), \omega(\sigma(x_{n+1}, Fx_{n+1})), \\ &\qquad \omega\left(\frac{\sigma(x_{n+1}, Fx_n) + \sigma(x_n, Fx_{n+1})}{2}\right)\Big\} \\ &= \max\Big\{\omega(\sigma(x_n, x_{n+1})), \omega(\sigma(x_{n+1}, x_{n+2})), \\ &\qquad \omega\left(\frac{\sigma(x_{n+1}, x_{n+1}) + \sigma(x_n, x_{n+2})}{2}\right)\Big\} \end{aligned}$$

and

$$\begin{aligned} \Theta(x_{n+1}, x_n) = \min\{&\omega(\sigma(x_{n+1}, x_{n+2})), \omega(\sigma(x_n, x_{n+1})), \\ &\omega(\sigma(x_{n+1}, x_{n+1})), \omega(\sigma(x_n, x_{n+2}))\}. \end{aligned}$$

Consider the following possible cases.

(1) If $\Delta(x_{n+1}, x_n) = \omega(\sigma(x_{n+1}, x_{n+2}))$ and $\Theta(x_{n+1}, x_n) = \omega(\sigma(x_{n+1}, x_{n+2}))$, by (3.6) and using the fact that $\omega(t) < t$ for all $t > 0$, we have

$$\sigma(x_{n+1}, x_{n+2}) < (\alpha + \beta + \gamma)\sigma(x_{n+1}, x_{n+2}) < \sigma(x_{n+1}, x_{n+2}),$$

a contradiction.

(2) If $\Delta(x_{n+1}, x_n) = \omega(\sigma(x_n, x_{n+1}))$ and $\Theta(x_{n+1}, x_n) = \omega(\sigma(x_{n+1}, x_{n+2}))$, we get

$$\sigma(x_{n+1}, x_{n+2}) < (\alpha + \gamma)\sigma(x_{n+1}, x_{n+2}) + \beta\sigma(x_n, x_{n+1}),$$

that is

$$\sigma(x_{n+1}, x_{n+2}) < \frac{\beta}{1 - \alpha - \gamma}\sigma(x_n, x_{n+1}).$$

(3) If $\Delta(x_{n+1},x_n)=\omega\left(\frac{\sigma(x_{n+1},x_{n+1})+\sigma(x_n,x_{n+2})}{2}\right)$ and $\Theta(x_{n+1},x_n)=\omega(\sigma(x_{n+1},x_{n+2}))$, we get

$$\sigma(x_{n+1},x_{n+2})<(\alpha+\gamma)\sigma(x_{n+1},x_{n+2})+\frac{\beta}{2}(\sigma(x_{n+1},x_{n+1})+\sigma(x_n,x_{n+2})).$$

By (σ_3), we have

$$\sigma(x_n,x_{n+2})\le\sigma(x_n,x_{n+1})+\sigma(x_{n+1},x_{n+2}).$$

Therefore we have

$$\sigma(x_{n+1},x_{n+2})<(\alpha+\gamma)\sigma(x_{n+1},x_{n+2})+\frac{\beta}{2}(3\sigma(x_{n+1},x_{n+2})+\sigma(x_n,x_{n+1}))$$

which implies that

$$\sigma(x_{n+1},x_{n+2})<\frac{\beta}{2-2\alpha-3\beta-2\gamma}\sigma(x_n,x_{n+1}).$$

(4) If $\Delta(x_{n+1},x_n)=\omega(\sigma(x_{n+1},x_{n+2}))$ and $\Theta(x_{n+1},x_n)=\omega(\sigma(x_n,x_{n+1}))$, by (3.6) and using the fact that $\omega(t)<t$ for all $t>0$, we have

$$\sigma(x_{n+1},x_{n+2})<(\alpha+\beta)\sigma(x_{n+1},x_{n+2})+\gamma\sigma(x_n,x_{n+1}),$$

that is,

$$\sigma(x_{n+1},x_{n+2})<\frac{\gamma}{1-\alpha-\beta}\sigma(x_n,x_{n+1}).$$

(5) If $\Delta(x_{n+1},x_n)=\omega(\sigma(x_n,x_{n+1}))$ and $\Theta(x_{n+1},x_n)=\omega(\sigma(x_n,x_{n+1}))$, we get

$$\sigma(x_{n+1},x_{n+2})<\alpha\sigma(x_{n+1},x_{n+2})+(\beta+\gamma)\sigma(x_n,x_{n+1}),$$

that is

$$\sigma(x_{n+1},x_{n+2})<\frac{\beta+\gamma}{1-\alpha}\sigma(x_n,x_{n+1}).$$

(6) If $\Delta(x_{n+1},x_n)=\omega\left(\frac{\sigma(x_{n+1},x_{n+1})+\sigma(x_n,x_{n+2})}{2}\right)$ and $\Theta(x_{n+1},x_n)=\omega(\sigma(x_n,x_{n+1}))$, we get

$$\sigma(x_{n+1},x_{n+2})<\alpha\sigma(x_{n+1},x_{n+2})+\frac{\beta}{2}(\sigma(x_{n+1},x_{n+1})+\sigma(x_n,x_{n+2}))$$
$$+\gamma\sigma(x_n,x_{n+1}).$$

By (σ_3), we have

$$\sigma(x_n,x_{n+2})\le\sigma(x_n,x_{n+1})+\sigma(x_{n+1},x_{n+2}).$$

Therefore we have

$$\sigma(x_{n+1},x_{n+2})<\alpha\sigma(x_{n+1},x_{n+2})+\frac{\beta}{2}(\sigma(x_n,x_{n+1})+3\sigma(x_{n+1},x_{n+2}))$$
$$+\gamma\sigma(x_n,x_{n+1})$$

which implies that

$$\sigma(x_{n+1}, x_{n+2}) < \frac{\beta + 2\gamma}{2 - 2\alpha - 3\beta}\sigma(x_n, x_{n+1}).$$

