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A NOTE ON LOCALLY η-INVEX SET

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Abstract. In this paper, we propose the concept of locally invexity property of a set, ρ-projective map and studied various basic results of it via a operator functional to extend the results studied by Ramík and Vlach [12] in starshaped region. The Lipschitz property of a multivalued map is studied under certain conditions.

1. Introduction

Earlier convex analysis, convex set was the important domain for the researchers to study the optimization problems and various type of applicational problems. Later the researchers, such as Ramík and Vlach [12] concentrated on starshaped region. In 1981, the notion of invex function was introduced by Hanson [9] to generalize the concept of convex function. He has applied the property of invex function to generalize the results studied in optimization theory. Later the researchers has included various type of function of invexity concepts to study the optimization problems. From the property of invex function, the researchers developed the concept of invex set. For reference, we refer Behera and Das [2], Das and Behera [6], Weir and Jeyakumar [13], Weir and Mond [14], to name only a few.

In this paper we have defined the concept of locally invex set via a operator functional and proved some basic properties of it to extend the results for starshaped region proved by Ramík and Vlach [12].

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For our need we recall the definition of starshaped region.

Definition 1.1. ([12]) Let X be a subset of $\mathbb{R}^n, y \in X$. The set X is *starshaped* from y if for every $x \in X$, we have the line segment joining x and y, i.e., $tx + (1-t)y \in X$. The set of all points $y \in X$ such that X is starshaped from y is called the *Kernel* of X and it is denoted by $\mathrm{Ker}(X)$. The set X is said to be a *starshaped region* if $\mathrm{Ker}(X)$ is nonempty, or X empty.

2. Results on locally η-invex set

Hanson [9] has defined the invex set as follows.

Definition 2.1. (η-invex set) The set $K \subset \mathbb{R}^n$ is said to be invex with respect to η if there exists a vector function $\eta : K \times K \to \mathbb{R}^n$ such that for all $x, y \in K$,

$$y + t\eta(x, y) \in K,$$

for all $t \in [0, 1]$.

Let X be subset of a topological vector space E and $\eta : X \times X \to X$ be a vector function. We define the following definitions for our need.

Definition 2.2. Let X be subset of a topological vector space E and $\eta : X \times X \to X$ be a vector valued function. Let

$$l(x, y) = \{\ell \in (0, 1] : \langle S(x, y), \eta(x, y)\rangle \leq \ell,\ S \in X^*,\ x, y \in X\} \neq \varnothing.$$

X is said to be

1. η-locally invex at $y \in X$, there exists a positive number $\ell \in l(x, y)$ such that
$$y + t\eta(x, y) \in K \quad \text{for all} \quad t \in (0, \ell),$$
2. η-locally invex if for all $x, y \in X$, there exists a positive number $\ell \in l(x, y)$ such that
$$y + t\eta(x, y) \in K \quad \text{for all} \quad t \in (0, \ell).$$

Definition 2.3. Let X be subset of a topological vector space E and $\eta : X \times X \to X$ be a vector valued function. Let

$$l(x, y) = \{\ell \in (0, 1] : \langle S(x, y), \eta(x, y)\rangle \leq \ell,\ S \in X^*,\ x, y \in X\} \neq \varnothing.$$

Assume $\phi : K \to l(x, y)$ such that $\phi(y) \in (0, \ell)$. Then define

1. The η-locally invex path joining x and y in X is defined by
$$IP_\eta(x, y) = \{z : z = y + \phi(y)\eta(x, y)\},$$

2. the η-locally kernel of X is defined by

$$Ker_\eta(X) = \{y \in X : IP_\eta(x, y) \subset X,\ x \in X\},$$

3. (a) X is locally η-starshaped invex from $y \in X$ if for every $x \in X$ if for every $x \in X$, we have $IP_\eta(x, y) \subset X$ for all $x \in X$,
   (b) X is locally η-starshaped invex, if there exists a $y \in X$ such that $IP_\eta(x, y) \subset X$ for all $x \in X$; Alternatively, X is η-invex if $Ker_\eta(X) \neq \varnothing$,
   (c) X is locally η-invex if $IP_\eta(x, y) \subset X$ for all $x, y \in X$.

