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STABILITY OF PEXIDERIZED CAUCHY FUNCTIONAL EQUATION IN FELBIN'S TYPE FUZZY NORMED LINEAR SPACES

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of Pexiderized Cauchy functional equation in Felbin's type fuzzy normed linear spaces and some applications of the main result to Banach spaces are also given.

1. Introduction and preliminaries

Speaking of the stability of a functional equation, we follow the question raised in 1940 by Ulam:

When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?

The first partial answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by Hyers (see [6]). This result was generalized by Aoki [1] for additive mappings and independently by Rassias [12] for linear mappings by considering an *unbounded Cauchy difference*. In 1994, a further generalization was obtained by Găvruta [5]. Rassias [10, 11] generalized Hyers result. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [3, 8, 9, 13]). We also refer the readers to the books: Czerwik [2] and Hyers, Isac and Rassias [7].

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We consider some basic concepts concerning in the theory of fuzzy real numbers. Let η be a fuzzy subset on $\mathbb{R}$, i.e., a mapping $\eta : \mathbb{R} \longrightarrow [0, 1]$ associating with each real number t its grade of membership $\eta(t)$.

Definition 1.1. A fuzzy subset η on $\mathbb{R}$ is called a fuzzy real number, whose α-level set is denoted by $[\eta]_\alpha$; i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:

(N1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.
(N2) For each $\alpha \in (0, 1]$; $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha]$ where $-\infty < \eta^-_\alpha \leq \eta^+_\alpha < +\infty$.

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. The number $\overline{0}$ stands for the fuzzy real number as:

$$\overline{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Clearly, $\overline{0} \in F^*(\mathbb{R})$. Also the set of all real numbers can be embedded in $F(\mathbb{R})$ because if $r \in (-\infty, +\infty)$, then $\overline{r} \in F(R)$ satisfies $\overline{r}(t) = \overline{0}(t - r)$.

Arithmetic operations $\oplus, \ominus, \otimes$ and $\oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as in [4]:

$$(\eta \oplus \delta)(t) = \sup_{s\in\mathbb{R}}\{\eta(s) \wedge \delta(t - s)\}, \qquad (t \in \mathbb{R}), \tag{1.1}$$

$$(\eta \ominus \delta)(t) = \sup_{s\in\mathbb{R}}\{\eta(s) \wedge \delta(s - t)\}, \qquad (t \in \mathbb{R}), \tag{1.2}$$

$$(\eta \otimes \delta)(t) = \sup_{s\in\mathbb{R}-\{\circ\}}\{\eta(s) \wedge \delta(\frac{t}{s})\}, \qquad (t \in \mathbb{R}), \tag{1.3}$$

$$(\eta \oslash \delta)(t) = \sup_{s\in\mathbb{R}}\{\eta(st) \wedge \delta(s)\}, \qquad (t \in \mathbb{R}). \tag{1.4}$$

Definition 1.2. For $k \in \mathbb{R}$-$\{0\}$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta)(t) = \eta(\frac{t}{k})$ and $0 \odot \eta$ is defined to be $\overline{0}$.

Definition 1.3. Let η be a non-negative fuzzy real number and $p \neq 0$ be a real number. Define η^p as:

$$\eta^p(t) = \begin{cases} \eta(t^{\frac{1}{p}}), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and $\eta^0 = \overline{1}$.

It is well-known that η^p is a non-negative fuzzy real number and $[\eta^p]_\alpha = [(\eta_\alpha^-)^p, (\eta_\alpha^+)^p]$ if $p > 0$ and $[\eta^p]_\alpha = [(\eta_\alpha^+)^p, (\eta_\alpha^-)^p]$ if $p < 0$.

Definition 1.4. Define a partial ordering $\preceq$ in $F(R)$ by $\eta \preceq \delta$ if and only if $\eta_\alpha^- \leq \delta_\alpha^-$ and $\eta_\alpha^+ \leq \delta_\alpha^+$ for all $\alpha \in (0,1]$. The strict inequality in $F(R)$ is defined by $\eta \prec \delta$ if and only if $\eta_\alpha^- < \delta_\alpha^-$ and $\eta_\alpha^- < \delta_\alpha^-$ for all $\alpha \in (0,1]$.