(7) If $\Delta(x_{n+1}, x_n) = \omega(\sigma(x_{n+1}, x_{n+2}))$ and $\Theta(x_{n+1}, x_n) = \omega(\sigma(x_n, x_{n+2}))$, by (3.6) and (σ_3), and using the fact that $\omega(t) < t$ for all $t > 0$, we have

$$\begin{aligned}\sigma(x_{n+1}, x_{n+2}) &< (\alpha + \beta)\sigma(x_{n+1}, x_{n+2}) + \gamma(\sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}))\\ &= (\alpha + \beta + \gamma)\sigma(x_{n+1}, x_{n+2}) + \gamma\sigma(x_n, x_{n+1})\end{aligned}$$

that is,

$$\sigma(x_{n+1}, x_{n+2}) < \frac{\gamma}{1 - \alpha - \beta - \gamma}\sigma(x_n, x_{n+1}).$$

(8) If $\Delta(x_{n+1}, x_n) = \omega(\sigma(x_n, x_{n+1}))$ and $\Theta(x_{n+1}, x_n) = \omega(\sigma(x_n, x_{n+2}))$, we get

$$\begin{aligned}\sigma(x_{n+1}, x_{n+2}) &< \alpha\sigma(x_{n+1}, x_{n+2}) + \beta\sigma(x_n, x_{n+1})\\ &\quad + \gamma(\sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}))\\ &= (\alpha + \gamma)\sigma(x_{n+1}, x_{n+2}) + (\beta + \gamma)\sigma(x_n, x_{n+1}),\end{aligned}$$

that is

$$\sigma(x_{n+1}, x_{n+2}) < \frac{\beta + \gamma}{1 - \alpha - \gamma}\sigma(x_n, x_{n+1}).$$

(9) If $\Delta(x_{n+1}, x_n) = \omega\left(\frac{\sigma(x_{n+1}, x_{n+1}) + \sigma(x_n, x_{n+2})}{2}\right)$ and $\Theta(x_{n+1}, x_n) = \omega(\sigma(x_n, x_{n+2}))$, we get

$$\begin{aligned}\sigma(x_{n+1}, x_{n+2}) &< \alpha\sigma(x_{n+1}, x_{n+2}) + \beta\left(\frac{\sigma(x_{n+1}, x_{n+1}) + \sigma(x_n, x_{n+2})}{2}\right)\\ &\quad + \gamma\sigma(x_n, x_{n+2})\\ &\le \alpha\sigma(x_{n+1}, x_{n+2}) + \frac{\beta}{2}(\sigma(x_n, x_{n+1}) + 3\sigma(x_{n+1}, x_{n+2}))\\ &\quad + \gamma(\sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}))\\ &= (\alpha + \gamma + 3\frac{\beta}{2})\sigma(x_{n+1}, x_{n+2}) + (\gamma + \frac{\beta}{2})\sigma(x_n, x_{n+1}).\end{aligned}$$

This implies that

$$\sigma(x_{n+1}, x_{n+2}) < \frac{\beta + 2\gamma}{2 - 2\alpha - 3\beta - 2\gamma}\sigma(x_n, x_{n+1}).$$

Similarly the cases for $\Theta(x_{n+1}, x_n) = \omega(\sigma(x_{n+1}, x_{n+1}))$ can be discussed using property (σ_3).

Denote

$$h := \max\Big\{ \frac{\beta}{1-\alpha-\gamma}, \frac{\beta}{2-2\alpha-3\beta-2\gamma}, \frac{\gamma}{1-\alpha-\beta}, \frac{\beta+\gamma}{1-\alpha}, \frac{\beta+2\gamma}{2-2\alpha-3\beta}, \frac{\gamma}{1-\alpha-\beta-\gamma}, \frac{\beta+\gamma}{1-\alpha-\gamma}, \frac{\beta+2\gamma}{2-2\alpha-3\beta-2\gamma} \Big\} < \frac{1}{2} \quad \left(\text{since } \alpha + \frac{5}{2}\beta + 3\gamma < 1, \text{ by assumption}\right). \tag{3.7}$$

Thus we have

$$\sigma(x_{n+1}, x_{n+2}) \le h\sigma(x_n, x_{n+1}) \le h^2\sigma(x_{n-1}, x_n) \le \cdots \le h^{n+1}\sigma(x_0, x_1). \tag{3.8}$$

We will show that $\{x_n\}$ is a 0-σ-Cauchy sequence. Take any $m, n \in \mathbb{N}$. Then, using (3.8) and the triangle inequality (σ_3) for metric-like σ we have

$$\begin{aligned} \sigma(x_n, x_{n+m}) &\le \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+m}) \\ &\le \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \sigma(x_{n+2}, x_{n+m}). \end{aligned}$$

Inductively, we have

$$\begin{aligned} 0 &\le \sigma(x_n, x_{n+m}) \le \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \cdots + \sigma(x_{n+m-1}, x_{n+m}) \\ &\le (h^n + h^{n+1} + \cdots + h^{n+m-1})\sigma(x_0, Fx_0) \\ &= h^n(1 + h + \cdots + h^{m-1})\sigma(x_0, Fx_0) \\ &\le \frac{h^n}{1-h}\sigma(x_0, Fx_0). \end{aligned}$$

Therefore, since $0 \le h < 1$, taking the limit as $n \to \infty$, we have

$$\lim_{m,n\to\infty} \sigma(x_n, x_m) = 0.$$

Hence, we conclude that $\{x_n\} = \{F^n x_0\}$ is a 0-σ-Cauchy sequence in $\mathcal{O}(x_0, F)$. Since (X, σ) is F-orbitally 0-σ-complete (for x_0), the sequence $\{F^n x_0\}$ converges, say

$$\lim_{n\to\infty} \sigma(F^n x_0, z) = \sigma(z, z) = 0. \tag{3.9}$$

We shall prove that z is a fixed point of F.

