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ON A HIGH ORDER ITERATIVE SCHEME FOR A NONLINEAR WAVE EQUATION WITH THE SOURCE TERM CONTAINING A NONLINEAR INTEGRAL

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Abstract. In this paper, we consider the initial and boundary value problem for a nonlinear wave equation, with the source term containing a nonlinear integral, associated with homogeneous Dirichlet boundary conditions. We establish here a high order iterative scheme in order to get a convergent sequence at a rate of order N to a local unique weak solution of the above problem. This scheme shows that the convergence can be obtained with a high rate if the nonlinear term in the original equation is smooth enough.

1. Introduction

In this paper, we consider the initial and boundary value problem for a nonlinear wave equation, with the source term containing a nonlinear integral,

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associated with homogeneous Dirichlet boundary conditions as follows

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) + \lambda u_t = f(x,t,u) + \int_0^t g(x,t,s,u(x,s))ds, \quad 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $\mu, f, g, \tilde{u}_0, \tilde{u}_1$ are given functions and $\lambda \neq 0$ is a given constant.

Eq. (1.1) constitutes a case, relatively simpler, of a more general equation, namely

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t,u,u_x)u_x\right) = F(x,t,u,u_x,u_t),\ \ 0 < x < 1,\ \ 0 < t < T. \tag{1.4}$$

In the special cases, when the functions $\mu,\ F$ have the simple forms, Eq. (1.4) with various initial-boundary conditions has been studied by many authors, see [1], [2], [4]-[11], [13]-[15] and references therein. In these works, the existence and properties of solutions have received much attention.

In [13], Santos studied the asymptotic behavior of solution of Eq. (1.4) with $F(x,t,u,u_x,u_t) \equiv 0$, $\mu(x,t,u,u_x) = \mu(t)$, associated with the Dirichlet boundary condition at $x = 0$ and a boundary condition of memory type at $x = 1$, that is $u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)ds = 0$, $t > 0$.

In [8], by combining the linearization method for the nonlinear term, the Faedo-Galerkin method and the weak compactness method, the existence of a unique weak solution of an initial and boundary value problem for the nonlinear wave equation $u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t,u,\|u_x\|^2)u_x\right) = F(x,t,u,u_x,u_t)$ with the nonhomogeneous boundary conditions is proved. We note, however, the recurrent sequence obtained here converges only at a rate of order 1. It is well known that, Newton's method and its variants are used to solve nonlinear operator equations or systems of nonlinear equations, see [12] and references therein. In case $\lim_{n\to\infty} u_n = u$, one speaks of *convergence of order* N if

$$|u_{n+1} - u| \leq C|u_n - u|^N$$

for some $C > 0$ and all large N. In the special cases $N = 1$ with $C < 1$ and $N = 2$ one also speaks of linear and quadratic convergence, respectively, see [3]. Based on the ideas about recurrence relations of these methods, a high order iterative scheme can be constructed for solving the nonlinear operator equation, see [9]-[11], [14], [15].

Motivated by results for wave equations in [6]-[8], and based on the use of a high order iterative scheme in [9]-[11], [14], [15], in this paper, we will establish a similar scheme to get the convergence of order N for Prob. (1.1)-(1.3). To achieve this purpose, we define a recurrent sequence $\{u_m\}$ associated with Eq.

(1.1) as follows

$$
\begin{aligned}
&\tfrac{\partial^2 u_m}{\partial t^2} - \tfrac{\partial}{\partial x}\left(\mu(x,t)\tfrac{\partial u_m}{\partial x}\right) + \lambda \tfrac{\partial u_m}{\partial t} \\
&= \textstyle\sum_{k=0}^{N-1} \frac{1}{k!} \frac{\partial^k f}{\partial u^k}(x,t,u_{m-1})\left(u_m - u_{m-1}\right)^k \\
&\quad + \textstyle\sum_{k=0}^{N-1} \frac{1}{k!} \int_0^t \left[\frac{\partial^k g}{\partial u^k}(x,t,s,u_{m-1}(x,s))\right]\left(u_m(x,s) - u_{m-1}(x,s)\right)^k ds,
\end{aligned}
\tag{1.5}
$$

$0 < x < 1$, $0 < t < T$, with u_m satisfying (1.2), (1.3). The first term u_0 is chosen as $u_0 \equiv 0$. If $\mu \in C^1([0,1] \times \mathbb{R}_+)$, $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, and $g \in C^k([0,1] \times \Delta \times \mathbb{R})$, with $\Delta = \{(t,s) \in \mathbb{R}_+^2 : s \leq t\}$, we prove that the sequence $\{u_m\}$ converges at rate of order N to a weak unique solution of Prob. (1.1)-(1.3). The main result is given in Theorems 2.1 and 2.3. In our proofs, the fixed point method and Faedo-Galerkin method are used.

2. A high order iterative scheme

First, we put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2, $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X.

We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$
\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty
$$

and

$$
\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \ \text{ for } p = \infty.
$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $f = f(x,t,u)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$ and $D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} f$; $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = k$, $D^{(0,0,0)} f = D^{(0)} f = f$.

Similarly, with $g \in C^k([0,1] \times \Delta \times \mathbb{R})$, $\Delta = \{(t,s) \in \mathbb{R}_+^2 : s \leq t\}$, $g = g(x,t,s,u)$, we put $D_1 g = \frac{\partial g}{\partial x}$, $D_2 g = \frac{\partial g}{\partial t}$, $D_3 g = \frac{\partial g}{\partial s}$, $D_4 g = \frac{\partial g}{\partial u}$ and $D^\beta g = D_1^{\beta_1} ... D_4^{\beta_4} g$; $\beta = (\beta_1, ..., \beta_4) \in \mathbb{Z}^4$, $|\beta| = \beta_1 + ... + \beta_4 = k$, $D^{(0,0,0,0)} g = D^{(0)} g = g$. With $\mu = \mu(x,t)$, we also put $D_1 \mu = \frac{\partial \mu}{\partial x}$, $D_2 \mu = \frac{\partial \mu}{\partial t}$.

We shall use the following norm on H^1

$$
\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.
$$

It is well known that the imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and for all $v \in H^1$,

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\,\|v\|_{H^1}\,.$$

Furthermore, on $H_0^1 = \{v \in H^1 : v(0) = v(1) = 0\}$, two norms $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v_x\|$ are equivalent and

$$\|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \quad \text{for all} \quad v \in H_0^1. \tag{2.1}$$

We make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in \left(H_0^1 \cap H^2\right) \times H_0^1$;

(H_2) $f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R})$ such that

(i) $f(0,t,0) = f(1,t,0) = 0,\ \forall t \geq 0$,

(ii) $D_3^i f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}),\ 0 \leq i \leq N$,

(iii) $D_1 D_3^i f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}),\ 1 \leq i \leq N-1$;

(H_3) $g \in C^N([0,1] \times \Delta \times \mathbb{R})$ such that

(i) $g(0,t,s,0) = g(1,t,s,0) = 0,\ \forall (t,s) \in \Delta = \{(t,s) \in \mathbb{R}_+^2 : s \leq t\}$,