Definition 1.5. Let X be a real linear space; L and R (respectively, left norm and right norm) be symmetric and non-decreasing mappings in both arguments from $[0,1] \times [0,1]$ into $[0,1]$ satisfying $L(0,0) = 0$ and $R(1,1) = 1$. The mapping $\|.\|$ from X into $F^*(R)$ is called a fuzzy norm if for $x \in X$ and $\alpha \in (0,1]$:

(A1) $\|x\| = 0$ if and only if $x = 0$;
(A2) $\|rx\| = \mid r \mid \bigodot \|x\|$ for all $x \in X$ and $r \in (-\infty; +\infty)$;
(A3) For all $x, y \in X$:
  (A3L) if $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$, then $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$;
  (A3R) if $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$, then $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$.

The quaternary $(X, \|.\|, L, R)$ is called a fuzzy normed linear space (abbreviated to FNLS).

Definition 1.6. Let $(X, \|.\|, L, R)$ be a fuzzy normed linear space and $\lim_{a\to 0^+} R(a,a) = 0$. A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$ and is called a Cauchy sequence if $\lim_{m,n\to\infty} \|x_n - x_m\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$. A subset $A \subseteq X$ is said to be complete if every Cauchy sequence in A, converges in A. The fuzzy normed space $(X, \|.\|, L, R)$ is said to be a fuzzy Banach space if it is complete.

Theorem 1.7. ([15]) *Let $(X, \|.\|, L, R)$ be an FNLS and suppose that:*

(R-1) $R(a,b) \leq \max(a,b)$,
(R-2) *For any $\alpha \in (0,1]$, there exists $\beta \in (0,\alpha]$ such that $R(\beta, y) \leq \alpha$ for all $y \in (0,\alpha)$,*
(R-3) $\lim_{a\to 0^+} R(a,a) = 0$,

then

$$(R-1) \quad \Rightarrow \quad (R-2) \quad \Rightarrow \quad (R-3).$$

Theorem 1.8. ([15]) *Let $(X, \|.\|, L, R)$ be an FNLS. Then we have the following:*

(1) *If* $R(a,b) \leq \max(a,b)$*, then for any* $\alpha \in (0,1]$*,* $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ *for all* $x, y \in X$*.*
(2) *If* $(R-2)$*, then for any* $\alpha \in (0,1]$*, there exists* $\beta \in (0,\alpha]$ *such that* $\|x+y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\alpha^+$ *for all* $x, y \in X$*.*
(3) *If* $\lim_{a \to 0^+} R(a,a) = 0$*, then for any* $\alpha \in (0,1]$*, there exists* $\beta \in (0,\alpha]$ *such that* $\|x+y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\beta^+$ *for all* $x, y \in X$*.*

Theorem 1.9. ([15]) *Let* $(X, \|.\|, L, R)$ *be an FNLS satisfying* $(R-2)$*. Then:*

(1) *For each* $\alpha \in (0,1]$*,* $\| . \|^+$ *is a continuous mapping from* X *into* $\mathbb{R}$*.*
(2) *For any* $n \in \mathbb{Z}^+$*,* $x_i \in X$ $(i = 1, 2, ..., n)$ *and* $\alpha \in (0,1]$ *there exists* $\beta \in (0,\alpha]$ *such that*

$$\left\| \sum_{i=1}^{n} x_i \right\|_\alpha^+ \leq \left\| \sum_{i=1}^{n} x_i \right\|_\beta^+ .$$

In this paper, we investigate the generalized Hyers-Ulam stability of Pexiderized Cauchy functional equation in Felbin's type fuzzy normed linear spaces and some applications of our results in the stability of Pexiderized Cauchy functional equation from a linear space to a Banach space will be exhibited. Throughout this paper, assume that k is a fixed positive integer greater than 1.

2. Generalized Hyers-Ulam stability of Pexiderized Cauchy functional equation

In this section, we prove the generalized Hyers–Ulam stability of Pexiderized Cauchy functional equation in Felbin's type fuzzy normed linear spaces.