Suppose that $\sigma(z, Fz) > 0$. Since F is orbitally continuous, we have

$$\lim_{n\to\infty} \sigma(FF^n x_0, Fz) = \sigma(Fz, Fz). \tag{3.10}$$

By the triangle inequality (σ_3), we have

$$\begin{aligned} \sigma(z, Fz) &\le \sigma(z, F^{n+1}x_0) + \sigma(F^{n+1}x_0, Fz) \\ &= \sigma(z, x_{n+1}) + \sigma(F^{n+1}x_0, Fz). \end{aligned}$$

Passing to the limit as $n \to \infty$ and using Lemma 2.6 with (3.9) and (3.10) we obtain

$$\begin{aligned}\sigma(z, Fz) &\le \lim_{n\to\infty} \sigma(z, x_{n+1}) + \lim_{n\to\infty} \sigma(F^{n+1}x_0, Fz) \\ &= \sigma(Fz, Fz).\end{aligned} \tag{3.11}$$

Using (3.1) with $x = Fz$ and $y = z$, we get

$$\begin{aligned}\sigma(F^2z, Fz) &\le \alpha\, \omega\left(\frac{\sigma(z, Fz)\sigma(Fz, F^2z)}{1+\sigma(Fz, z)}\right) + \beta\, \Delta(Fz, z) + \gamma\, \Theta(Fz, z) \\ &\le \alpha\, \sigma(Fz, F^2z) + \beta\, \Delta(Fz, z) + \gamma\, \Theta(Fz, z),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\Delta(Fz, z) &= \max\Big\{\omega(\sigma(Fz, z)), \omega(\sigma(Fz, F^2z)), \omega(\sigma(z, Fz)), \\ &\qquad \omega\left(\frac{\sigma(z, F^2z) + \sigma(Fz, Fz)}{2}\right)\Big\} \\ &< \max\left\{\sigma(Fz, z), \sigma(Fz, F^2z), \frac{\sigma(z, F^2z) + \sigma(Fz, Fz)}{2}\right\}\end{aligned}$$

and

$$\begin{aligned}\Theta(Fz, z) &= \min\{\omega(\sigma(Fz, F^2z)), \omega(\sigma(z, Fz)), \omega(\sigma(Fz, Fz)), \omega(\sigma(z, F^2z))\} \\ &< \min\{\sigma(Fz, F^2z), \sigma(z, Fz), \sigma(Fz, Fz), \sigma(z, F^2z)\}.\end{aligned}$$

Consider the following possible cases.

- If $\Delta(Fz, z) < \sigma(Fz, z)$ and $\Theta(Fz, z) < \sigma(Fz, F^2z)$, then we obtain from (3.12)

$$\sigma(F^2z, Fz) \le \alpha\, \sigma(Fz, F^2z) + \beta\, \sigma(Fz, z) + \gamma\, \sigma(Fz, F^2z),$$

that is,

$$\sigma(F^2z, Fz) \le \frac{\beta}{1-\alpha-\gamma}\sigma(Fz, z).$$

- If $\Delta(Fz, z) < \sigma(Fz, F^2z)$ and $\Theta(Fz, z) < \sigma(Fz, F^2z)$, then we obtain from (3.12)

$$\sigma(F^2z, Fz) \le \alpha\, \sigma(Fz, F^2z) + \beta\, \sigma(Fz, F^2z) + \gamma\, \sigma(Fz, F^2z),$$

a contradiction.

• If $\Delta(Fz,z) < \frac{\sigma(z,F^2z)+\sigma(Fz,Fz)}{2}$ and $\Theta(Fz,z) < \sigma(Fz,F^2z)$, then we obtain from (3.12) and (σ_3)

$$\begin{aligned}\sigma(F^2z,Fz) &\le \alpha\,\sigma(Fz,F^2z) + \beta\,\frac{\sigma(z,F^2z)+\sigma(Fz,Fz)}{2} + \gamma\,\sigma(Fz,F^2z),\\ &\le \alpha\,\sigma(Fz,F^2z) + \frac{\beta}{2}\left(\sigma(z,Fz)+3\sigma(Fz,F^2z)\right) + \gamma\,\sigma(Fz,F^2z),\end{aligned}$$

that is,

$$\sigma(F^2z,Fz) \le \frac{\beta}{2-2\alpha-3\beta-2\gamma}\sigma(Fz,z).$$

• If $\Delta(Fz,z) < \sigma(Fz,z)$ and $\Theta(Fz,z) < \sigma(z,Fz)$, then we obtain from (3.12)

$$\sigma(F^2z,Fz) \le \alpha\,\sigma(Fz,F^2z) + \beta\,\sigma(Fz,z) + \gamma\,\sigma(z,Fz),$$

that is,

$$\sigma(F^2z,Fz) \le \frac{\beta+\gamma}{1-\alpha}\sigma(Fz,z).$$

• If $\Delta(Fz,z) < \sigma(Fz,F^2z)$ and $\Theta(Fz,z) < \sigma(z,Fz)$, then we obtain from (3.12)

$$\sigma(F^2z,Fz) \le \alpha\,\sigma(Fz,F^2z) + \beta\,\sigma(Fz,F^2z) + \gamma\,\sigma(z,Fz),$$

that is,

$$\sigma(F^2z,Fz) \le \frac{\gamma}{1-\alpha-\beta}\sigma(Fz,z).$$

• If $\Delta(Fz,z) < \frac{\sigma(z,F^2z)+\sigma(Fz,Fz)}{2}$ and $\Theta(Fz,z) < \sigma(z,Fz)$, then we obtain from (3.12) and (σ_3)

$$\begin{aligned}\sigma(F^2z,Fz) &\le \alpha\,\sigma(Fz,F^2z) + \beta\,\frac{\sigma(z,F^2z)+\sigma(Fz,Fz)}{2} + \gamma\,\sigma(z,Fz)\\ &\le \alpha\,\sigma(Fz,F^2z) + \frac{\beta}{2}(\sigma(z,Fz)+3\sigma(Fz,F^2z)) + \gamma\,\sigma(z,Fz),\end{aligned}$$

that is,

$$\sigma(F^2z,Fz) \le \frac{\beta+2\gamma}{2-2\alpha-3\beta}\sigma(Fz,z).$$

• If $\Delta(Fz,z) < \sigma(Fz,z)$ and $\Theta(Fz,z) < \sigma(z,F^2z)$, then we obtain from (3.12) and (σ_3)

$$\begin{aligned}\sigma(F^2z,Fz) &\le \alpha\,\sigma(Fz,F^2z) + \beta\,\sigma(Fz,z) + \gamma\,\sigma(z,F^2z)\\ &\le \alpha\,\sigma(Fz,F^2z) + \beta\,\sigma(Fz,z) + \gamma\,(\sigma(z,Fz)+\sigma(Fz,F^2z)),\end{aligned}$$

that is,

$$\sigma(F^2z,Fz) \le \frac{\beta+\gamma}{1-\alpha-\gamma}\sigma(Fz,z).$$

• If $\Delta(Fz, z) < \sigma(Fz, F^2z)$ and $\Theta(Fz, z) < \sigma(z, F^2z)$, then we obtain from (3.12) and (σ_3)