(ii) $D_4^i g \in C^0([0,1] \times \Delta \times \mathbb{R}),\ 0 \leq i \leq N$,

(iii) $D_1 D_4^i g \in C^0([0,1] \times \Delta \times \mathbb{R}),\ 1 \leq i \leq N-1$;

(H_4) $\mu \in C^2([0,1] \times \mathbb{R}_+)$ and there exists constant $\mu_0 > 0$ such that

$$\mu(x,t) \geq \mu_0 \quad \text{for all} \quad (x,t) \in [0,1] \times \mathbb{R}_+.$$

Fix $T^* > 0$. For each $M > 0$ given, we set the constants $K_0(M,f)$, $K_M(f)$, $\bar{K}_0(M,g)$, $\bar{K}_M(g)$, $\tilde{K}_0(\mu)$, $\tilde{K}(\mu)$ as follows

$$\begin{cases} K_0(M,f) = \sup\{|f(x,t,u)| : 0 \leq x \leq 1,\ 0 \leq t \leq T^*,\ |u| \leq M\}, \\ K_M(f) = \sum_{i=0}^N K_0(M, D_3^i f) + \sum_{i=1}^{N-1} K_0(M, D_1 D_3^i f), \\ \bar{K}_0(M,g) = \sup\{|g(x,t,s,u)| : 0 \leq x \leq 1,\ 0 \leq s \leq t \leq T^*,\ |u| \leq M\}, \\ \bar{K}_M(g) = \sum_{i=0}^N \bar{K}_0(M, D_4^i g) + \sum_{i=1}^{N-1} \bar{K}_0(M, D_1 D_4^i g), \\ \tilde{K}_0(\mu) = \|\mu\|_{C^0([0,1]\times[0,T^*])} = \sup\limits_{(x,t)\in[0,1]\times[0,T^*]} |\mu(x,t)|\,, \\ \tilde{K}(\mu) = \|\mu\|_{C^2([0,1]\times[0,T^*])} = \sum\limits_{i+j\leq 2} \tilde{K}_0\left(D_1^i D_2^j \mu\right). \end{cases}$$

For every $T \in (0, T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T; H_0^1 \cap H^2) : v_t \in L^\infty(0,T; H_0^1),\ v_{tt} \in L^2(Q_T), \\ \qquad\qquad \text{with } \|v\|_{L^\infty(0,T;H_0^1\cap H^2)}\,,\ \|v_t\|_{L^\infty(0,T;H_0^1)}\,,\ \|v_{tt}\|_{L^2(Q_T)} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T; L^2)\}, \end{cases}$$

in which $Q_T = \Omega \times (0,T)$.

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{2.2}$$

we associate problem (1.1)-(1.3) with the following problem.

Find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + \langle \mu(t) u_{mx}(t), w_x\rangle + \lambda \langle u_m'(t), w\rangle \\ = \langle \Phi_m(t), w\rangle, \quad \forall w \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} &\Phi_m(x,t) \\ &= \textstyle\sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1})(u_m - u_{m-1})^i \\ &\quad + \textstyle\sum_{i=0}^{N-1} \frac{1}{i!} \int_0^t \left[D_4^i g(x,t,s,u_{m-1}(x,s))\right] (u_m(x,s) - u_{m-1}(x,s))^i\, ds \\ &= \textstyle\sum_{j=0}^{N-1} \left[A_j(x,t,u_{m-1}) u_m^j + \int_0^t B_j(x,t,s,u_{m-1}(x,s)) u_m^j(x,s) ds\right] \end{aligned} \tag{2.4}$$

and

$$\begin{cases} A_j(x,t,u_{m-1}) = \sum_{i=j}^{N-1} \frac{(-1)^{i-j}}{j!(i-j)!} D_3^i f(x,t,u_{m-1}) u_{m-1}^{i-j}, \\ B_j(x,t,s,u_{m-1}) = \sum_{i=j}^{N-1} \frac{(-1)^{i-j}}{j!(i-j)!} D_4^i g(x,t,s,u_{m-1}) u_{m-1}^{i-j}. \end{cases} \tag{2.5}$$

Then we have the following theorem.

Theorem 2.1. *Let* (H_1)-(H_4) *hold. Then there exist a constant* $M > 0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, μ *and a constant* $T > 0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, μ, f, g *such that, for* $u_0 \equiv 0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (2.3) *and* (2.4).

Proof. The proof of Theorem 2.1 consists three steps.

Step 1. (The Faedo-Galerkin approximation) Let $\{w_j\}$ be a basis of H_0^1, formed by eigenfunction w_j of the operator $-\Delta = -\frac{\partial^2}{\partial x^2} : -\Delta w_j = \lambda_j w_j,\ w_j \in H_0^1 \cap H^2,\ w_j(x) = \sqrt{2}\sin(j\pi x),\ \lambda_j = (j\pi)^2,\ j = 1,\ 2,\ 3, \cdots$.

We find an approximate solution of Prob. (2.3), (2.4) in the form

$$u_m^{(k)}(t) = \textstyle\sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{2.6}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the following system of linear differential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \left\langle \mu(t) u_{mx}^{(k)}(t), w_{jx} \right\rangle + \lambda \left\langle \dot{u}_m^{(k)}(t), w_j \right\rangle \\ = \left\langle \Phi_m^{(k)}(t), w_j \right\rangle, \ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{2.7}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \text{ strongly in } H_0^1 \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \tilde{u}_1 \text{ strongly in } H_0^1, \end{cases} \tag{2.8}$$

and

$$\begin{aligned} &\Phi_m^{(k)}(x,t) \\ &= \sum_{i=0}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_{m-1})(u_m^{(k)} - u_{m-1})^i \\ &\quad + \sum_{i=0}^{N-1} \tfrac{1}{i!} \int_0^t \left[D_4^i g(x,t,s,u_{m-1}(x,s)) \right] \left(u_m^{(k)}(x,s) - u_{m-1}(x,s) \right)^i ds \\ &= \sum_{j=0}^{N-1} \Big[A_j(x,t,u_{m-1}) \left(u_m^{(k)} \right)^j \\ &\qquad + \int_0^t B_j(x,t,s,u_{m-1}(x,s)) \left(u_m^{(k)}(x,s) \right)^j ds \Big]. \end{aligned} \tag{2.9}$$

The system (2.7) can be written in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \sum_{i=1}^{k} \mu_{ij}(t) c_{mi}^{(k)}(t) + \lambda \dot{c}_{mj}^{(k)}(t) = \Phi_{mj}^{(k)}(t), \ 1 \le j \le k, \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \end{cases} \tag{2.10}$$

where

$$\mu_{ij}(t) = \left\langle \mu(t) w_{ix}, w_{jx} \right\rangle, \ \Phi_{mj}^{(k)}(t) = \left\langle \Phi_m^{(k)}(t), w_j \right\rangle, \ 1 \le i, \ j \le k. \tag{2.11}$$

Using the Banach's contraction principle, it is not difficult to show that (2.10) has a unique solution $c_{mj}^{(k)}(t)$ in $[0, T_m^{(k)}]$, with certain $T_m^{(k)} \in (0, T]$. Therefore, (2.7) has a unique solution $u_m^{(k)}(t)$ in $[0, T_m^{(k)}]$.