Theorem 2.1. *Let* X *be a linear space and* $(Y, \|.\|, L, R)$ *be a fuzzy Banach space satisfying* $(R-1)$*. Let* $f, g, h : X \longrightarrow Y$ *be mappings such that* $g(0) = h(0) = 0$ *and there exists a function* $\varphi : X \times X \longrightarrow F^*(\mathbb{R})$ *such that*

$$\lim_{n\to\infty} \frac{1}{k^n} \varphi(k^n x, k^n y)_\alpha^+ = 0, \tag{2.1}$$

$$\sum_{i=0}^{\infty} \frac{1}{k^i} \vartheta_{k,\alpha}(k^i x) < \infty \tag{2.2}$$

and

$$\|f(x+y) - g(x) - h(y)\| \preceq \varphi(x,y), \tag{2.3}$$

for all $x, y \in X$ and $\alpha \in (0,1]$. Then there exists a unique additive mapping $A_k : X \longrightarrow Y$ such that for all $\alpha \in (0,1]$ there exists $\beta \in (0,\alpha]$ such that

$$\Big\| f(x) - A_k(x) \Big\|_\alpha^+ \le \frac{1}{k} \sum_{i=0}^{\infty} \frac{1}{k^i} \vartheta_{k,\beta}(k^i x), \tag{2.4}$$

$$\Big\| g(x) - A_k(x) \Big\|_\alpha^+ \le \frac{1}{k} \sum_{i=0}^{\infty} \frac{1}{k^i} \vartheta_{k,\beta}(k^i x) + \varphi(x,0)_\alpha^+,$$

$$\Big\| h(x) - A_k(x) \Big\|_\alpha^+ \le \frac{1}{k} \sum_{i=0}^{\infty} \frac{1}{k^i} \vartheta_{k,\beta}(k^i x) + \varphi(0,x)_\alpha^+,$$

for all $x \in X$, where

$$\vartheta_{k,\alpha}(x) =: \sum_{i=1}^{k-1} \Big[\varphi(ix, x)_\alpha^+ + \varphi(0, ix)_\alpha^+ \Big] + (k-1)\varphi(x,0)_\alpha^+.$$

Proof. Replacing $y = 0$ in (2.3), we get

$$\|f(x) - g(x)\| \preceq \varphi(x,0). \tag{2.5}$$

Replacing $x = 0$ in (2.3), we get

$$\|f(y) - h(y)\| \preceq \varphi(0,y). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\|f(x) - g(x)\|_\alpha^+ \le \varphi(x,0)_\alpha^+, \tag{2.7}$$

$$\|f(y) - h(y)\|_\alpha^+ \le \varphi(0,y)_\alpha^+, \tag{2.8}$$

for all $\alpha \in (0,1]$. It follows from (2.3), (2.7), (2.8) and Theorem 1.8 that

$$\|f(x+y) - f(x) - f(y)\|_\alpha^+ \le \varphi(x,y)_\alpha^+ + \varphi(x,0)_\alpha^+ + \varphi(0,y)_\alpha^+, \tag{2.9}$$

for all $x, y \in X$ and $\alpha \in (0,1]$. Replacing y by x in (2.9), we get

$$\|f(2x) - 2f(x)\|_\alpha^+ \le \varphi(x,x)_\alpha^+ + \varphi(x,0)_\alpha^+ + \varphi(0,x)_\alpha^+, \tag{2.10}$$

for all $x \in X$ and $\alpha \in (0,1]$. Replacing y by $2x$ in (2.9), we get

$$\|f(3x) - f(x) - f(2x)\|_\alpha^+ \le \varphi(x,2x)_\alpha^+ + \varphi(x,0)_\alpha^+ + \varphi(0,2x)_\alpha^+, \tag{2.11}$$

for all $x \in X$ and $\alpha \in (0,1]$. It follows from (2.10) and (2.11) that

$$\begin{aligned} \|f(3x) - 3f(x)\|_\alpha^+ \le{}& \varphi(x,x)_\alpha^+ + \varphi(x,2x)_\alpha^+ + \varphi(0,x)_\alpha^+ \\ &+ \varphi(0,2x)_\alpha^+ + 2\varphi(x,0)_\alpha^+, \end{aligned}$$

for all $x \in X$ and $\alpha \in (0,1]$. By induction on k, we have

$$\|f(kx) - kf(x)\|_\alpha^+ \le \vartheta_{k,\alpha}(x), \tag{2.12}$$

where

$$\vartheta_{k,\alpha}(x) =: \sum_{i=1}^{k-1} \Big[\varphi(ix,x)^+_\alpha + \varphi(0,ix)^+_\alpha\Big] + (k-1)\varphi(x,0)^+_\alpha.$$