$$\sigma(F^2z, Fz) \leq \alpha\, \sigma(Fz, F^2z) + \beta\, \sigma(Fz, F^2z) + \gamma\, (\sigma(z, Fz) + \sigma(Fz, F^2z)),$$

that is,

$$\sigma(F^2z, Fz) \leq \frac{\gamma}{1-\alpha-\beta-\gamma}\sigma(Fz, z).$$

• If $\Delta(Fz, z) < \frac{\sigma(z,F^2z)+\sigma(Fz,Fz)}{2}$ and $\Theta(Fz, z) < \sigma(z, F^2z)$, then we obtain from (3.12) and (σ_3)

$$\begin{aligned}\sigma(F^2z, Fz) &\leq \alpha\, \sigma(Fz, F^2z) + \beta\, \frac{\sigma(z, F^2z) + \sigma(Fz, Fz)}{2} \\ &\quad + \gamma\, (\sigma(z, Fz) + \sigma(Fz, F^2z)) \\ &\leq \alpha\, \sigma(Fz, F^2z) + \frac{\beta}{2}(\sigma(z, Fz) + 3\sigma(Fz, F^2z)) \\ &\quad + \gamma\, (\sigma(z, Fz) + \sigma(Fz, F^2z)),\end{aligned}$$

that is,

$$\sigma(F^2z, Fz) \leq \frac{\beta+2\gamma}{2-2\alpha-3\beta-2\gamma}\sigma(Fz, z).$$

Similarly the cases for $\Theta(x_{n+1}, x_n) < \omega(\sigma(x_{n+1}, x_{n+1}))$ can be discussed using property (σ_3). Using (3.7) we conclude that

$$\sigma(Fz, F^2z) \leq h\sigma(z, Fz). \tag{3.13}$$

Notice that due to (σ_3) we have

$$\sigma(Fz, Fz) \leq 2\sigma(F^2z, Fz). \tag{3.14}$$

Combining (3.11), (3.13) and (3.14), we get

$$\sigma(Fz, z) \leq \sigma(Fz, Fz) \leq 2\sigma(F^2z, Fz) \leq 2h\sigma(Fz, z) < \sigma(Fz, z),$$

since $2h < 1$ and $\sigma(Fz, z) > 0$. A contradiction.

Hence, $\sigma(Fz, z) = 0$. Therefore z is a fixed point of F and, by Lemma 2.6, $\sigma(Fz, Fz) = 0$. The proof is complete. □

The uniqueness of fixed point can be obtained under the following conditions.

Proposition 3.2. *Suppose that (X, σ) and F satisfy the conditions of Theorem 3.1 and, moreover, $\sigma(z, Fz) = 0$ for each fixed point z of F in $\overline{\mathcal{O}(x_0; F)}$. Then the fixed point of F is unique in $\overline{\mathcal{O}(x_0; F)}$.*

Proof. Assume that for some $u, v \in \overline{\mathcal{O}(x_0; F)}$, $\sigma(u, Fu) = \sigma(v, Fv) = 0$. Replace x by u and y by v in (3.1) to obtain

$$\sigma(u, v) = \sigma(Fu, Fv) \le \alpha\,\Lambda(u, v) + \beta\,\Delta(u, v) + \gamma\,\Theta(u, v),$$

where

$$\begin{aligned}
\Lambda(u, v) &= \omega\left(\frac{\sigma(v, Fv)\sigma(u, Fu)}{1 + \sigma(u, v)}\right) = 0,\\
\Delta(u, v) &= \max\left\{\omega(\sigma(u, v)), \omega(\sigma(u, Fu)), \omega(\sigma(v, Fv)), \omega\left(\frac{\sigma(v, Fu)+\sigma(u, Fv)}{2}\right)\right\}\\
&= \max\left\{\omega(\sigma(u, v)), 0, 0, \omega\left(\frac{\sigma(v, u) + \sigma(u, v)}{2}\right)\right\}\\
&< \sigma(u, v),\\
\Theta(u, v) &= \min\{\omega(\sigma(u, Fu)), \omega(\sigma(v, Fv)), \omega(\sigma(u, Fv)), \omega(\sigma(v, Fu))\} = 0.
\end{aligned}$$

Therefore we get

$$\sigma(u, v) \le \beta\,\sigma(u, v),$$

which is only possible if $\sigma(u, v) = 0$, that is $u = v$. □

We demonstrate the use of Theorem 3.1 with the help of the following example.

Example 3.3. Let $X = [0, +\infty)$ be equipped with the metric-like σ given by

$$\sigma(x, y) = \begin{cases} 3x, & \text{if } x = y,\\ \max\{x, y\}, & \text{otherwise.}\end{cases}$$

for $x, y \in X$. A mapping $F : X \to X$ is given by

$$Fx = \begin{cases} \frac{1}{5}x^2, & x \in [0, 1],\\ 5x, & x > 1,\end{cases}$$

and $\omega \in \Omega$ is given by $\omega(t) = \frac{1}{5}t$. Let $\alpha = \frac{1}{10}$, $\beta = \frac{1}{5}$ and $\gamma = \frac{1}{10}$ (hence, $\alpha + \frac{5}{2}\beta + 3\gamma < 1$). Taking $x_0 = 1$ we have that

$$\mathcal{O}(x_0; F) \subset \left\{\frac{1}{5^k} : k \in \mathbb{N} \cup \{0\}\right\}, \qquad \overline{\mathcal{O}(x_0; F)} = \mathcal{O}(x_0; F) \cup \{0\}.$$

All the conditions of Theorem 3.1 and Proposition 3.2 are satisfied. In particular, we shall check the contractive condition (3.1).

