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A STUDY OF GENERALIZED PD-OPERATORS ON PROBABILISTIC METRIC SPACE

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Abstract. In this paper, we introduce the notion of generalized PD-operator pair with order n for single valued mappings and obtain some common fixed point theorems for generalized PD-operators on a set X equipped with the function $F : X \times X \to \Delta$ without using the triangle inequality besides relaxing symmetric condition. Our results extend the results of Pathak and Rai [Common fixed points for PD-operator pairs under relaxed conditions with applications, Journal of Computational and Applied Mathematics, 239(1) (2013), 103–113], Hussain et al.[Common fixed points for JH-operators and occasionally weakly biased pairs under relaxed conditions, Nonlinear Anal., 74 (2011), 2133-2140], Sintunavarat and Poom [Common fixed point theorems for generalized JH-operator classes and invariant approximations, Journal of ineqalities and Applications, 67 (2011)] and several others.

1. Introduction

Probabilistic metric spaces were first introduced by Menger in 1942 and reconsidered by him in the early 1950's [21, 22, 23]. Since 1958, Schweizer and Sklar [34] have been studying these spaces, and have developed their theory in depth [10, 35, 36]. These spaces have also been considered by several other authors e.g., [24, 25]. An extensive, detailed up-to-date presentation may be found in [11]. Sehgal and Bharucha-Reid [37] obtained a generalization of Banach contraction principle on a complete Menger space. Ciric [8] defined the generalized contractions on probabilistic metric space which is an important step in the development of fixed point theorems in probabilistic metric space. Over the years, the theory has found several important applications in the

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investigation of physical quantities in quantum particle physics and string theory as studied by Naschie [26, 27]. The area of probabilistic metric spaces is also of fundamental importance in probabilistic functional analysis.

In 1976, Jungck [13] initiated a study of common fixed points of commuting maps. On the other hand in 1982, Sessa [38] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [14] soon enlarged this concept to compatible maps. The notion of compatible mappings in a Mengar space has been introduced by Mishra [25]. After this, Jungck and Rhoades [16] gave the concept of weakly compatible maps. Aamri and El Moutawakil [1] introduced the $(E.A.)$ property and thus generalized the concept of non-compatible maps. The results obtained in the metric fixed point theory by using the notion of non-compatible maps or the $(E.A)$ property are very interesting. Al-Thagafi and Shahzad [40](Main Results also, Jungck and Rhoades [18]) defined the concept of occasionally weakly compatible mappings which is more general than the concept of weakly compatible maps. Bhatt et. al. [4] have given application of occasionally weakly compatible mappings in dynamical programming. Pathak and Hussain [31] defined the concept of P-operators. Hussain et. al. [12] gave the concepts of JH-operators and occasionally weakly g-biased. Sintunavarat and Poom defined the concept of generalized JH-operators. Recently Pathak and Rai [32] proved some common fixed point theorems for more generalized non commuting notion, namely, PD-operators and gave some applications in variational inequalities and dynamical programming.

In this paper, we extend some common fixed point theorems for generalized PD-operators under relaxed condition on probabilistic metric space. Our results extend the results of Pathak and Rai [32], Hussain et. al. [12], Bhatt et. al. [4] and others [5, 6, 7, 17, 19, 30, 33, 39].

2. Preliminaries

We begin with the following basic definitions of concepts relating to probabilistic metric spaces for ready reference and also for the sake of completeness.

Definition 2.1. ([11]) A distribution function (on $[-\infty,+\infty]$) is a function $F : [-\infty,+\infty] \to [0,1]$ which is left-continuous on R, non-decreasing and $F(-\infty)=0, F(+\infty)=1$. The Heaviside function H is a distribution function defined by,

$$H(t)=\begin{cases}0, & \text{if} \quad t\le 0,\\ 1, & \text{if} \quad t>0.\end{cases}$$

Definition 2.2. ([11]) A distance distribution function $F : [-\infty, +\infty] \to [0, 1]$ is a distribution function with support contained in $[0, \infty]$. The family of all distance distribution functions will be denoted by Δ^+. We denote

$$D^+ = \left\{ F : F \in \Delta^+, \lim_{x\to\infty} F(x) = 1 \right\}.$$

Definition 2.3. ([35]) A probabilistic metric space in the sense of Schweizer and Sklar is an ordered pair (X, F), where X is a nonempty set and $F : X \times X \to \Delta^+$,if and only if the following conditions are satisfied ($F(x, y) = F_{x,y}$, for every $(x, y) \in X \times X$) :

