Nonlinear Functional Analysis and Applications
Vol. 21, No. 1 (2016), pp. 105-120

http://nfaa.kyungnam.ac.kr/journal-nfaa

ON THE CONVERGENCE OF ITERATIVE METHODS WITH APPLICATIONS IN GENERALIZED FRACTIONAL CALCULUS

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Abstract. We present a semilocal convergence study of some iterative methods on a generalized Banach space setting to approximate a locally unique zero of an operator. Earlier studies such as [7, 8, 9, 14] require that the operator involved is Fréchet-differentiable. In the present study we assume that the operator is only continuous. This way we extend the applicability of these methods to include generalized fractional calculus and problems from other areas. Some applications include generalized fractional calculus involving the Riemann-Liouville fractional integral and the Caputo fractional derivative. Fractional calculus is very important for its applications in many applied sciences.

1. Introduction

Many problems in Computational sciences can be formulated as an operator equation using Mathematical Modelling [4, 9, 11, 15]. The zeros of these operators can rarely be found in closed form. That is why most solution methods are usually iterative.

The semilocal convergence is, based on the information around an initial point, to give conditions ensuring the convergence of the method.

[0]Received July 13, 2015. Revised October 5, 2015.
[0]2010 Mathematics Subject Classification: 26A33, 65G99, 47J25.
[0]Keywords: Generalized Banach space, Newton-type method, semilocal convergence, Riemann-Liouville fractional integral, Caputo fractional derivative.

We present a semilocal convergence analysis for some iterative methods on a generalized Banach space setting to approximate a zero of an operator. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in section 2). Earlier studies such as [7, 8, 9, 14] for Newton's method have shown that a more precise convergence analysis is obtained when compared to the real norm theory. However, the main assumption is that the operator involved is Fréchet-differentiable. This hypothesis limits the applicability of Newton's method. In the present study we only assume the continuity of the operator. This may be expand the applicability of these methods.

The rest of the paper is organized as follows: section 2 contains the basic concepts on generalized Banach spaces and auxiliary results on inequalities and fixed points. In section 3 we present the semilocal convergence analysis of Newton-type methods. Finally, in the concluding sections 4-5, we present special cases and applications in generalized fractional calculus.

2. Generalized Banach spaces

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [7, 8, 9, 14], and the references there in.

Definition 2.1. A generalized Banach space is a triplet $(x, E, /\cdot/)$ such that

(i) X is a linear space over $\mathbb{R}(\mathbb{C})$.
(ii) $E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e.
  (ii$_1$) $(E, \|\cdot\|)$ is a real Banach space,
  (ii$_2$) E is partially ordered by a closed convex cone K,
  (ii$_3$) The norm $\|\cdot\|$ is monotone on K.
(iii) The operator $/\cdot/ : X \to K$ satisfies
  $/x/ = 0 \Leftrightarrow x = 0$, $/\theta x/ = |\theta| /x/$,
  $/x + y/ \leq /x/ + /y/$ for each $x, y \in X$, $\theta \in \mathbb{R}(\mathbb{C})$.
(iv) X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|\cdot\| \cdot /\cdot/$.

Remark 2.2. The operator $/\cdot/$ is called a generalized norm. In view of (iii) and (ii$_3$) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm.

Let $L\left(X^j, Y\right)$ stand for the space of j-linear symmetric and bounded operators from X^j to Y, where X and Y are Banach spaces. For X, Y partially ordered $L_+\left(X^j, Y\right)$ stands for the subset of monotone operators P such that

$$0 \leq a_i \leq b_i \Rightarrow P(a_1, ..., a_j) \leq P(b_1, ..., b_j). \tag{2.1}$$

Definition 2.3. The set of bounds for an operator $Q \in L(X,X)$ on a generalized Banach space $(X, E, /\cdot/)$ the set of bounds is defined to be:

$$B(Q) := \{P \in L_+(E,E),\ /Qx/ \leq P/x/ \text{ for each } x \in X\}. \tag{2.2}$$

Let $D \subset X$ and $T : D \to D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0) \tag{2.3}$$

is well defined. We write in case of convergence

$$T^{\infty}(x_0) := \lim (T^n(x_0)) = \lim_{n\to\infty} x_n. \tag{2.4}$$

We need some auxiliary results on inequations.