Suppose first that $x = \frac{1}{5^m}$, $y = \frac{1}{5^n}$ and, e.g., $n > m$. Then

$$\Lambda(x,y) = \frac{1}{5} \cdot \frac{\frac{1}{5^n} \cdot \frac{1}{5^m}}{1 + \frac{1}{5^m}} = \frac{1}{5} \cdot \frac{1}{5^n(5^m+1)},$$
$$\Delta(x,y) = \frac{1}{5} \max\left\{\frac{1}{5^m}, \frac{1}{5^m}, \frac{1}{5^n}, \frac{1}{2}\left(\frac{1}{5^n} + \max\left\{\frac{1}{5^m}, \frac{1}{5^{2n+1}}\right\}\right)\right\} = \frac{1}{5} \cdot \frac{1}{5^m},$$
$$\Theta(x,y) = \frac{1}{5} \min\left\{\frac{1}{5^m}, \frac{1}{5^n}, \frac{1}{5^m}, \max\left\{\frac{1}{5^n}, \frac{1}{5^{2m+1}}\right\}\right\} = \frac{1}{5} \cdot \frac{1}{5^n}$$

and

$$\sigma(Fx,Fy) = \sigma\left(\frac{1}{5^{2m+1}}, \frac{1}{5^{2n+1}}\right) = \frac{1}{5^{2m+1}} \leq \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5^m} = \beta\,\Delta(x,y)$$
$$< \alpha\,\Lambda(x,y) + \beta\,\Delta(x,y) + \gamma\,\Theta(x,y).$$

Suppose now that $x = y = \frac{1}{5^n}$. Then

$$\Lambda(x,y) = \frac{1}{5} \cdot \frac{\frac{1}{5^n} \cdot \frac{1}{5^n}}{1 + \frac{3}{5^n}} = \frac{1}{5^{n+1}(5^n+3)},$$
$$\Delta(x,y) = \frac{1}{5} \max\left\{\frac{3}{5^n}, \frac{1}{5^n}, \frac{1}{5^n}, \frac{1}{5^n}\right\} = \frac{3}{5^{n+1}},$$
$$\Theta(x,y) = \frac{1}{5} \min\left\{\frac{1}{5^n}, \frac{1}{5^n}, \frac{1}{5^n}, \frac{1}{5^n}\right\} = \frac{1}{5^{n+1}}$$

and

$$\sigma(Fx,Fy) = \frac{3}{5^{2n+1}} \leq \frac{1}{5} \cdot \frac{3}{5^{n+1}} = \beta\,\Delta(x,y)$$
$$< \alpha\,\Lambda(x,y) + \beta\,\Delta(x,y) + \gamma\,\Theta(x,y).$$

If x or y is equal to 0, (3.1) also holds. Thus, condition (3.1) is fulfilled for all $x, y \in \overline{\mathcal{O}(x_0;F)}$.

Therefore, Theorem 3.1 can be applied to conclude that F has a fixed point (which is $z = 0$). Moreover, it is unique in $\overline{\mathcal{O}(x_0;F)}$.

Note that the contractive condition (3.1) is not satisfied for all $x, y \in X$ (e.g., it does not hold if $x = 2$, $y = 3$). Hence, this example shows that our results can be applied when, e.g., the results of paper [20] cannot.

3.1. **Consequences.** In this subsection, we derive some fixed point results from our main result given by Theorem 3.1.

Corollary 3.4. *Let* (X, σ) *and* $F : X \to X$ *satisfy all the conditions of Theorem* 3.1*, except that condition* (3.1) *is replaced by*

$$\sigma(Fx,Fy) \leq \alpha\,\Lambda(x,y) + \beta\,\Delta_1(x,y) + \gamma\,\Theta_1(x,y),$$

for all $x, y \in \overline{\mathcal{O}(x_0; F)}$, *where*

$$\Delta_1(x,y) = \omega\Bigg(\max \Bigg\{ \sigma(x,y), \sigma(x,Fx), \sigma(y,Fy), \frac{\sigma(y,Fx)+\sigma(x,Fy)}{2} \Bigg\} \Bigg),$$

$$\Theta_1(x,y) = \omega\Bigg(\min \Bigg\{ \sigma(x,Fx), \sigma(y,Fy), \sigma(x,Fy), \sigma(y,Fx) \Bigg\} \Bigg)$$

and $\omega \in \Omega$ *is nondecreasing. Then the same conclusions hold as in Theorem* 3.1.

Proof. It follows from Theorem 3.1 by observing that if ω is nondecreasing, we have

$$\Delta(x,y) = \omega\Bigg(\max \Bigg\{ \sigma(x,y), \sigma(x,Fx), \sigma(y,Fy), \frac{\sigma(y,Fx)+\sigma(x,Fy)}{2} \Bigg\} \Bigg)$$

and

$$\Theta(x,y) = \omega\Bigg(\min \Bigg\{ \sigma(x,Fx), \sigma(y,Fy), \sigma(x,Fy), \sigma(y,Fx) \Bigg\} \Bigg).$$

□

Corollary 3.5. *Let* (X, σ) *and* $F : X \to X$ *satisfy all the conditions of Theorem* 3.1, *except that condition* (3.1) *is replaced by*

$$\sigma(Fx,Fy) \leq \alpha\, \omega\left(\frac{\sigma(y,Fy)\sigma(x,Fx)}{1+\sigma(x,y)} \right) + \beta\, \omega(\sigma(x,y)) + \gamma \min\{\omega(\sigma(x,Fx)), \omega(\sigma(y,Fy)), \omega(\sigma(x,Fy)), \omega(\sigma(y,Fx))\},$$

for all $x, y \in \overline{\mathcal{O}(x_0; F)}$, *where* $\omega \in \Omega$. *Then the same conclusions hold as in Theorem* 3.1.

Corollary 3.6. *Let* (X, σ) *and* $F : X \to X$ *satisfy all the conditions of Theorem* 3.1, *except that condition* (3.1) *is replaced by*

$$\sigma(Fx,Fy) \leq \alpha\, \omega\left(\frac{\sigma(y,Fy)\sigma(x,Fx)}{1+\sigma(x,y)} \right) + \beta \max\{\omega(\sigma(x,y)), \omega(\sigma(x,Fx)), \omega(\sigma(y,Fy))\},$$

for all $x, y \in \overline{\mathcal{O}(x_0; F)}$ *and some* α, β *satisfying* $\alpha + \frac{5}{2}\beta < 1$ *and* $\omega \in \Omega$. *Then the same conclusions as in Theorem* 3.1 *hold.*

Corollary 3.7. *Let* (X, σ) *and* $F : X \to X$ *satisfy all the conditions of Theorem* 3.1, *except that condition* (3.1) *is replaced by*

$$\sigma(Fx,Fy) \leq a_1 \left(\frac{\sigma(y,Fy)\sigma(x,Fx)}{1+\sigma(x,y)} \right) + a_2\sigma(x,y) + a_3\sigma(x,Fx) + a_4\sigma(y,Fy),$$

for all $x, y \in \overline{\mathcal{O}(x_0; F)}$ *and some positive constants* a_j *with* $a_1 + \frac{5}{2}(a_2 + a_3 + a_4) < 1$. *Then the same conclusions as in Theorem* 3.1 *hold.*