(i) for every $(x, y) \in X \times X, F_{x,y}(0) = 0$;
(ii) for every $(x, y) \in X \times X, F_{x,y} = F_{y,x}$;
(iii) $F_{x,y} = 1$, for every $t > 0 \Leftrightarrow x = y$;
(iv) for every $(x, y, z) \in X \times X \times X$ and for every $t_1, t_2 > 0$,

$$F_{x,y}(t_1) = 1, \;\; F_{y,z}(t_2) = 1 \quad \Rightarrow \quad F_{x,z}(t_1 + t_2) = 1.$$

For each x and y in X and for each real number $t \geq 0$, $F_{x,y}(t)$ is to be thought of as the probability that the distance between x and y is less than t. Indeed, if (X, d) is a metric space, then the distribution function $F_{x,y}(t)$ defined by the relation $F_{x,\,y}(t) = H(t - d(x,\,y))$ induces a probabilistic metric space.

Definition 2.4. Let an ordered pair (X, F), where X is a nonempty set and F is a mapping from $X \times X$ into Δ^+ satisfying the following condition:

$$F_{x,\,y}(t) = 1, \;\; \forall\, t > 0 \quad \Leftrightarrow \quad x = y. \tag{2.1}$$

Where $F : X \times X \longrightarrow \Delta$ defined by $F_{x,\,y}(t) = H(t - d(x,\,y))$ for all x, $y \in X$ and d be a function $d : X \times X \to [0, \infty)$ such that $d(x,\,y) = 0$ iff $x = y, \forall x, y \in X$(symmetric and triangle conditions are not required). A topology $\tau(d)$ on X is given by $U \in \tau(d)$ if and only if for each $x \in U$, $B(x, \epsilon) \subset U$ for some $\epsilon > 0$, where $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

Remark 2.5. We note that every symmetric(semi-metric) space (X, d) [41] can be realized as a probabilistic semi-metric space by taking $F : X \times X \longrightarrow \Delta^+$ defined by $F_{x,\,y}(t) = H(t - d(x, y))$ for all x, y in X. So probabilistic semi-metric spaces provide a wider framework than that of the symmetric spaces and are better suited in many situations.

In this paper we have relaxed the symmetric condition from probabilistic semi metric space.

Definition 2.6. ([10, 24]) Let (X, F) be a probabilistic metric space and A be a nonempty subset of X. The probabilistic diameter $\delta_A : [0, \infty) \to [0, 1]$ is defined by,

$$\delta_A(x) = \sup_{t<x} \inf_{p,q \in A} F_{p,q}(t).$$

If (X, F) satisfies condition (2.1), the probabilistic diameter is defined by,

$$\delta_A(x) = \sup_{t<x} \inf_{p,q \in A} \{F_{p,q}(t), F_{q,p}(t)\}.$$

Let X be a non-empty set together with the function $F : X \times X \to \Delta$ satisfying the condition (2.1). A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case $w = fx = gx$ is called a point of coincidence of f and g.

Let $C(f, g)$ and $PC(f, g)$ denote the sets of coincidence points and points of coincidence, respectively, of the pair (f, g).

Definition 2.7. Let X be a non-empty set together with the function $F : X \times X \to \Delta$ satisfying the condition (2.1), two selfmaps f and g of a space (X, F) are called generalized PD-operators with order n iff for all t and for some $(\delta_{PC(f,g)}(t))^{\frac{1}{n}}$ there is a point $x \in C(f, g)$ such that

$$F_{fgx,\, gfx}(t) \geq (\delta_{PC(f,g)}(t))^{\frac{1}{n}} \text{ and } F_{gfx,\, fgx}(t) \geq (\delta_{PC(f,g)}(t))^{\frac{1}{n}},$$

for some $n \in N$.

Remark 2.8. It is obvious that the (f, g) is generalizedPD-operators with order n, but not commuting, not weakly compatible and not occasionally weakly compatible not PD-operator. It is also clear that the generalized PD-operators is different from P-operator pair, generalized-JH-operator pair.