Remark 2.4. Let $E = \mathbb{R}^n$. The starshaped set reduces from η-invex set and $Ker(X)$ reduces from $Ker_\eta(X)$ if $\eta(x, y) = x - y$.

Proposition 2.5. *Let X be subset of a topological vector space and $\eta : X \times X \to X$ be any vector valued map. Then the kernel $Ker_\eta(X)$ is η-invex.*

Proof. If X is not η-invex, then there is no $y \in X$ such that $IP_\eta(x, y) \subset X$ for any $x \in X$, i.e.,

$$Ker_\eta(X) = \{y \in X : IP_\eta(x) \subset X, x \in X\} = \varnothing,$$

so is η-invex. Let X be η-invex, $x_1, x_2 \in Ker_\eta(X)$ and $\phi(x_2) \in l(x_1, x_2)$, then

$$z = x_2 + \phi(x_2)\eta(x_1, x_2) \in X.$$

By contradiction, we show that $z \in Ker_\eta(X)$. On the opposite, let there is a $u \in X$ and $\phi(z) \in l(u, z)$ such that

$$v = z + \phi(z)\eta(u, z) \notin X.$$

Since $u \in X, x_2 \in Ker_\eta(X)$, we have $x_2 + \phi(x_3)\eta(u, x_2) \in X$ for $\phi(x_3) \in l(u, x_2)$. Let there exist a $\phi(x_4) \in l(u, x_2)$ such that

$$w = x_2 + \phi(x_4)\eta(u, x_2) \in X,$$

then

$$v = x_1 + \phi(y_1)\eta(w, x_1) \in X$$

for some $\phi(y_1) \in l(w, x_1)$ because $w \in X$ and $x_1 \in Ken_\eta(X)$ which leads to a contradiction. This completes the proof. □

Definition 2.6. Let X be subset of a topological vector space and $\eta : X \times X \to X$ be any vector valued map. Let A and B be two locally η-invex subsets of X. Then η is said to be locally distributive on $A + B$ if for $x, y \in A + B$ with $x = u + v$ and $y = a + b$ where $u, a \in A$ and $v, b \in B$, we have

$$\eta(x, y) = \eta(u, a) + \eta(v, b) \in X.$$

Example 2.7. Let X be subset of a topological vector space and $\eta : X \times X \to X$ be any vector valued map. Let A and B be two subsets of X. Let $P : X \to X$ be a linear map and ρ-projective [10], i.e., $P^2 = \rho P$ where $\rho > 0$. Let η be defined by $\eta(x, y) = P^2(x) - P^2(y)$ for $x, y \in X$, then η is locally distributive on $A + B$ which is shown as follows.

For $x, y \in A + B$, there exist $u, a \in A$ and $v, b \in B$ such that $x = u + v$ and $y = a + b$. Since

$$\begin{aligned}
\eta(x, y) &= \eta(u + v, a + b) \\
&= P^2(u + v) - P^2(a + b) \\
&= \rho P(u + v) - \rho P(a + b) \\
&= \rho[P(u) + P(v) - P(a) - P(b)] \\
&= P^2(u) - P^2(a) + P^2(v) - P^2(b) \\
&= \eta(u, a) + \eta(v, b),
\end{aligned}$$

η is distributive on $A + B$.

Now we define the following definition:

Definition 2.8. Let X be subset of a topological vector space and $\eta_i : X \times X \to X$ be the family of vector valued maps where I is the index set. The vector valued map $\eta : X \times X \to X$ covers $\eta_i, i \in I$, if X is locally η-invex with respect to $\eta_i, i \in I$ then X is locally η-invex with respect to η.