Replacing x by $k^n x$ and dividing both sides of (2.12) by $k^{(n+1)}$, we get

$$\Big\|\frac{1}{k^{(n+1)}} f(k^{n+1}x) - \frac{1}{k^n} f(k^n x)\Big\|^+_\alpha \le \frac{1}{k^{(n+1)}}\vartheta_{k,\alpha}(k^n x), \tag{2.13}$$

for all $x \in X$, $\alpha \in (0,1]$ and all non-negative integers n. By Theorem 1.9 and inequality (2.13), we conclude that for all $\alpha \in (0,1]$ there exists $\beta \in (0,\alpha]$ such that

$$\begin{aligned} &\Big\|\frac{1}{k^{(n+1)}} f(k^{n+1}x) - \frac{1}{k^m} f(k^m x)\Big\|^+_\alpha \\ &\le \sum_{i=m}^{n} \Big\|\frac{1}{k^{(i+1)}} f(k^{i+1}x) - \frac{1}{k^i} f(k^i x)\Big\|^+_\beta \le \frac{1}{k}\sum_{i=m}^{n}\frac{1}{k^i}\vartheta_{k,\beta}(k^i x), \end{aligned} \tag{2.14}$$

for all $x \in X$ and all non-negative integers m and n with $n \ge m$. Passing the limit $n, m \to \infty$ in (2.14), we get

$$\lim_{n,m\to\infty} \Big\|\frac{1}{k^n} f(k^n x) - \frac{1}{k^m} f(k^m x)\Big\|^+_\alpha = 0.$$

Therefore, the sequence $\{\frac{1}{k^n} f(k^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{k^n} f(k^n x)\}$ converges for all $x \in X$. So one can define the mapping $A_k : X \to Y$ by

$$A_k(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x), \tag{2.15}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.14) by continuity of $\| \, . \, \|^+$, we get (2.4).

Now, we show that A_k is additive. It follows from (3.1), (2.9) and (2.15) that

$$\begin{aligned} &\|A_k(x+y) - A_k(x) - A_k(y)\|^+_\alpha \\ &= \lim_{n\to\infty} \frac{1}{k^n}\Big\|f(k^n x + k^n y) - f(k^n x) - f(k^n y)\Big\|^+_\alpha \\ &\le \lim_{n\to\infty} \frac{1}{k^n}[\varphi(k^n x, k^n y)^+_\alpha + \varphi(k^n x, 0)^+_\alpha + +\varphi(0, k^n x)^+_\alpha] = 0, \end{aligned} \tag{2.16}$$

for all $x, y \in X$. Therefore, we get that the mapping $A_k : X \to Y$ is additive.

To prove the uniqueness of A_k, let $A' : X \to Y$ be another additive mapping which satisfies inequality (2.4), Then we have

$$\begin{aligned}\|A_k(x) - A'(x)\|_\alpha^+ &= \lim_{n\to\infty} \frac{1}{k^n}\big\|f(k^n x) - A'(k^n x)\big\|_\alpha^+ \\ &\le \lim_{n\to\infty} \frac{1}{k^{n+1}} \sum_{i=0}^{\infty} \frac{1}{k^i}\vartheta_{k,\beta}(k^i x) = 0.\end{aligned}$$

So, $A = A'$. □

Remark 2.2. We can formulate a similar theorem to Theorem 2.1 in which we can define the sequence $A_k(x) := \lim_{n\to\infty} k^n f(\frac{x}{k^n})$ under suitable conditions on the function φ.

We will use following lemma [4]:

Lemma 2.3. *Let $\eta, \delta \in F(\mathbb{R})$. Then*
(1) $(\eta \oplus \delta)_\alpha^+ = \eta_\alpha^+ + \delta_\alpha^+$,
(2) $(\eta \otimes \delta)_\alpha^+ = \eta_\alpha^+ \times \delta_\alpha^+, \quad \eta, \delta \in F^*(\mathbb{R})$.

Corollary 2.4. *Let μ be a nonnegative fuzzy real number and p be a nonnegative real number such that $p < 1$. Let X be a linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfy the $(R-1)$. Suppose that $f, g, h : X \longrightarrow Y$ are mappings such that $g(0) = h(0) = 0$ and satisfy the inequality*

$$\|f(x+y) - g(x) - h(y)\| \preceq \mu \otimes (\|x\|^p \oplus \|y\|^p),$$

for all $x, y \in X$. Then there exists a unique additive mapping $A_k : X \to Y$ such that