The following estimates allow one to take $T_m^{(k)} = T$ independent of m and k [2]. By such a priori estimates of $u_m^{(k)}(t)$, it can be extended outside $[0, T_m^{(k)}]$ and then, a solution defined in $[0, T]$ will be obtained.

Step 2. (A priori estimates) First, for all $j = 1, \cdots, k$, multiplying $(2.7)_1$ by $\dot{c}_{mj}^{(k)}(t)$, summing on j, and integrating with respect to the time variable from

0 to t, we have

$$\begin{aligned} X_m^{(k)}(t) &= X_m^{(k)}(0) - 2\lambda \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|^2 ds + 2\int_0^t \left\langle \Phi_m^{(k)}(s), \dot{u}_m^{(k)}(s) \right\rangle ds \\ &\quad + \int_0^t ds \int_0^1 \mu'(x,s) \left| u_{mx}^{(k)}(x,s) \right|^2 dx, \end{aligned} \tag{2.12}$$

where

$$X_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu(t)} u_{mx}^{(k)}(t) \right\|^2 .$$

Next, by replacing w_j in $(2.7)_1$ by $-w_{jxx}$, we obtain that

$$\begin{aligned} &\left\langle \ddot{u}_{mx}^{(k)}(t), w_{jx} \right\rangle + \left\langle \frac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t) \right), w_{jxx} \right\rangle + \lambda \left\langle \dot{u}_{mx}^{(k)}(t),\ w_{jx} \right\rangle \\ &= \left\langle \Phi_{mx}^{(k)}(t), w_{jx} \right\rangle, \quad 1 \le j \le k, \end{aligned}$$

similar to $(2.7)_1$, it gives

$$\begin{aligned} &Y_m^{(k)}(t) \\ &= Y_m^{(k)}(0) - 2\lambda \int_0^t \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 ds + \int_0^t ds \int_0^1 \mu'(x,s) \left| u_{mxx}^{(k)}(x,s) \right|^2 dx \\ &\quad -2\int_0^t \left\langle \mu_x(s), u_{mxx}^{(k)}(s) \right\rangle ds + 2\int_0^t \left\langle \Phi_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s) \right\rangle ds, \end{aligned} \tag{2.13}$$

where

$$Y_m^{(k)}(t) = \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu(t)} u_{mxx}^{(k)}(t) \right\|^2 .$$

We note that the equation (2.7) can be written as follows

$$\begin{aligned} &\left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle - \left\langle \frac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t) \right), w_j \right\rangle + \lambda \left\langle \dot{u}_m^{(k)}(t),\ w_j \right\rangle \\ &= \left\langle \Phi_m^{(k)}(t), w_j \right\rangle, \quad 1 \le j \le k. \end{aligned}$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\begin{aligned} \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds &\le 3\int_0^t \left\| \frac{\partial}{\partial x} \left(\mu(s) u_{mx}^{(k)}(s) \right) \right\|^2 ds \\ &\quad + 3\lambda^2 \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|^2 ds + 3\int_0^t \left\| \Phi_m^{(k)}(s) \right\|^2 ds. \end{aligned} \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) lead to

$$\begin{aligned}
&S_m^{(k)}(t) \\
&= X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \\
&= S_m^{(k)}(0) + \int_0^t ds \int_0^1 \mu'(x,s) \left[\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| u_{mxx}^{(k)}(x,s) \right|^2 \right] dx \\
&\quad +3\lambda^2 \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|^2 ds - 2\lambda \int_0^t \left(\left\| \dot{u}_m^{(k)}(s) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 \right) ds \\
&\quad -2 \int_0^t \left\langle \mu_x(s) \dot{u}_{mx}^{(k)}(s),\ \dot{u}_{mxx}^{(k)}(s) \right\rangle ds + 3 \int_0^t \left\| \frac{\partial}{\partial x} \left(\mu(s) u_{mx}^{(k)}(s) \right) \right\|^2 ds \\
&\quad +3 \int_0^t \left\| \Phi_m^{(k)}(s) \right\|^2 ds + 2 \int_0^t \left\langle \Phi_m^{(k)}(s), \dot{u}_m^{(k)}(s) \right\rangle ds \\
&\quad +2 \int_0^t \left\langle \Phi_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s) \right\rangle ds \\
&= S_m^{(k)}(0) + \sum_{j=1}^8 I_j.
\end{aligned} \tag{2.15}$$

We shall estimate, respectively the following integrals and $S_m^{(k)}(0)$ on the right-hand side of (2.15).

First integral I_1: By

$$S_m^{(k)}(t) \geq \mu_0 \left(\left\| u_{mx}^{(k)}(t) \right\|^2 + \left\| u_{mxx}^{(k)}(t) \right\|^2 \right), \tag{2.16}$$

we have

$$\begin{aligned}
I_1 &= \int_0^t ds \int_0^1 \mu'(x,s) \left[\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| u_{mxx}^{(k)}(x,s) \right|^2 \right] dx \\
&\leq \frac{1}{\mu_0} \tilde{K}(\mu) \int_0^t \left(\left\| \sqrt{\mu(s)} u_{mx}^{(k)}(s) \right\|^2 + \left\| \sqrt{\mu(s)} u_{mxx}^{(k)}(s) \right\|^2 \right) ds \\
&\leq \frac{1}{\mu_0} \tilde{K}(\mu) \int_0^t S_m^{(k)}(s) ds \leq \frac{1}{\mu_0} \tilde{K}(\mu) \int_0^t \left[1 + \left(S_m^{(k)}(s) \right)^{N-1} \right] ds.
\end{aligned} \tag{2.17}$$

Second integral I_2 :

$$\begin{aligned}
I_2 &= 3\lambda^2 \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|^2 ds \\
&\leq 3\lambda^2 \int_0^t S_m^{(k)}(s) ds \leq 3\lambda^2 \int_0^t \left[1 + \left(S_m^{(k)}(s) \right)^{N-1} \right] ds.
\end{aligned} \tag{2.18}$$

Third integral I_3 :

$$\begin{aligned}
I_3 &= -2\lambda \int_0^t \left(\left\| \dot{u}_m^{(k)}(s) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 \right) ds \\
&\leq 2 |\lambda| \int_0^t S_m^{(k)}(s) ds \leq 2 |\lambda| \int_0^t \left[1 + \left(S_m^{(k)}(s) \right)^{N-1} \right] ds.
\end{aligned} \tag{2.19}$$

Fourth integral I_4 :

$$\begin{aligned} I_4 &= -2\int_0^t \left\langle \mu_x(s)\dot{u}_{mx}^{(k)}(s),\ \dot{u}_{mxx}^{(k)}(s)\right\rangle ds \\ &= 2\left\langle \mu_x(0)u_{mx}^{(k)}(0),\ u_{mxx}^{(k)}(0)\right\rangle - 2\left\langle \mu_x(t)u_{mx}^{(k)}(t),\ u_{mxx}^{(k)}(t)\right\rangle \\ &\quad +2\int_0^t \left\langle \frac{\partial}{\partial s}\left(\mu_x(s)u_{mx}^{(k)}(s)\right),\ u_{mxx}^{(k)}(s)\right\rangle ds \\ &= 2\left\langle \mu_x(0)\tilde{u}_{0kx},\ \tilde{u}_{0kxx}\right\rangle + I_4^{(1)} + I_4^{(2)}. \end{aligned} \tag{2.20}$$