Theorem 2.9. *Let X be subset of a topological vector space. Let $\eta_1 : X \times X \to X$ and $\eta_2 : X \times X \to X$ be any vector valued maps. Let A and B be two locally invex subsets of X with respect to η_1 and η_2 respectively. Let $\eta : X \times X \to X$ be the vector valued map covers both η_1 and η_2. If η is distributive on $A + B$, then*

(a) $Ker_\eta(A) + Ker_\eta(B) \subseteq Ker_\eta(A + B)$,
(b) $Ker_\eta(\alpha A) = \alpha Ker_\eta(A)$ *for all* $\alpha \in \mathbb{R}$ *if* η *satisfying* $\eta(\alpha v, \alpha u) = \alpha\eta(v, u)$ *for all* $u, v \in A$.

Proof. Since η covers both η_1 and η_2, so both A and B are locally η-invex on X. Firstly, if A is nonempty and B is empty, the proof is trivial because since $A + B = A$, $Ker_\eta(A) + Ker_\eta(B) = Ker_\eta(A) + \varnothing = Ker_\eta(A) = Ker_\eta(A + B)$. For inclusion, let us take $y \in Ker_\eta(A) + Ker_\eta(B)$, then there exists a $a \in Ker_\eta(A)$ and $b \in Ker_\eta(B)$ such that $y = a + b$. Let $\phi : X \to l(x, y)$ such that $\phi(y) \in (0, \ell)$ (Definition 2.2). Since $a \in Ker_\eta(A)$ and $b \in Ker_\eta(B)$, we have $a + \phi(z)\eta(u, a) \in A$ for all $u \in A$ and $z \in X$, and $b + \phi(z)\eta(v, b) \in B$ for all $v \in B$ and $z \in X$. To prove $y \in Ker_\eta(A + B)$, we show that $y + \phi(z)\eta(x, y) \in$

$A+B$ for all $z \in X$, and $x = u+v \in A+B$ taken arbitrarily. Since η is distributive on $A+B$, we have

$$\begin{aligned} y+\phi(z)\eta(x,y) &= (a+b)+\phi(z)\eta(u+v,a+b) \\ &= a+b+\phi(z)(\eta(u+a)+\eta(v+b)) \\ &= a+\phi(z)\eta(u,a)+b+\phi(z)\eta(v,b) \in A+B, \end{aligned}$$

for all $x = u+v \in A+B$ and $z \in X$, implying $y \in Ker_\eta(A+B)$. Hence

$$Ker_\eta(A)+Ker_\eta(B) \subseteq Ker_\eta(A+B).$$

Secondly, let $y \in \alpha Ker_\eta(A)$, then there exists a $u \in Ker_\eta(A)$ such that $y = \alpha u$. Since $u \in Ker_\eta(A)$, we have $u+\phi(z)\eta(v,u)$ for all $v \in A$ and $z \in X$. Let $x \in \alpha A$, then $x = \alpha v$ for all $v \in A$. Since α satisfying $\eta(\alpha v, \alpha u) = \alpha\eta(v,u)$ for all $u, v \in A$, we have

$$\begin{aligned} y+\phi(z)\eta(x,y) &= \alpha u+\phi(z)\eta(\alpha v,\alpha u) \\ &= \alpha u+\alpha\phi(z)\eta(v,u) \\ &= \alpha(u+\phi(z)\eta(v,u) \in \alpha A, \end{aligned}$$

for all $x \in \alpha A$ and $z \in X$. Thus $y \in Ker_\eta(\alpha A)$, implying

$$\alpha Ker_\eta(A) \subset Ker_\eta(\alpha A).$$

Let $\alpha \neq 0$. Replacing α by $\frac{1}{\alpha}$ and A by αA in the above inclusion equation, we get

$$\frac{1}{\alpha}Ker_\eta(\alpha A) \subset Ker_\eta(\frac{1}{\alpha}\alpha A).$$

For $y \in Ker_\eta(\alpha A)$, we have

$$u = \frac{1}{\alpha}y \in \frac{1}{\alpha}Ker_\eta(\alpha A) \subset Ker_\eta(A).$$