Corollary 3.8. *Let* (X, σ) *and* $F : X \to X$ *satisfy all the conditions of Theorem* 3.1*, except that condition* (3.1) *is replaced by*

$$\begin{aligned}\sigma(Fx, Fy) \le a_1 \left(\frac{\sigma(y, Fy)\sigma(x, Fx)}{1 + \sigma(x, y)}\right) + a_2\sigma(x, y) + a_3\sigma(x, Fx) + a_4\sigma(y, Fy) \\ + a_5 \left(\frac{\sigma(y, Fx) + \sigma(x, Fy)}{2}\right),\end{aligned}$$

for all $x, y \in \overline{\mathcal{O}(x_0; F)}$ *and some positive constants* a_j *with* $a_1 + \frac{5}{2}(a_2 + a_3 + a_4 + a_5) < 1$. *Then the same conclusions as in Theorem* 3.1 *hold.*

4. Common fixed point results for a pair of mappings

Sastry et al. [18] extended the concepts mentioned in the Preliminaries to two and three mappings and employed them to prove common fixed point results for commuting mappings. In what follows, we collect such definitions for two maps in metric-like spaces.

Definition 4.1. Let S, F be two self-mappings defined on a metric-like space (X, σ).

(1) If for a point $x_0 \in X$, a sequence $\{x_n\}$ in X is such that $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Fx_{2n+1}$, $n = 0, 1, 2, \dots$, then the set $\mathcal{O}(x_0; S, F) = \{x_n : n = 1, 2, \dots\}$ is called the orbit of (S, F) at x_0.
(2) The space (X, σ) is said to be (S, F)-orbitally 0-σ-complete at x_0 if every 0-σ-Cauchy sequence in $\mathcal{O}(x_0; S, F)$ converges to a point z in X such that $\sigma(z, z) = 0$.
(3) The maps S, F are said to be orbitally continuous at x_0 if they are continuous on $\mathcal{O}(x_0; S, F)$.

The main result of this section is the following:

Theorem 4.2. *Let* (X, σ) *be a metric-like space. Let* $S, F : X \to X$ *be given two mappings satisfying*

$$\begin{aligned}\sigma(Fx, Sy) \le \alpha \frac{\sigma(x, Fx)\sigma(y, Sy)}{1 + \sigma(x, y)} + \beta\, \sigma(x, y) \\ + \gamma\, [\sigma(x, Fx) + \sigma(y, Sy)] + \delta\, [\sigma(x, Sy) + \sigma(y, Fx)],\end{aligned} \tag{4.1}$$

for all $x, y \in \overline{\mathcal{O}(x_0; S, F)}$ *(for some* x_0*), where* $\alpha, \beta, \gamma, \delta$ *are nonnegative reals with* $\alpha + \beta + 2\gamma + 4\delta < 1$. *We assume that* (X, σ) *is* (S, F)*-orbitally* 0-σ*-complete* *at x_0. Then S and F have a common fixed point $z \in X$ such that $\sigma(z,z)=0$. If, moreover, each common fixed point z of S and F in $\overline{\mathcal{O}(x_0;F,S)}$ satisfies that $\sigma(z,z)=0$, then the common fixed point of S and F in $\overline{\mathcal{O}(x_0;F,S)}$ is unique.*

Proof. First of all we show that, if S or F has a fixed point z such that $\sigma(z,z)=0$, then it is a common fixed point of S and T. Indeed, let $z \in X$ be such that $Sz = z$ and $\sigma(z,z)=0$ and assume $\sigma(z,Fz)>0$. If we use the inequality (4.1) with $x=y=z$ we have

$$\sigma(Fz,z)=\sigma(Fz,Sz)\le(\gamma+\delta)\sigma(z,Fz),$$

a contradiction, since $\gamma+\delta<1$. Thus $\sigma(z,Fz)=0$ and so z is a common fixed point of S and F.

Starting with the given point x_0, consider the sequence $\{x_n\}$ in X given by

$$x_{2n+1}=Sx_{2n} \quad \text{and} \quad x_{2n+2}=Fx_{2n+1} \quad \text{for} \quad n\in\{0,1,\dots\}.$$

If $\sigma(x_{n_0},Sx_{n_0})=0$ or $\sigma(x_{n_0},Fx_{n_0})=0$ for some n_0, then the proof is finished. So without loss of generality we can suppose that the successive terms of $\{x_n\}$ are distinct.

Next, we claim that $\{x_n\}$ is a 0-σ-Cauchy sequence in the metric-like space $\mathcal{O}(x_0;S,F)$. For this we show that

$$\sigma(x_{n+1},x_n)\le\left(\frac{\beta+\gamma+\delta}{1-\alpha-\gamma-3\delta}\right)^n\sigma(x_1,x_0),\ \text{for } n\in\mathbb{N}. \tag{4.2}$$

Indeed, from (4.1) and using property (σ_3) of the metric-like, we have

$$\begin{aligned}
\sigma(x_2,x_1)&=\sigma(Fx_1,Sx_0)\\
&\le\frac{\alpha\sigma(x_1,Fx_1)\sigma(x_0,Sx_0)}{1+\sigma(x_1,x_0)}+\beta\sigma(x_1,x_0)\\
&\quad+\gamma[\sigma(x_1,Fx_1)+\sigma(x_0,Sx_0)]+\delta[\sigma(x_1,Sx_0)+\sigma(x_0,Fx_1)]\\
&=\frac{\alpha\sigma(x_1,x_2)\sigma(x_0,x_1)}{1+\sigma(x_1,x_0)}+\beta\sigma(x_1,x_0)\\
&\quad+\gamma[\sigma(x_1,x_2)+\sigma(x_0,x_1)]+\delta[\sigma(x_1,x_1)+\sigma(x_0,x_2)]\\
&\le\alpha\sigma(x_1,x_2)+\beta\sigma(x_1,x_0)+\gamma[\sigma(x_1,x_2)+\sigma(x_0,x_1)]\\
&\quad+\delta[3\sigma(x_1,x_2)+\sigma(x_0,x_1)].
\end{aligned}$$