We shall estimate $I_4^{(1)}$ and $I_4^{(2)}$ as follows:

Estimate $I_4^{(1)}$:

$$\begin{aligned} I_4^{(1)} &= -2\left\langle \mu_x(t)u_{mx}^{(k)}(t),\ u_{mxx}^{(k)}(t)\right\rangle \le 2\tilde{K}(\mu)\left\|u_{mx}^{(k)}(t)\right\|\left\|u_{mxx}^{(k)}(t)\right\| \\ &\le \tfrac{1}{2}\left\|u_{mxx}^{(k)}(t)\right\|^2 + 2\tilde{K}^2(\mu)\left\|u_{mx}^{(k)}(t)\right\|^2 \\ &\le \tfrac{1}{2}\left\|u_{mxx}^{(k)}(t)\right\|^2 + 2\tilde{K}^2(\mu)\left[\left\|\tilde{u}_{0kx}\right\| + \int_0^t\left\|\dot{u}_{mx}^{(k)}(s)\right\| ds\right]^2 \\ &\le \tfrac{1}{2}\left\|u_{mxx}^{(k)}(t)\right\|^2 + 2\tilde{K}^2(\mu)\left[2\left\|\tilde{u}_{0kx}\right\|^2 + 2t\int_0^t\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 ds\right] \\ &\le \tfrac{1}{2}S_m^{(k)}(t) + 2\tilde{K}^2(\mu)\left[2\left\|\tilde{u}_{0kx}\right\|^2 + 2t\int_0^t S_m^{(k)}(s)ds\right] \\ &\le 4\tilde{K}^2(\mu)\left\|\tilde{u}_{0kx}\right\|^2 + \tfrac{1}{2}S_m^{(k)}(t) + 4\tilde{K}^2(\mu)T^*\int_0^t S_m^{(k)}(s)ds. \end{aligned} \tag{2.21}$$

Estimate $I_4^{(2)}$:

$$\begin{aligned} I_4^{(2)} &= 2\int_0^t \left\langle \frac{\partial}{\partial s}\left(\mu_x(s)u_{mx}^{(k)}(s)\right),\ u_{mxx}^{(k)}(s)\right\rangle ds \\ &= 2\int_0^t \left\langle \dot{\mu}_x(s)u_{mx}^{(k)}(s) + \mu_x(s)\dot{u}_{mx}^{(k)}(s),\ u_{mxx}^{(k)}(s)\right\rangle ds \\ &\le 2\tilde{K}(\mu)\int_0^t \left(\left\|u_{mx}^{(k)}(s)\right\| + \left\|\dot{u}_{mx}^{(k)}(s)\right\|\right)\left\|u_{mxx}^{(k)}(s)\right\| ds \\ &\le 2\tilde{K}(\mu)\int_0^t \left(\sqrt{\frac{S_m^{(k)}(s)}{\mu_0}} + \sqrt{S_m^{(k)}(s)}\right)\sqrt{\frac{S_m^{(k)}(s)}{\mu_0}}ds \\ &= 2\tilde{K}(\mu)\frac{1+\sqrt{\mu_0}}{\mu_0}\int_0^t S_m^{(k)}(s)ds. \end{aligned} \tag{2.22}$$

Hence, we deduce from (2.20)-(2.22) that

$$\begin{aligned} I_4 &\le 2\left\langle \mu_x(0)\tilde{u}_{0kx}, \tilde{u}_{0kxx}\right\rangle + 4\tilde{K}^2(\mu)\left\|\tilde{u}_{0kx}\right\|^2 + \tfrac{1}{2}S_m^{(k)}(t) \\ &\quad +2\tilde{K}(\mu)\left(2T^*\tilde{K}(\mu) + \frac{1+\sqrt{\mu_0}}{\mu_0}\right)\int_0^t \left[1 + \left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right] ds. \end{aligned} \tag{2.23}$$

Fifth integral I_5 :

$$\begin{aligned} I_5 &= 3\int_0^t \left\| \frac{\partial}{\partial x}\left(\mu(s) u_{mx}^{(k)}(s)\right)\right\|^2 ds \\ &\leq 3\tilde{K}^2(\mu)\int_0^t \left[\left\|u_{mx}^{(k)}(s)\right\| + \left\|u_{mxx}^{(k)}(s)\right\|\right]^2 ds \\ &\leq \frac{6}{\mu_0}\tilde{K}^2(\mu)\int_0^t \left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right] ds. \end{aligned} \tag{2.24}$$

To estimate integrals $I_6,\ I_7,\ I_8$, we use the following Lemma.

Lemma 2.2. *We have*

$$\begin{aligned} &\text{(i)}\ \left\|\Phi_m^{(k)}(t)\right\|_{L^\infty} \\ &\quad \leq \tilde{\alpha}_M\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] + \tilde{\beta}_M\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right] ds, \\ &\text{(ii)}\ \left\|\Phi_{mx}^{(k)}(t)\right\| \\ &\quad \leq \tilde{\alpha}_M\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] + \tilde{\beta}_M\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right] ds, \end{aligned} \tag{2.25}$$

where $\tilde{\alpha}_M$ and $\tilde{\beta}_M$ are defined as follows

$$\begin{cases} \tilde{\alpha}_M = K_M(f)\sum_{i=0}^{N-1}\tilde{b}_i,\ \tilde{\beta}_M = \bar{K}_M(g)\sum_{i=0}^{N-1}\tilde{b}_i, \\ \tilde{b}_i = \begin{cases} 1+M+\sum_{i=1}^{N-1}\frac{2^{i-1}}{i!}(1+M+i)M^i, & i=0, \\ \frac{2^{i-1}}{i!}\frac{1+M+i}{\sqrt{\mu_0^i}}, & 1\leq i\leq N-1. \end{cases} \end{cases} \tag{2.26}$$

Proof. (i) Using the inequalities $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, for all $a,\, b>0,\ p\geq 1$ and $s^i \leq 1+s^q,\ \forall s\geq 0,\ \forall i,\ q,\ 0\leq i\leq q$, we have

$$\begin{aligned} &\sum_{i=0}^{N-1}\frac{1}{i!}\left(\left|u_m^{(k)}\right| + |u_{m-1}|\right)^i \\ &\leq 1+\sum_{i=1}^{N-1}\frac{1}{i!}\left(\left\|u_{mx}^{(k)}(t)\right\| + M\right)^i \leq 1+\sum_{i=1}^{N-1}\frac{2^{i-1}}{i!}\left(\left\|u_{mx}^{(k)}(t)\right\|^i + M^i\right) \\ &\leq 1+\sum_{i=1}^{N-1}\frac{2^{i-1}}{i!}\left[\left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}}\right)^i + M^i\right] \\ &= 1+\sum_{i=1}^{N-1}\frac{2^{i-1}}{i!}M^i + \sum_{i=1}^{N-1}\frac{2^{i-1}}{i!}\frac{1}{\sqrt{\mu_0^i}}\left(\sqrt{S_m^{(k)}(t)}\right)^i \end{aligned}$$