Therefore, $u \in Ker_\eta(A)$, implying $y = \alpha u \in \alpha Ker_\eta(A)$. Thus

$$Ker_\eta(\alpha A) \subset \alpha Ker_\eta(A).$$

Hence

$$Ker_\eta(\alpha A) = \alpha Ker_\eta(A).$$

This completes the proof of the theorem. □

Example 2.10. Let X be subset of a topological vector space and $\eta : X \times X \to X$ be any vector valued map. Let A and B be two subset of X. Let $P : X \to X$ be a linear map and ρ-projective, i.e., $P^2 = \rho P$ where $\rho > 0$. Let η be defined by $\eta(x_1, x_2) = P^2(x_1) - P^2(x_2)$ for $x_1, x_2 \in X$, then

$$\begin{aligned} \eta(\alpha u, \alpha v) = P^2(\alpha u) - P^2(\alpha v) = \rho(\alpha P(u) - \alpha P(v)) \\ = \alpha(P^2(u) - P^2(v)) = \alpha\eta(u,v), \end{aligned}$$

since P is linear. By the above example η is distributive on $A+B$. Since all the conditions of η satisfies, by Theorem 2.9, we have

(a) $Ker_\eta(A) + Ker_\eta(B) \subseteq Ker_\eta(A+B)$,
(b) $Ker_\eta(\alpha A) = \alpha Ker_\eta(A)$ for all $\alpha \in \mathbb{R}$.

Corollary 2.11. *Let X be subset of a topological vector space and $\eta_i : X \times X \to X$ be the family of vector valued maps where I is the index set. Let $K_i, i \in I$ be the family of η_i-invex subsets of X. Let $\eta : X \times X \to X$ be the vector valued map covers each $\eta_i, i \in I$. If η is distributive on $\sum_{i\in I} K_i$, then*

$$\sum_{i\in I} Ker_\eta(K_i) \subseteq Ker_\eta\left(\sum_{i\in I} K_i\right).$$

Proof. By Theorem 2.9, if A and B are η_1-invex and η_2-invex on X respectively, and η covers both η_1 and η_2, then

$$Ker_\eta(A) + Ker_\eta(B) \subseteq Ker_\eta(A+B).$$

Taking $A = K_1$ and $B = K_2$, we have

$$\sum_{i=1}^{2} Ker_\eta(K_i) \subseteq Ker_\eta\left(\sum_{i=1}^{2} K_i\right).$$

For generality, assume that $y \in \sum_{i\in I} K_i$, then there exist a $y_i \in \sum_{i\in I} K_i$ such that $y = \sum_{i\in I} y_i$. Since for each $i \in I, y_i \in Ker_\eta(K_i)$, we have $y_i + \phi(z)\eta(x_i, y_i) \in K_i$ for all $x_i \in K_i$ and $\phi : X \to l(x,y)$ such that $\phi(y) \in (0, \ell)$, by Definition 2.2. To prove $y \in Ker_\eta(\sum_{i\in I} K_i)$, we show that

$$y + \phi(z)\eta(x,y) \in \sum_{i\in I} K_i,$$

for all $z \in X$, and $x = \sum_{i\in I} x_i \in \sum_{i\in I} K_i$, taken arbitrarily. Since η is distributive on $A+B$, we have

$$\begin{aligned} y + \phi(z)\eta(x,y) &= \sum_{i\in I} y_i + \phi(z)\eta\left(\sum_{i\in I} x_i, \sum_{i\in I} y_i\right) \\ &= \sum_{i\in I}(y_i + \phi(z)\eta(x_i, y_i)) \in \sum_{i\in I} K_i, \end{aligned}$$

for all $x \in \sum_{i\in I} K_i$ and $z \in X$, implying $y \in Ker_\eta(\sum_{i\in I} K_i)$. Hence

$$\sum_{i\in I}(Ker_\eta(K_i) \subseteq Ker_\eta\left(\sum_{i\in I} K_i\right).$$

This completes the proof. $\square$

An intersection of locally η-invex (non-convex) sets do not need to be η-invex as the intersection may be disconnected. However, the following result holds.