This implies that

$$\sigma(x_2,x_1)\le\left(\frac{\beta+\gamma+\delta}{1-\alpha-\gamma-3\delta}\right)\sigma(x_1,x_0).$$

In a similar way one can prove that

$$\sigma(x_3, x_2) \le \left(\frac{\beta+\gamma+\delta}{1-\alpha-\gamma-3\delta}\right)\sigma(x_2, x_1).$$

It follows by induction that (4.2) holds for each $n \in \mathbb{N}$. Setting $\lambda_n := \sigma(x_n, x_{n+1})$, $n \in \mathbb{N}_0$ and $h := \frac{\beta+\gamma+\delta}{1-\alpha-\gamma-3\delta} < 1$, we get that the sequence $\{\lambda_n\}$ is decreasing and

$$\lambda_n \le h\lambda_{n-1} \le h^2\lambda_{n-2} \le \cdots \le h^n\lambda_0. \tag{4.3}$$

If $\lambda_0 = 0$ then $\sigma(x_0, x_1) = 0$, which is excluded. Therefore, let $\lambda_0 > 0$. Then, for each $n \ge m$ we have, in view of the triangular inequality (σ_3) and using (4.3), that

$$\begin{aligned} 0 \le \sigma(x_n, x_{n+m}) &\le \sigma(x_n, x_{n-1}) + \sigma(x_{n-1}, x_{n-2}) + \cdots + \sigma(x_{m+1}, x_m) \\ &\le (h^{n-1} + h^{n-2} + \cdots + h^m)\sigma(x_0, x_1) \\ &\le \frac{h^m}{1-h}\sigma(x_0, x_1). \end{aligned}$$

Thus, $\lim_{m,n\to\infty}\sigma(x_n, x_m) = 0$. This implies that $\{x_n\}$ is a 0-σ-Cauchy sequence in the metric-like space $\mathcal{O}(x_0; S, F)$. Since X is (S, F)-orbitally 0-σ-complete at x_0, there exists a $z \in X$ with $\lim_{n\to\infty} x_n = z$.

Applying the contractive condition (4.1), putting the values for $x = x_{2n-1}$ and $y = z$ we have:

$$\begin{aligned} \sigma(z, Sz) &\le \sigma(z, x_{2n}) + \sigma(Fx_{2n-1}, Sz) \\ &\le \sigma(z, x_{2n}) + \frac{\alpha\sigma(x_{2n-1}, Fx_{2n-1})\sigma(z, Sz)}{1+\sigma(x_{2n-1}, z)} + \beta\sigma(x_{2n-1}, z) \\ &\quad + \gamma[\sigma(x_{2n-1}, Fx_{2n-1}) + \sigma(z, Sz)] + \delta[\sigma(z, Fx_{2n-1}) + \sigma(x_{2n-1}, Sz)] \\ &= \sigma(z, x_{2n}) + \frac{\alpha\sigma(x_{2n-1}, x_{2n})\sigma(z, Sz)}{1+\sigma(z, x_n)} + \beta\sigma(x_{2n-1}, z) \\ &\quad + \gamma[\sigma(x_{2n-1}, x_{2n}) + \sigma(z, Sz)] + \delta[\sigma(z, x_{2n}) + \sigma(x_{2n-1}, Sz)]. \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality and using Lemma 2.6, we get

$$\sigma(z, Sz) \le (\gamma+\delta)\sigma(z, Sz) \le (\alpha+\beta+2\gamma+4\delta)\sigma(z, Sz),$$

that is, $\sigma(z, Sz) = 0$ (since $\alpha+\beta+2\gamma+4\delta < 1$) and then $Sz = z$. Hence, z is a fixed point of S such that $\sigma(z, z) = 0$ and so also a common fixed point of F and S.

Now, suppose that the each common fixed point z of F and S in $\overline{\mathcal{O}(x_0; S, F)}$ satisfies that $\sigma(z, z) = 0$. We claim that there is a unique common fixed point of T and S in $\overline{\mathcal{O}(x_0; S, F)}$. Assume to the contrary that $\sigma(u, Su) = \sigma(u, Fu) =$

0 and $\sigma(v,Sv)=\sigma(v,Fv)=0$ but $u\neq v$. By supposition, we can replace x by u and y by v in (4.1) to obtain

$$\begin{aligned}\sigma(u,v)&=\sigma(Fu,Sv)\\&\le\frac{\alpha\sigma(u,Fu)\sigma(v,Sv)}{1+\sigma(u,v)}+\beta\sigma(u,v)\\&\quad+\gamma[\sigma(u,Fu)+\sigma(v,Sv)]+\delta[\sigma(u,Sv)+\sigma(v,Tu)]\\&=(\beta+2\delta)\sigma(u,v),\end{aligned}$$

a contradiction, since $\beta+2\delta<1$. Hence, $u=v$. □

We state the following consequence of Theorem 4.2.

Corollary 4.3. *Let (X,σ) be a metric-like space such that X is F-orbitally 0-σ-complete at some point x_0 of X, where $F:X\to X$ is a given mapping satisfying*

$$\begin{aligned}\sigma(Fx,Fy)\le{}&\frac{\alpha\sigma(x,Fx)\sigma(y,Fy)}{1+\sigma(x,y)}+\beta\sigma(x,y)\\&+\gamma[\sigma(x,Fx)+\sigma(y,Fy)]+\delta[\sigma(x,Fy)+\sigma(y,Fx)],\end{aligned}$$

for all $x,y\in\overline{\mathcal{O}(x_0;F)}$, where $\alpha,\beta,\gamma,\delta$ are nonnegative reals with $\alpha+\beta+2\gamma+4\delta<1$. Then F has a fixed point. If, moreover, each fixed point z of F in $\overline{\mathcal{O}(x_0;F)}$ satisfies that $\sigma(z,z)=0$, then such fixed point is unique.

The following example shows how Theorem 4.2 can be used.