$$\leq \sum_{i=0}^{N-1} \tilde{b}_i \left(\sqrt{S_m^{(k)}(t)}\right)^i \leq \sum_{i=0}^{N-1} \tilde{b}_i \left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right], \tag{2.27}$$

with $\tilde{b}_i$, $0 \leq i \leq N-1$ defined by (2.26). Hence

$$\begin{aligned}
&\left|\Phi_m^{(k)}(x,t)\right| \\
&\leq \sum_{i=0}^{N-1} \left|\tfrac{1}{i!} D_3^i f(x,t,u_{m-1})(u_m^{(k)} - u_{m-1})^i\right| \\
&\quad + \sum_{i=0}^{N-1} \tfrac{1}{i!} \int_0^t \left| \left[D_4^i g(x,t,s,u_{m-1})\right] \left(u_m^{(k)}(x,s) - u_{m-1}(x,s)\right)^i \right| ds \\
&\leq K_M(f) \left[\sum_{i=0}^{N-1} \tfrac{1}{i!} \left(\left|u_m^{(k)}\right| + |u_{m-1}|\right)^i\right] \\
&\quad + \bar{K}_M(g) \int_0^t \left[\sum_{i=0}^{N-1} \tfrac{1}{i!} \left(\left|u_m^{(k)}(x,s)\right| + |u_{m-1}(x,s)|\right)^i\right] ds \qquad (2.28) \\
&\leq K_M(f) \sum_{i=0}^{N-1} \tilde{b}_i \left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] \\
&\quad + \bar{K}_M(g) \sum_{i=0}^{N-1} \tilde{b}_i \int_0^t \left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right] ds \\
&\leq \tilde{\alpha}_M \left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] + \tilde{\beta}_M \int_0^t \left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right] ds,
\end{aligned}$$

it implies that Lemma 2.2 (i) holds.

(ii) We also have

$$\begin{aligned}
&\Phi_{mx}^{(k)}(x,t) \\
&= D_1 f(x,t,u_{m-1}) + D_3 f(x,t,u_{m-1}) \nabla u_{m-1} \\
&\quad + \sum_{i=1}^{N-1} \tfrac{1}{i!} \Big\{ \left[D_1 D_3^i f(x,t,u_{m-1}) + D_3^{i+1} f(x,t,u_{m-1}) \nabla u_{m-1}\right] (u_m^{(k)} - u_{m-1})^i \\
&\quad + f(x,t,u_{m-1}) i (u_m^{(k)} - u_{m-1})^{i-1} (u_{mx}^{(k)} - \nabla u_{m-1}) \Big\} \\
&\quad + \int_0^t \left[D_1 g(.) + D_4 g(.) \nabla u_{m-1}(x,s)\right] ds \\
&\quad + \sum_{i=1}^{N-1} \tfrac{1}{i!} \int_0^t \Big\{ \left[D_1 D_4^i g(.) + D_4^{i+1} g(.) \nabla u_{m-1}(x,s)\right] \left(u_m^{(k)}(x,s) - u_{m-1}(x,s)\right)^i \\
&\quad + D_4^i g(.) i \left(u_m^{(k)}(x,s) - u_{m-1}(x,s)\right)^{i-1} \left(u_{mx}^{(k)}(x,s) - \nabla u_{m-1}(x,s)\right) \Big\} ds,
\end{aligned} \tag{2.29}$$

in which $g(.) = g(x,t,s,u_{m-1}(x,s))$, hence

$$\begin{aligned}
&\left\|\Phi_{mx}^{(k)}(t)\right\| \\
&\leq K_M(f)(1+\|\nabla u_{m-1}\|)+\sum_{i=1}^{N-1}\tfrac{1}{i!}\Big\{K_M(f)(1+\|\nabla u_{m-1}\|)\left\|u_{mx}^{(k)}-\nabla u_{m-1}\right\|^i \\
&\quad +iK_M(f)\left\|u_{mx}^{(k)}-\nabla u_{m-1}\right\|^i\Big\} + \textstyle\int_0^t \bar{K}_M(g)\,(1+\|\nabla u_{m-1}(s)\|)\,ds \\
&\quad +\sum_{i=1}^{N-1}\tfrac{1}{i!}\textstyle\int_0^t\Big\{\bar{K}_M(g)\,(1+\|\nabla u_{m-1}\|)\left\|u_{mx}^{(k)}(s)-\nabla u_{m-1}\right\|^i \\
&\quad +i\bar{K}_M(g)\left\|u_{mx}^{(k)}(s)-\nabla u_{m-1}\right\|^i\Big\}\,ds \\
&\leq K_M(f)\,(1+M)+\sum_{i=1}^{N-1}\tfrac{1}{i!}\Big\{K_M(f)\,(1+M)\left(\left\|u_{mx}^{(k)}(t)\right\|+M\right)^i \qquad (2.30)\\
&\quad +iK_M(f)\left(\left\|u_{mx}^{(k)}(t)\right\|+M\right)^i\Big\} + \textstyle\int_0^t \bar{K}_M(g)\,(1+M)\,ds \\
&\quad +\sum_{i=1}^{N-1}\tfrac{1}{i!}\textstyle\int_0^t\Big\{\bar{K}_M(g)\,(1+M)\left(\left\|u_{mx}^{(k)}(s)\right\|+M\right)^i \\
&\quad +i\bar{K}_M(g)\left(\left\|u_{mx}^{(k)}(s)\right\|+M\right)^i\Big\}\,ds \\
&= K_M(f)\sum_{i=0}^{N-1}\tfrac{1}{i!}\,(1+M+i)\left(\left\|u_{mx}^{(k)}(t)\right\|+M\right)^i \\
&\quad +\bar{K}_M(g)\textstyle\int_0^t\sum_{i=0}^{N-1}\tfrac{1}{i!}\,(1+M+i)\left(\left\|u_{mx}^{(k)}(s)\right\|+M\right)^i ds,
\end{aligned}$$

where $\nabla u_{m-1} = \nabla u_{m-1}(t)$, or $\nabla u_{m-1} = \nabla u_{m-1}(s)$. Note that

$$\begin{aligned}
&\sum_{i=0}^{N-1}\tfrac{1}{i!}\,(1+M+i)\left(\left\|u_{mx}^{(k)}(t)\right\|+M\right)^i \\
&\leq \sum_{i=0}^{N-1}\tfrac{1}{i!}\,(1+M+i)\left[\left(\sqrt{\tfrac{S_m^{(k)}(t)}{\mu_0}}\right)+M\right]^i \\
&\leq \sum_{i=0}^{N-1}\tfrac{1}{i!}\,(1+M+i)\,2^{i-1}\left[\left(\sqrt{\tfrac{S_m^{(k)}(t)}{\mu_0}}\right)^i+M^i\right] \qquad (2.31)\\
&= \sum_{i=0}^{N-1}\tfrac{1}{i!}\,(1+M+i)\,2^{i-1}M^i+\sum_{i=0}^{N-1}\tfrac{1}{i!}\,(1+M+i)\,2^{i-1}\tfrac{1}{\sqrt{\mu_0^i}}\left(\sqrt{S_m^{(k)}(t)}\right)^i \\
&= 1+M+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}(1+M+i)\,M^i+\sum_{i=1}^{N-1}\tfrac{1}{i!}(1+M+i)\,2^{i-1}\tfrac{1}{\sqrt{\mu_0^i}}\left(\sqrt{S_m^{(k)}(t)}\right)^i \\
&= \sum_{i=0}^{N-1}\tilde{b}_i\left(\sqrt{S_m^{(k)}(t)}\right)^i \leq \sum_{i=0}^{N-1}\tilde{b}_i\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right],
\end{aligned}$$

hence

$$\begin{aligned}
&\left\|\Phi_{mx}^{(k)}(t)\right\| \\
&\leq K_M(f)\sum_{i=0}^{N-1}\tilde{b}_i\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] \\
&\quad+\bar{K}_M(g)\sum_{i=0}^{N-1}\tilde{b}_i\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]ds \\
&=\tilde{\alpha}_M\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right]+\tilde{\beta}_M\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]ds.
\end{aligned} \tag{2.32}$$