Example 2.9. Let $X = [0, \infty)$ and $F_{x,\, y}(t) = H(t - d(x,\, y))$, where,

$$d(x, y) = \begin{cases} e^{x-y} - 1, & \text{if } x \geq y, \\ e^{y-x}, & \text{otherwise.} \end{cases}$$

Define $f, g : X \to X$ by

$$f(x) = \begin{cases} 3, & \text{if } x = 0, \\ 5, & \text{if } x = 2, \\ 2x, & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} 3, & \text{if } x = 0, \\ 5, & \text{if } x = 2, \\ x^2, & \text{otherwise.} \end{cases}$$

Here $C(f, g) = \{0, 2\}$ and $PC(f, g) = \{3, 5\}$.

In this example (f, g) is generalized PD-operators with order $n \geq 2$ but not a PD-operator but not commuting, not weakly compatible and occasionally weakly compatible and JH-operator not a Banach operator pair.

Remark 2.10. It is obvious that the (f, g) is PD-operator, but not commuting, not weakly compatible and not occasionally weakly compatible. It is also clear that the PD-operator is different from P-operator pair and JH-operator pair.

Example 2.11. Let $X = [0, 1]$ and $F_{x,y}(t) = H(t - d(x, y))$, where,

$$d(x,y) = \begin{cases} e^{x-y} - 1, & \text{if } x \geq y, \\ e^{y-x}, & \text{otherwise.} \end{cases}$$

Define $f, g : X \to X$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ x^2, & \text{if } 0 < x \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \qquad g(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{x}{2}, & \text{if } 0 < x \leq 1. \end{cases}$$

Here $C(f, g) = \left\{0, \frac{1}{2}\right\}$ and $PC(f, g) = \left\{1, \frac{1}{4}\right\}$.

In this example (f, g) is (PD)-operator but not commuting, not weakly compatible and not occasionally weakly compatible.

Example 2.12. Let $X = [0, \infty)$ and $F_{x,y}(t) = H(t - d(x, y))$, where,

$$d(x,y) = \begin{cases} e^{x-y} - 1, & \text{if } x \geq y, \\ e^{y-x}, & \text{otherwise.} \end{cases}$$

Define $f, g : X \to X$ by

$f(x) = 2x$ and $g(x) = 2x^2$, for all $x \neq 0$ and $f(0) = g(0) = 1$.

Here $C(f, g) = \{0, 1\}$, $PC(f, g) = \{1, 2\}$.

In this example (f, g) is P-operator and JH-operator pair, but not PD-operator pair. We also observe that if (f, g) is PD-operator then it is not necessary that (f, g) be P-operator and JH-operator pair.

3. Main results

In this section, we prove some fixed point theorems for a pair of generalized PD-operators with order n on space (X, F) without imposing the restriction of the triangle inequality or symmetry on F.

Theorem 3.1. *Let X be a non-empty set together with the function $F : X\times X \to \Delta$ satisfying the condition (2.1). Suppose f and g are generalizedPD-operators with order n satisfying the following condition:*

$$F_{fx,fy}(t) \geq F_{gx,gy}\left(\frac{t}{a}\right) + \min\left\{F_{fx,gx}\left(\frac{t}{b}\right), F_{fy,gy}\left(\frac{t}{b}\right)\right\} + \min\left\{F_{gx,gy}\left(\frac{t}{c}\right), F_{gx,fx}\left(\frac{t}{c}\right), F_{gy,fy}\left(\frac{t}{c}\right)\right\}, \tag{3.1}$$

for all $x, y \in X$ with $f(x) \neq fy$ and $t > 0$ where $0 < a < 1$, $0 < b < 1$ and $0 < c < 1$. Then f and g have a unique common fixed point.

Proof. Since (f, g) pair is generalizedPD-operators with order n, hence for all t and for some $(\delta_{PC(f,g)}(t))^{\frac{1}{n}}$ there exist a point u in X such that $fu = gu$ and

$$F_{fgu,\,gfu}(t) \geq (\delta_{PC(f,\,g)}(t))^{\frac{1}{n}}. \tag{3.2}$$

First, we claim that $PC(f,\, g)$ is singleton. If possible, suppose w and w_1 be two distinct points in X such that $fu = gu = w$ and $fv = gv = w_1$ for some $u, v \in C(f, g)$. Then from (3.1), we get,

$$\begin{aligned} &F_{w,w_1}(t) = F_{fu,fv}(t) \\ &\geq F_{gu,gv}\left(\frac{t}{a}\right) + 1 + \min\left\{F_{gu,gv}\left(\frac{t}{c}\right), F_{gu,fu}\left(\frac{t}{c}\right), F_{gv,fv}\left(\frac{t}{c}\right)\right\} \\ &= F_{gu,fv}\left(\frac{t}{a}\right) + 1 + F_{gu,fv}\left(\frac{t}{c}\right) > 1, \end{aligned}$$