Proposition 2.12. *Let X be subset of a topological vector space and $\eta_i : X \times X \to X$ be the family of vector valued maps where I is the index set. Let $K_i, i \in I$ be the family of η_i-invex subsets of X. Let $\eta : X \times X \to X$ be the vector valued map covers each $\eta_i, i \in I$. Then*

$$\bigcap_{i\in I} Ker_\eta(K_i) \subset Ker_\eta\left(\bigcap_{i\in I} K_i\right).$$

Proof. Since $K_i, i \in I$ is the family of η_i-invex subsets of X and η covers η_i for each $i \in I$, so each K_i is η-invex on X. Let $y \in \bigcap_{i\in I} Ker_\eta(K_i)$, then $y \in Ker_\eta(K_i)$ for each $i \in I$. Since $Ker_\eta(K_i) \subset K_i$ for each $i \in I$, we have

$$\bigcap_{i\in I} Ker_\eta(K_i) \subset K_i.$$

Thus $y \in \bigcap_{i\in I}(K_i)$. Thus for $\phi : X \to l(x, y)$ such that $\phi(y) \in (0, \ell)$, by Definition 2.2, and for all $x \in \bigcap_{i\in I}(K_i)$,

$$y + \phi(z)\eta(x, y) \in \bigcap_{i\in I} K_i,$$

for all $z \in X$, implying $y \in Ker_\eta(\bigcap_{i\in I}(K_i))$. Hence

$$\bigcap_{i\in I} Ker_\eta(K_i) \subset Ker_\eta\left(\bigcap_{i\in I} K_i\right).$$

This completes the proof. □

Corollary 2.13. *Let X be subset of a topological vector space and $\eta_i : X\times X \to X$ be the family of vector valued maps where I is the index set. Let $K_i, i \in I$ be the family of η_i-invex subsets of X such that $\bigcap_{i\in I} K_i \neq \varnothing$. Let $\eta : X \times X \to X$ be the vector valued map covers each $\eta_i, i \in I$. Then $\bigcap_{i\in I} K_i$ is η-invex in X if $\bigcap_{i\in I} Ker_\eta(K_i) \neq \varnothing$.*

Proof. Since $\bigcap_{i\in I} Ker_\eta(K_i) \neq \varnothing$, assume $y \in \bigcap_{i\in I} Ker_\eta(K_i)$. By Proposition 2.12, we have

$$\bigcap_{i\in I} Ker_\eta(K_i) \subset Ker_\eta\left(\bigcap_{i\in I} K_i\right),$$

so $y \in Ker_\eta(\bigcap_{i\in I} K_i)$. For $\phi : X \to l(x,y)$ such that $\phi(z) \in (0,\ell)$, by Definition 2.2, and for all $x \in \bigcap_{i\in I} K_i$, we have

$$y + \phi(z)\eta(x,y) \in \bigcap_{i\in I} K_i.$$

Hence $\bigcap_{i\in I} K_i$ is η-invex on X. This completes the proof. $\square$

3. η-Lipschitz

Lipschitz functions appear nearly everywhere in mathematics. Typically, the Lipschitz condition is first encountered in the elementary theory of ordinary differential equations, where it is used in existence theorems. For our convenience we define the Lipschitz condition on η-invex sets.

Definition 3.1. Let X and Y be topologival vector spaces. Let $S : X \times X \to L(X,Y)$ be multilinear map and $\eta : X \times X \to X$ be vector valued. Then S is said to be η-Lipschitz with rank $k \in \mathbb{R}_+$ if

$$\|\langle S(z,x) - S(z,y), \eta(x,y)\rangle\| \leq k \cdot \|\eta(x,y)\|$$

and

$$\|\langle S(x,z) - S(y,z), \eta(x,y)\rangle\| \leq k \cdot \|\eta(x,y)|\|$$

for all $z \in IP_\eta(X)$ and $x, y \in X$.