It implies that Lemma 2.2 (ii) holds. Therefore, Lemma 2.2 is proved. □

Now, the integrals $I_6,\ I_7,\ I_8$ are estimated as follows.

$$\begin{aligned}
&I_6=3\int_0^t\left\|\Phi_m^{(k)}(s)\right\|^2ds \\
&\leq 6\tilde{\alpha}_M^2\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]^2ds \\
&\quad+6\tilde{\beta}_M^2\int_0^t\left[\int_0^s\left(1+\left(\sqrt{S_m^{(k)}(\tau)}\right)^{N-1}\right)d\tau\right]^2ds \\
&\leq 12\tilde{\alpha}_M^2\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds+12\tilde{\beta}_M^2\int_0^t\left[s\int_0^s\left(1+\left(S_m^{(k)}(\tau)\right)^{N-1}\right)d\tau\right]ds \\
&\leq 12\tilde{\alpha}_M^2\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds+6\tilde{\beta}_M^2T_*^2\int_0^t\left(1+\left(S_m^{(k)}(\tau)\right)^{N-1}\right)d\tau \\
&\leq 6\left(2\tilde{\alpha}_M^2+\tilde{\beta}_M^2T_*^2\right)\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds;
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
&I_7=2\int_0^t\left\langle\Phi_m^{(k)}(s),\dot{u}_m^{(k)}(s)\right\rangle ds\leq 2\int_0^t\left\|\Phi_m^{(k)}(s)\right\|\left\|\dot{u}_m^{(k)}(s)\right\|ds \\
&\leq 2\tilde{\alpha}_M\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{S_m^{(k)}(s)}ds \\
&\quad+2\tilde{\beta}_M\int_0^t\sqrt{S_m^{(k)}(s)}ds\int_0^s\left[1+\left(\sqrt{S_m^{(k)}(\tau)}\right)^{N-1}\right]d\tau \\
&\leq 2\tilde{\alpha}_M\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]^2ds+2\tilde{\beta}_M\left(\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]ds\right)^2
\end{aligned}$$

$$\begin{aligned}&\le 4\tilde{\alpha}_M \int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds+4\tilde{\beta}_M t\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds\\&\le 4\left(\tilde{\alpha}_M+\tilde{\beta}_M T_*\right)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}I_8 &= 2\int_0^t \left\langle \Phi_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s)\right\rangle ds \le 2\int_0^t \left\|\Phi_{mx}^{(k)}(s)\right\| \left\|\dot{u}_{mx}^{(k)}(s)\right\| ds\\&\le 4\left(\tilde{\alpha}_M+\tilde{\beta}_M T_*\right)\int_0^t \left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds.\end{aligned} \tag{2.35}$$

Combining (2.15)-(2.19), (2.23), (2.24) and (2.23)-(2.35), we have

$$\begin{aligned}S_m^{(k)}(t) &\le 2S_m^{(k)}(0)+4\left\langle \mu_x(0)\tilde{u}_{0kx},\ \tilde{u}_{0kxx}\right\rangle+8\tilde{K}^2(\mu)\left\|\tilde{u}_{0kx}\right\|^2\\&\quad +TC_1(M)+C_1(M)\int_0^t \left(S_m^{(k)}(s)\right)^N ds,\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}C_1(M) &= \tfrac{1}{\mu_0}\tilde{K}(\mu)+3\lambda^2+2\left|\lambda\right|+2\tilde{K}(\mu)\left(2T^*\tilde{K}(\mu)+\tfrac{1+\sqrt{\mu_0}}{\mu_0}\right)\\&\quad +\tfrac{6}{\mu_0}\tilde{K}^2(\mu)+6\left(2\tilde{\alpha}_M^2+\tilde{\beta}_M^2 T_*^2\right)+8\left(\tilde{\alpha}_M+\tilde{\beta}_M T_*\right).\end{aligned} \tag{2.37}$$

By means of the convergences (2.8) we can deduce the existence of a constant $M>0$ independent of k and m such that

$$2S_m^{(k)}(0)+4\left\langle \mu_x(0)\tilde{u}_{0kx},\ \tilde{u}_{0kxx}\right\rangle+8\tilde{K}^2(\mu)\left\|\tilde{u}_{0kx}\right\|^2 \le \tfrac{M^2}{4}, \tag{2.38}$$

for all $m, k\in\mathbb{N}$. Finally, it follows from (2.36) and (2.38) that

$$S_m^{(k)}(t) \le \tfrac{M^2}{4}+TC_1(M)+C_1(M)\int_0^t \left(S_m^{(k)}(s)\right)^N ds, \tag{2.39}$$

for $0\le t\le T_m^{(k)}\le T$. Then, by solving a nonlinear Volterra integral inequality (2.39) (based on the methods in [4]), the following lemma is proved.

Lemma 2.3. *There exist a constant $T>0$ independent of k and m such that*

$$S_m^{(k)}(t)\le M^2,\quad \forall\, t\in[0,T],\ \forall\, m,\ k\in\mathbb{N}. \tag{2.40}$$

By Lemma 2.3, we can take constant $T_m^{(k)}=T$ for all k and $m\in\mathbb{N}$. Thus, we have

$$u_m^{(k)}\in W(M,T),\ \ \forall\, m,\ k\in\mathbb{N}. \tag{2.41}$$

Step 3. (Convergence) Thanks to (2.41), there exists a subsequence $\{u_m^{(k_j)}\}$ of $\{u_m^{(k)}\}$, still denoted by $\{u_m^{(k)}\}$ such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in } \ L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in } \ L^\infty(0,T;H_0^1) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in } \ L^2(Q_T) \text{ weakly}, \\ u_m \in W(M,T). \end{cases} \tag{2.42}$$

Using the compactness lemma of Lions ([4], p.57) and applying Fischer - Riesz theorem, from (2.42), there exists a subsequence of $\{u_m^{(k)}\}$, denoted by the same symbol satisfying

$$\begin{cases} u_m^{(k)} \to u_m & \text{strong in } \ L^2(0,T;H_0^1) & \text{and a.e. in } Q_T, \\ \dot{u}_m^{(k)} \to u_m' & \text{strong in } \ L^2(Q_T) & \text{and a.e. in } Q_T. \end{cases} \tag{2.43}$$