$$\forall\alpha \in (0,1],\ \exists\beta \in (0,\alpha];\ \ \|f(x) - A_k(x)\|_\alpha^+ \le \frac{2\mu_\beta^+}{k - k^p}(\|x\|_\beta^+)^p\Big(\sum_{j=1}^{k-1} j^p + (k-1)\Big),$$

$$\|g(x) - A_k(x)\|_\alpha^+ \le \frac{2\mu_\beta^+}{k - k^p}(\|x\|_\beta^+)^p\Big(\sum_{j=1}^{k-1} j^p + (k-1)\Big) + \mu_\alpha^+\|x\|_\alpha^+,$$

$$\|h(x) - A_k(x)\|_\alpha^+ \le \frac{2\mu_\beta^+}{k - k^p}(\|x\|_\beta^+)^p\Big(\sum_{j=1}^{k-1} j^p + (k-1)\Big) + \mu_\alpha^+\|x\|_\alpha^+,$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking:

$$\varphi(x, y) = \mu \otimes (\|x\|^p \oplus \|y\|^p)$$

and using Lemma 2.3. □

3. Application of fuzzy stability to the stability of Pexiderized Cauchy functional equation in normed spaces

In this section by using Theorem 2.1, we prove the generalized Hyers-Ulam stability of Pexiderized Cauchy functional equation in normed spaces. Throughout this section, let X be a linear space and Y be a Banach space with norm $\|.\|_Y$.

Theorem 3.1. *Let $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{k^n}\phi(k^n x, k^n y) = 0, \tag{3.1}$$

for all $x, y \in X$, and

$$\sum_{i=0}^{\infty} \frac{1}{k^i}\vartheta_k(k^i x) < \infty, \tag{3.2}$$

for all $x, y \in X$. Let $f, g, h : X \longrightarrow Y$ be mappings such that $g(0) = h(0) = 0$ and satisfy the inequality

$$\|f(x + y) - g(x) - h(y)\|_Y \le \phi(x, y), \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A_k : X \to Y$ such that

$$\Big\|f(x) - A_k(x)\Big\|_Y \le \frac{1}{k}\sum_{i=0}^{\infty} \frac{1}{k^i}\vartheta_k(k^i x), \tag{3.4}$$

where

$$\vartheta_k(x) =: \sum_{i=1}^{k-1}\Big[\varphi(ix, x) + \varphi(x, ix)\Big] + (k - 1)\varphi(x, 0).$$

for all $x \in X$.

Proof. Set $\|y\|(t) = \overline{0}(t- \parallel y \parallel_Y)$, $R(a, b) = \max\{a, b\}$ and $L(a, b) = \min\{a, b\}$. It is easy to see that $(Y, \|.\|, R, L)$ is a fuzzy normed linear space and $\|y\|^+_\alpha = \|y\|_Y$ for all $y \in Y$ and $\alpha \in (0, 1]$. It follows from (3.1), (3.3) and (3.2) that

$$\lim_{n\to\infty} \frac{1}{k^n}\overline{\phi(k^n x, k^n y)}^+_\alpha = 0,$$

$$\|f(x + y) - g(x) - h(y)\| \preceq \overline{\phi(x, y)}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{k^i}\overline{\vartheta_k(k^i x)} < \infty.$$

The proof follows from Theorem 2.1 by taking:

$$\varphi(x, y) = \overline{\phi(x, y)}.$$

□

Corollary 3.2. *Let θ and p be nonnegative real numbers such that $p < 1$. Let X be a linear space and $(Y, \|.\|)$ be a Banach space. Suppose that $f, g, h : X \longrightarrow Y$ are mappings such that $g(0) = h(0) = 0$ and satisfy the inequality*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in X$. Then there exists a unique additive mapping $A_k : X \to Y$ such that

$$\|f(x) - A_k(x)\| \leq \frac{2\theta}{k - k^p}\|x\|^p\Big(\sum_{j=1}^{k-1} j^p + (k-1)\Big),$$

$$\|g(x) - A_k(x)\| \leq \frac{2\theta}{k - k^p}\|x\|^p\Big(\sum_{j=1}^{k-1} j^p + (k-1)\Big) + \theta\|x\|^p,$$

$$\|g(x) - A_k(x)\| \leq \frac{2\theta}{k - k^p}\|x\|^p\Big(\sum_{j=1}^{k-1} j^p + (k-1)\Big) + \theta\|x\|^p,$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking:

$$\phi(x, y) = \theta(\|x\|^p + \|y\|^p).$$

□

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