Example 4.4. Let the set $X=[0,+\infty)$ be equipped with the metric-like

$$\sigma(x,y)=\begin{cases}2x, & \text{if } x=y,\\ \max\{x,y\}, & \text{otherwise.}\end{cases}$$

Consider the following self-mappings on X:

$$Fx=\begin{cases}\frac{1}{4}x, & 0\le x\le 1,\\ 4x, & x>1,\end{cases}\qquad Sx=\begin{cases}\frac{1}{5}x, & 0\le x\le 1,\\ 5x, & x>1.\end{cases}$$

Take $x_0=1$. Then it is easy to show that

$$\mathcal{O}(x_0;S,F)\subset\left\{\frac{1}{4^k\cdot 5^l}:k,l\in\mathbb{N}\cup\{0\}\right\}\quad\text{and}\quad\overline{\mathcal{O}(x_0;S,F)}=\mathcal{O}(x_0;S,F)\cup\{0\}.$$

Take $\alpha = \frac{1}{6}$, $\beta = \gamma = \delta = \frac{1}{12}$. Then $\alpha + \beta + 2\gamma + 4\delta < 1$ and the contractive condition (4.1), for $x, y \in \mathcal{O}(x_0; S, F)$, takes the form

$$\begin{aligned}\sigma(Fx, Sy) \leq \frac{1}{6}\frac{xy}{1+\sigma(x,y)} + \frac{1}{12}\sigma(x,y) + \frac{1}{12}(x+y) \\ + \frac{1}{12}[\sigma(x, Sy) + \sigma(y, Fx)].\end{aligned} \tag{4.4}$$

Denote by R the right-hand side of (4.4) and consider the following possible cases.

Case I: If $x \geq y$. Then

$$\begin{aligned}R &= \frac{1}{6}\frac{xy}{1+x} + \frac{1}{12}x + \frac{1}{12}(x+y) + \frac{1}{12}(x + \max\{y, \tfrac{x}{3}\}) \\ &\geq 3 \cdot \frac{1}{12}x = \frac{1}{4}x = \sigma(Fx, Sy).\end{aligned}$$

Case II: $\frac{3}{4}y \leq x < y$. Then

$$R = \frac{1}{6}\frac{xy}{1+y} + \frac{1}{12}y + \frac{1}{12}(x+y) + \frac{1}{12}(x+y) \geq \frac{y}{4} > \frac{x}{4} = \sigma(Fx, Sy).$$

Case III: $x < \frac{4}{5}y$. Then

$$\begin{aligned}R &= \frac{1}{6}\frac{xy}{1+y} + \frac{1}{12}y + \frac{1}{12}(x+y) + \frac{1}{12}(\max\{x, \tfrac{y}{5}\} + y) \\ &\geq \frac{y}{4} > \frac{y}{5} = \sigma(Fx, Sy).\end{aligned}$$

Case IV: If $y \geq \frac{x}{3}$. Then

$$\begin{aligned}R &= \frac{1}{6}\frac{xy}{1+x} + \frac{1}{12}y + \frac{1}{12}(x+y) + \frac{1}{12}(\max\{x, \tfrac{y}{5}\} + y) \\ &\geq 3 \cdot \frac{1}{12}y \geq \frac{y}{4} > \frac{x}{4} = \sigma(Fx, Sy).\end{aligned}$$

Similarly, for $x = y$ one gets that

$$\begin{aligned}\sigma(Fx, Sx) \leq \frac{1}{6}\frac{\sigma(x,Fx)\sigma(x,Sx)}{1+\sigma(x,x)} + \frac{1}{12}\sigma(x,x) + \frac{1}{12}[\sigma(x,Fx) + \sigma(x,Sx)] \\ + \frac{1}{12}[\sigma(x, Sx) + \sigma(x, Fx)],\end{aligned}$$

$$\sigma(\frac{1}{4}x,\frac{1}{5}x) \le \frac{1}{6}\frac{\sigma(x,\frac{1}{4}x)\sigma(x,\frac{1}{5}x)}{1+\sigma(x,x)} + \frac{1}{12}\cdot 2x + \frac{1}{12}[\sigma(x,\frac{1}{4}x)+\sigma(x,\frac{1}{5}x)],$$
$$+\frac{1}{12}[\sigma(x,\frac{1}{5}x)+\sigma(x,\frac{1}{4}x)],$$
$$\frac{1}{4}x \le \frac{1}{6}\frac{x\cdot x}{1+2x} + \frac{1}{12}\cdot 2x + \frac{1}{12}\cdot 2x + \frac{1}{12}\cdot 2x,$$
$$\frac{1}{4}x \le \frac{1}{6}\frac{x^2}{1+2x} + \frac{x}{2}.$$

If x or y is equal to 0, (4.4) also holds. Thus, condition (4.4) is fulfilled for all $x, y \in \overline{\mathcal{O}(x_0; S, F)}$. Hence, all the conditions of Theorem 4.2 are satisfied and S, F have a unique common fixed point (which is $z = 0$).

Note that S and F do not satisfy the contractive condition (4.1) in the whole X. Also, this condition is not satisfied in X equipped with the standard metric $d(x, y) = |x - y|$ (and with the same values for $\alpha, \beta, \gamma, \delta$). Indeed, in this case the condition (4.1) takes the form

$$d(Fx,Sy) \le \frac{1}{6}\frac{d(x,Fx)d(y,Sy)}{1+d(x,y)} + \frac{1}{12}d(x,y) + \frac{1}{12}[d(x,Fx)+d(y,Sy)] \quad (4.5)$$
$$+\frac{1}{12}[d(x,Sy)+d(y,Fx)].$$

Denote by L and R, respectively, the left-hand and right-hand sides of the contraction condition (4.5) and take $x, y > 1$, i.e.,

$$L = d(Fx,Sy) = |4x-5y|$$

and

$$R = \frac{1}{6}\frac{|x-Fx||y-Sy|}{1+|x-y|} + \frac{1}{12}|x-y| + \frac{1}{12}[|x-Fx|+|y-Sy|]$$
$$+\frac{1}{12}[|x-Sy|+|y-Fx|].$$

Then

$$R = \frac{1}{6}\frac{|x-4x||y-5y|}{1+|x-y|} + \frac{1}{12}|x-y| + \frac{1}{12}[|x-4x|+|y-5y|]$$
$$+\frac{1}{12}[|x-5y|+|y-4x|].$$
$$\le \frac{2xy}{1+|x-y|} + \frac{1}{12}|x-y| + \frac{1}{12}[3x+4y] + \frac{1}{12}[5x+6y].$$
$$\le \frac{2xy}{1+|x-y|} + \frac{1}{12}[9x+11y].$$

This shows that $L \not\le R$ and the contraction condition (4.1) is not satisfied in the metric space (X, d).

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