On the other hand, by $L^\infty(0,T;H_0^1 \cap H^2) \hookrightarrow L^\infty(Q_T)$ and using the inequality

$$\left|a^j - b^j\right| \le jM^{j-1}\left|a-b\right|, \quad \forall\, a,b \in [-M,M], \quad \forall\, M>0, \quad \forall\, j \in \mathbb{N}, \tag{2.44}$$

we deduce from (2.41) that

$$\left|(u_m^{(k)})^j - u_m^j\right| \le jM^{j-1}\left|u_m^{(k)} - u_m\right|, \quad 0 \le j \le N-1. \tag{2.45}$$

Therefore, (2.43) and (2.45) give

$$(u_m^{(k)})^j \to u_m^j \quad \text{strong in} \quad L^2(Q_T). \tag{2.46}$$

We note that

$$\begin{aligned} &|A_j(x,t,u_{m-1}(t))| \le K_M(f)\textstyle\sum_{i=j}^{N-1} \frac{M^{i-j}}{j!(i-j)!} \equiv \bar{D}_j(M), \\ &|B_j(x,t,s,u_{m-1}(s))| \le \bar{K}_M(g)\textstyle\sum_{i=j}^{N-1} \frac{M^{i-j}}{j!(i-j)!} \\ &\qquad\qquad \equiv \tilde{D}_j(M), \quad 0 \le j \le N-1. \end{aligned} \tag{2.47}$$

By (2.4), (2.9) and (2.47), we obtain

$$\begin{aligned} \left\|\Phi_m^{(k)}(t) - \Phi_m(t)\right\| &\le \textstyle\sum_{j=0}^{N-1} \bar{D}_j(M)\left\|(u_m^{(k)}(t))^j - u_m^j(t)\right\| \\ &\quad + \textstyle\sum_{j=0}^{N-1} \sqrt{T}\tilde{D}_j(M)\left\|(u_m^{(k)})^j - u_m^j\right\|_{L^2(Q_T)}. \end{aligned} \tag{2.48}$$

Hence, we have

$$\begin{aligned} &\left\|\Phi_m^{(k)} - \Phi_m\right\|_{L^2(Q_T)}^2 \\ &\le 2N\textstyle\sum_{j=0}^{N-1}\left(\bar{D}_j^2(M) + T^2\tilde{D}_j^2(M)\right)\left\|(u_m^{(k)})^j - u_m^j\right\|_{L^2(Q_T)}^2. \end{aligned} \tag{2.49}$$

It leads to

$$\Phi_m^{(k)} \to \Phi_m \quad \text{strong in} \quad L^2(Q_T). \tag{2.50}$$

Passing to limit in (2.7), (2.8), we have u_m satisfying (2.3), (2.4) in $L^2(0,T)$.

On the other hand, it follows from $(2.3)_1$ and $(2.42)_4$ that

$$u_m'' = \frac{\partial}{\partial x}\left(\mu(x,t)\frac{\partial u_m}{\partial x}\right) - \lambda u_m' + \Phi_m \in L^\infty(0,T;L^2).$$

Hence, $u_m \in W_1(M,T)$ and Theorem 2.1 is proved. □

Next, the main result is given by the following theorem. We consider the space $W_1(T)$, defined by

$$W_1(T) = \{v \in L^\infty(0,T;H_0^1) : v' \in L^\infty(0,T;L^2)\}, \tag{2.51}$$

then $W_1(T)$ is a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H_0^1)} + \|v'\|_{L^\infty(0,T;L^2)}\,. \tag{2.52}$$

Theorem 2.4. *Let (H_1)-(H_4) hold. Then, there exist constants $M>0$ and $T>0$ such that the problem (1.1)-(1.3) has a unique weak solution $u \in W_1(M,T)$ and the recurrent sequence $\{u_m\}$, defined by (2.3)-(2.4), converges at a rate of order N to the solution u strongly in the space $W_1(T)$ in sense*

$$\|u_m - u\|_{W_1(T)} \le C\,\|u_{m-1} - u\|_{W_1(T)}^N\,, \tag{2.53}$$

for all $m \ge 1$, where C is a suitable constant. On the other hand, the following estimate is fulfilled

$$\|u_m - u\|_{W_1(T)} \le C_T \beta^{N^m}, \quad \text{for all } m \in \mathbb{N}, \tag{2.54}$$

where C_T and $0<\beta<1$ are the constants depending only on T.

Proof. (Existence of a solution) We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Indeed, we put $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{cases} \langle v_m''(t), w\rangle + \langle \mu(t) v_{mx}(t), w_x\rangle + \lambda\langle v_m'(t), w\rangle \\ = \langle \Phi_{m+1}(t) - \Phi_m(t), w\rangle\,, \quad \forall\, w \in H_0^1, \\ v_m(0) = v_m'(0) = 0. \end{cases} \tag{2.55}$$

Taking $w = v_m'$ in (2.55), after integrating in t, we have

$$\begin{aligned} \rho_m(t) &\le 2\,|\lambda| \int_0^t \|v_m'(s)\|^2\, ds + \int_0^t ds \int_0^1 |\mu'(x,s)|\, v_{mx}^2(x,s) dx \\ &\quad + 2\int_0^t \|\Phi_{m+1}(s) - \Phi_m(s)\|\, \|v_m'(s)\|\, ds \equiv \sum_{k=1}^3 J_k, \end{aligned} \tag{2.56}$$

where

$$\rho_m(t) = \|v_m'(t)\|^2 + \left\|\sqrt{\mu(t)}\, v_{mx}(t)\right\|^2. \tag{2.57}$$

Next, we need to estimate the integrals on the right side of (2.56) as follows

$$J_1 = 2\left|\lambda\right| \int_0^t \left\|v_m'(s)\right\|^2 ds \leq 2\left|\lambda\right| \int_0^t \rho_m(s)ds, \tag{2.58}$$

$$J_2 = \int_0^t ds \int_0^1 \left|\mu'(x,s)\right| v_{mx}^2(x,s)dx \leq \frac{1}{\mu_0}\tilde{K}(\mu) \int_0^t \rho_m(s)ds. \tag{2.59}$$

Using Taylor's expansion of the functions $f(x,t,u_m) = f(x,t,u_{m-1}+v_{m-1})$ and $g(x,t,s,u_m) = g(x,t,s,u_{m-1}+v_{m-1})$ around the point u_{m-1} up to order N, we obtain

$$\begin{aligned} f(x,t,u_m) - f(x,t,u_{m-1}) &= \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1}) v_{m-1}^i \\ &\quad + \frac{1}{N!} D_3^N f(x,t,\delta_{m-1}) v_{m-1}^N, \\ g(x,t,s,u_m) - g(x,t,s,u_{m-1}) &= \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} D_4^i g(x,t,s,u_{m-1}) v_{m-1}^i \\ &\quad + \frac{1}{N!} D_4^N g(x,t,s,\tilde{\delta}_{m-1}) v_{m-1}^N, \end{aligned} \tag{2.60}$$

where

$$\delta_{m-1} = \delta_{m-1}(x,t) = u_{m-1} + \theta_1 v_{m-1},\ 0 < \theta_1 < 1$$

and

$$\tilde{\delta}_{m-1} = \tilde{\delta}_{m-1}(x,s) = u_{m-1} + \theta_2 v_{m-1},\ 0 < \theta_2 < 1.$$