a contradiction. Hence, $w = w_1$. Thus $PC(f, g)$ is singleton and w is the unique point of coincidence. This further implies $\delta(PC(f,\, g)) = 1$. Using (3.2), $fgu = gfu$ for some $u \in C(f, g)$, for some $(\delta_{PC(f,g)}(t))^{\frac{1}{n}}$ and for all t. Now by (3.1), we have

$$\begin{aligned} &F_{fu,ffu}(t) \\ &\geq F_{gu,gfu}\left(\frac{t}{a}\right) + 1 + \min\left\{F_{gu,gfu}\left(\frac{t}{c}\right), F_{gu,fu}\left(\frac{t}{c}\right), F_{gfu,ffu}\left(\frac{t}{c}\right)\right\} \\ &= F_{gu,ffu}\left(\frac{t}{a}\right) + 1 + F_{gu,ffu}\left(\frac{t}{c}\right) > 1, \end{aligned}$$

a contradiction. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness follows from (3.1). □

Let a function ϕ be defined by $\phi : [0, 1] \longrightarrow [0, 1]$ satisfying the condition $\phi(q) > q$, for all $0 \leq q < 1$.

Theorem 3.2. *Let X be a non-empty set together with the function $F : X \times X \to \Delta$ satisfying the condition (2.1). If (f, g) pair is generalized PD-operators with order n. Suppose*

$$F_{fx,fy}(t) \geq \phi\left[\min\left\{F_{gx,gy}(t), F_{gx,fy}(t), F_{fx,gy}(t), F_{gy,fy}(t)\right\}\right], \tag{3.3}$$

for all $x, y \in X$ and $t > 0$. Then f and g have a unique common fixed point.

Proof. Since (f, g) pair is generalizedPD-operators with order n, hence for all t and for some $(\delta_{PC(f,g)}(t))^{\frac{1}{n}}$ there exist a point u in X such that $fu = gu$ and

$$F_{fgu,\,gfu}(t) \geq (\delta_{PC(f,\,g)}(t))^{\frac{1}{n}}. \tag{3.4}$$

First, we claim that $PC(f,\,g)$ is singleton. If possible, suppose w and w_1 be two distinct points in X such that $fu = gu = w$ and $fv = gv = w_1$ for some $u, v \in C(f, g)$. Then from (3.3), we can easily get, $w = w_1$, i.e., $w = fu = gu = fv = gv = w_1$. Therefore $PC(f,\,g)$ is singleton i.e., $w = fu = gu$ is the unique point of coincidence. $\delta(PC(f,\,g)) = 1$. From (3.4), $fgu = gfu$, for some $u, v \in C(f, g)$, for some $(\delta_{PC(f,g)}(t))^{\frac{1}{n}}$ and for all t. Now, by (3.3), we have

$$\begin{aligned} F_{ffu,fu}(t) &\geq \phi\left[\min\left\{F_{gfu,gu}(t), F_{gfu,fu}(t), F_{ffu,gu}(t), F_{gu,fu}(t)\right\}\right], \\ &= \phi\left[\min\left\{F_{ffu,fu}(t), F_{ffu,fu}(t), F_{ffu,fu}(t), 1\right\}\right], \\ &= \phi\left[F_{ffu,fu}(t)\right]. \end{aligned}$$

Since $\phi : [0, 1] \longrightarrow [0, 1]$ satisfying the condition $\phi(q) > q$, for all $0 \leq q < 1$. Therefore, $F_{ffu,fu}(t) > F_{ffu,fu}(t)$. which is a contradiction. Therefore $ffu = fu = gfu$, f and g have a common fixed point. Uniqueness, is obvious. Therefore, f and g have a unique common fixed point. This completes the proof of the theorem. □

Corollary 3.3. *Let X be a non-empty set together with the function $F : X \times X \to \Delta$ satisfying the condition (2.1). If f and g are generalizedPD-operators with order n on X. Suppose*

$$F_{fx,fy}(t) \geq \phi\left[F_{gx,gy}(t)\right], \tag{3.5}$$

for some $x, y \in X$ and $t > 0$. Then f and g have a unique common fixed point.

Remark 3.4. As an application of Corollary 3.3, the existence and uniqueness of a common solution of the functional equations arising in dynamic programming can be established which extends Theorem 4.1 [4].

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