Example 3.2. Let $X = \mathbb{R}, Y = \mathbb{R}^2$. Let $S : X \times X \to L(X,Y)$ be defined by

$$S(u,v) = \begin{bmatrix} u \\ v \end{bmatrix}$$

for all $u, v \in X$ and $\langle S(u,v), z\rangle = S(u,v) \cdot z$, where $z \in X$.
Now

$$\begin{aligned} \|\langle S(z,x) - S(z,y), \eta(x,y)\rangle\| &\leq \|S(z,x) - S(z,y)\| \cdot \|\eta(x,y)\| \\ &= \left\| \begin{bmatrix} z \\ x \end{bmatrix} - \begin{bmatrix} z \\ y \end{bmatrix} \right\| \cdot \|\eta(x,y)\| \\ &= \left\| \begin{bmatrix} 0 \\ x-y \end{bmatrix} \right\| \cdot \|\eta(x,y)\| \\ &\leq k \cdot \|\eta(x,y)\|, \end{aligned}$$

where k is the matrix norm.

Theorem 3.3. *Let X and Y be topological vector spaces. Let $S : X \times X \to L(X,Y)$ be multilinear map and $\eta : X \times X \to X$ be vector valued. Let $|\,\|\langle S(z,y)\rangle\| \leq k$ for all $z \in IP_\eta(X)$ and for any fixed y, and similarly*

$\|\langle S(x,z)\rangle\| \leq k$ *for all* $z \in IP_\eta(X)$ *and for any fixed* x. *Then* S *is* η*-Lipschitz with respect to* η.

Proof. For any $x, y \in X$ and $z \in IP_\eta(X)$ then

$$\begin{aligned}\langle S(z,y), \eta(x,y)\rangle &\leq \|S(z,y)\| \cdot \|\eta(x,y)\| \\ &\leq k \cdot \|\eta(x,y)\|,\end{aligned}$$

implying S is η-Lipschitz on X. This completes the proof of the theorem. □

Theorem 3.4. *Let* X *and* Y *be topological vector spaces. Let* $S : X \times X \to L(X,Y)$ *be multilinear map and* $\eta : X \times X \to X$ *be vector valued. Let* $\|\langle S(z,y)\rangle\| \leq k$ *for all* $z \in IP_\eta(X)$ *and for any fixed* y*, and similarly* $\|\langle S(x,z)\rangle\| \leq k$ *for all* $z \in IP_\eta(X)$ *and for any fixed* x. *Then* $\|S\| \cdot \|\phi(y)\| \leq k$ *if* $\|\eta(x,y)\| < 1$ *for all* $x, y \in X$.

Proof. For any $x, y \in X$ and $z \in IP_\eta(X)$ then

$$\begin{aligned}\|\langle S(z,y), \eta(x,y)\rangle\| &\leq \|S(z,y)\| \cdot \|\eta(x,y)\| \\ &\leq \|S\| \cdot \|z-y\| \cdot \|\eta(x,y)\| \\ &\leq \|S\| \cdot \|\eta(x,y)\|^2 \cdot \|\phi(y)\| \\ \Rightarrow \|S\| \cdot \|\eta(x,y)\|^2 \cdot \|\phi(y)\| &\leq \|S\| \cdot \|\phi(y)\|.\end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.5. *Let* X *be a topological vector space. Assume* $S : X \times X \to X^*$ *be* η*-Lipschitz with rank* k *and* $\|\eta(x,y)\| \leq \frac{\ell}{k}$ *for any* $\ell \in (0,1)$ *if and only if* $\ell \in l(x,y)$.

Proof. Since S is η-lipschitz with rank k, so

$$\begin{aligned}\langle S(x,y), \eta(x,y)\rangle &\leq \|\langle S(x,y), \eta(x,y)\rangle\| \\ &\leq k\,\|\eta(x,y)\| \\ &\leq k \cdot \frac{\ell}{k} \\ &= \ell.\end{aligned}$$

This completes the necessary part. Similarly, assume $\ell \in l(x,y)$,
Then $\langle S(x,y), \eta(x,y)\rangle \leq \ell$, that is

$$\begin{aligned}\|\langle S(x,y), \eta(x,y)\rangle\| &\leq \ell \\ \Rightarrow \|S(x,y)\| \cdot \|\eta(x,y)\| &\leq \ell \\ \Rightarrow \|\eta(x,y)\| &\leq \frac{\ell}{k}.\end{aligned}$$

This completes the sufficient part and completes the proof of the theorem. □

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