Hence, it follows from (2.4) and (2.60) that

$$\begin{aligned} &\Phi_{m+1}(x,t) - \Phi_m(x,t) \\ &= \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_m) v_m^i + \frac{1}{N!} D_3^N f(x,t,\delta_m) v_{m-1}^N \\ &\quad + \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} \int_0^t D_4^i g(x,t,s,u_m) v_m^i ds + \frac{1}{N!} \int_0^t D_4^N g(x,t,s,\tilde{\delta}_m) v_{m-1}^N ds. \end{aligned} \tag{2.61}$$

Therefore, we have

$$\begin{aligned} &\left\|\Phi_{m+1}(t) - \Phi_m(t)\right\| \\ &\leq K_M(f) \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} \left\|v_{mx}(t)\right\|^i + \frac{1}{N!} K_M(f) \left\|v_{m-1,\,x}(t)\right\|^N \\ &\quad + \bar{K}_M(g) \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} \int_0^t \left\|v_{mx}(s)\right\|^i ds + \frac{1}{N!} \bar{K}_M(g) \int_0^t \left\|v_{m-1,\,x}(s)\right\|^N ds \\ &\leq \frac{1}{\sqrt{\mu_0}} K_M(f) \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} M^{i-1} \sqrt{\rho_m(t)} + \frac{1}{N!} K_M(f) \left\|v_{m-1}\right\|_{W_1(T)}^N \\ &\quad + \frac{1}{\sqrt{\mu_0}} \bar{K}_M(g) \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} M^{i-1} \int_0^t \sqrt{\rho_m(s)} ds + \frac{1}{N!} T \bar{K}_M(g) \left\|v_{m-1}\right\|_{W_1(T)}^N \\ &\leq \eta_T^{(1)} \sqrt{\rho_m(t)} + \eta_T^{(2)} \int_0^t \sqrt{\rho_m(s)} ds + \eta_T^{(3)} \left\|v_{m-1}\right\|_{W_1(T)}^N, \end{aligned} \tag{2.62}$$

where

$$\begin{aligned} \eta_T^{(1)} &= \frac{1}{\sqrt{\mu_0}} K_M(f) \sum_{i=1}^{N} \frac{1}{i!} M^{i-1}, \\ \eta_T^{(2)} &= \frac{1}{\sqrt{\mu_0}} \bar{K}_M(g) \sum_{i=1}^{N} \frac{1}{i!} M^{i-1}, \\ \eta_T^{(3)} &= \frac{1}{N!} \left(K_M(f) + T\bar{K}_M(g)\right). \end{aligned} \tag{2.63}$$

Then we deduce from (2.56), (2.58), (2.59) and (2.62) that

$$\begin{aligned}\rho_m(t) &\le \left(2|\lambda| + \tfrac{1}{\mu_0}\tilde{K}(\mu)\right)\int_0^t \rho_m(s)ds \\ &\quad +2\int_0^t \left(\eta_T^{(1)}\sqrt{\rho_m(s)} + \eta_T^{(2)}\int_0^s \sqrt{\rho_m(\tau)}d\tau \right.\\ &\quad \left. +\eta_T^{(3)}\|v_{m-1}\|_{W_1(T)}^N\right)\sqrt{\rho_m(s)}ds \\ &\le T\eta_T^{(3)}\|v_{m-1}\|_{W_1(T)}^{2N} + \eta_T^{(4)}\int_0^t \rho_m(s)ds,\end{aligned} \tag{2.64}$$

where $\eta_T^{(4)} = 2|\lambda| + \frac{1}{\mu_0}\tilde{K}(\mu) + 2\eta_T^{(1)} + 2T\eta_T^{(2)} + \eta_T^{(3)}$. By using Gronwall's lemma, (2.64) leads to

$$\|v_m\|_{W_1(T)} \le \mu_T \|v_{m-1}\|_{W_1(T)}^N, \tag{2.65}$$

where $\mu_T = \left(1 + \frac{1}{\sqrt{\mu_0}}\right)\sqrt{T\eta_T^{(3)}\exp\left(T\eta_T^{(4)}\right)}$.

Choosing $T > 0$ enough small such that $\beta = M\mu_T^{\frac{-1}{N-1}} < 1$, it follows from (2.65) that, for all m and p,

$$\|u_m - u_{m+p}\|_{W_1(T)} \le (1-\beta)^{-1}(\mu_T)^{\frac{-1}{N-1}}\beta^{N^m}. \tag{2.66}$$

Hence, $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strong in} \quad W_1(T). \tag{2.67}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;H_0^1) \text{ weakly*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ u \in W(M,T). \end{cases} \tag{2.68}$$

On the other hand

$$\begin{aligned}&\left\|\Phi_m(\cdot,t) - f(\cdot,t,u(t)) - \int_0^t g(\cdot,t,s,u(s))ds\right\| \\ &\le \|f(\cdot,t,u_{m-1}(t)) - f(\cdot,t,u(t))\| + \left\|\sum_{i=1}^{N-1}\frac{1}{i!}\frac{\partial^i f}{\partial u^i}(\cdot,t,u_{m-1})(u_m - u_{m-1})^i\right\| \\ &\quad + \left\|\int_0^t g(\cdot,t,s,u_{m-1}(s))ds - \int_0^t g(\cdot,t,s,u(s))ds\right\| \\ &\quad + \left\|\sum_{i=1}^{N-1}\frac{1}{i!}\int_0^t\left[\frac{\partial^i g}{\partial u^i}(\cdot,t,s,u_{m-1}(s))\right](u_m(s) - u_{m-1}(s))^i ds\right\| \\ &\le \left(K_M(f) + T\bar{K}_M(g)\right)\left[\|u_{m-1} - u\|_{W_1(T)} + \sum_{i=1}^{N-1}\frac{1}{i!}\|u_m - u_{m-1}\|_{W_1(T)}^i\right].\end{aligned} \tag{2.69}$$

Therefore, it implies from (2.67) and (2.69) that

$$\Phi_m(t) \to f(\cdot, t, u(t)) + \int_0^t g(\cdot, t, s, u(s))ds \quad \text{strong in} \quad L^\infty(0, T; L^2). \tag{2.70}$$

Finally, passing to limit in (2.3) and (2.4) as $m = m_j \to \infty$, there exists $u \in W(M, T)$ satisfying the equation

$$\begin{aligned} &\langle u''(t), w\rangle + \langle \mu(t) u_x(t), w_x\rangle + \lambda \langle u'(t), w\rangle \\ &= \langle f(\cdot, t, u(t)), w\rangle + \left\langle \int_0^t g(\cdot, t, s, u(s))ds, w \right\rangle, \end{aligned} \tag{2.71}$$

for all $w \in H_0^1$ and the initial condition

$$u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1.$$

(Uniqueness) Applying a similar argument used in the proof of Theorem 2.1, $u \in W_1(M, T)$ is a unique local weak solution of Pro. (1.1)-(1.3).

Passing to the limit in (2.66) as $p \to \infty$ for fixed m, we get (2.54). Also with a similar argument, (2.53) follows. Theorem 2.4 is proved completely. □

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