Lemma 2.4. *Let $(E, K, \|\cdot\|)$ be a partially ordered Banach space, $\xi \in K$ and $M, N \in L_+(E,E)$.*

(i) *Suppose there exists $r \in K$ such that*

$$R(r) := (M+N)r + \xi \leq r \tag{2.5}$$

and

$$(M+N)^k r \to 0 \quad \text{as} \quad k \to \infty. \tag{2.6}$$

Then $b := R^{\infty}(0)$ is well defined satisfies the equation $t = R(t)$ and is the smaller than any solution of the inequality $R(s) \leq s$.

(ii) *Suppose there exists $q \in K$ and $\theta \in (0,1)$ such that $R(q) \leq \theta q$, then there exists $r \leq q$ satisfying* (i).

Proof. (i) Define sequence $\{b_n\}$ by $b_n = R^n(0)$. Then, we have by (2.5) that $b_1 = R(0) = \xi \leq r \Rightarrow b_1 \leq r$. Suppose that $b_k \leq r$ for each $k = 1, 2, ..., n$. Then, we have by (2.5) and the inductive hypothesis that $b_{n+1} = R^{n+1}(0) = R(R^n(0)) = R(b_n) = (M+N)b_n + \xi \leq (M+N)r + \xi \leq r \Rightarrow b_{n+1} \leq r$. Hence, sequence $\{b_n\}$ is bounded above by r. Set $P_n = b_{n+1} - b_n$. We shall show that

$$P_n \leq (M+N)^n r \ \text{ for each } n = 1, 2, \cdots. \tag{2.7}$$

We have by the definition of P_n and (2.6) that

$$\begin{aligned} P_1 &= R^2(0) - R(0) = R(R(0)) - R(0) = R(\xi) - R(0) \\ &= \int_0^1 R'(t\xi)\xi dt \leq \int_0^1 R'(\xi)\xi dt \leq \int_0^1 R'(r) r dt \leq (M+N)r, \end{aligned}$$

which shows (2.7) for $n = 1$. Suppose that (2.7) is true for $k = 1, 2, ..., n$. Then, we have in turn by (2.6) and the inductive hypothesis that

$$\begin{aligned} P_{k+1} &= R^{k+2}(0) - R^{k+1}(0) = R^{k+1}(R(0)) - R^{k+1}(0) \\ &= R^{k+1}(\xi) - R^{k+1}(0) = R\left(R^k(\xi)\right) - R\left(R^k(0)\right) \end{aligned}$$

$$= \int_0^1 R'\left(R^k(0) + t\left(R^k(\xi) - R^k(0)\right)\right)\left(R^k(\xi) - R^k(0)\right) dt$$
$$\le R'\left(R^k(\xi)\right)\left(R^k(\xi) - R^k(0)\right) = R'\left(R^k(\xi)\right)\left(R^{k+1}(0) - R^k(0)\right)$$
$$\le R'(r)\left(R^{k+1}(0) - R^k(0)\right) \le (M+N)(M+N)^k r = (M+N)^{k+1} r,$$

which completes the induction for (2.7). It follows that $\{b_n\}$ is a complete sequence in a Banach space and as such it converges to some b. Notice that $R(b) = R\left(\lim_{n\to\infty} R^n(0)\right) = \lim_{n\to\infty} R^{n+1}(0) = b \Rightarrow b$ solves the equation $R(t) = t$. We have that $b_n \le r \Rightarrow b \le r$, where r a solution of $R(r) \le r$. Hence, b is smaller than any solution of $R(s) \le s$.

(ii) Define sequences $\{v_n\}$, $\{w_n\}$ by $v_0 = 0$, $v_{n+1} = R(v_n)$, $w_0 = q$, $w_{n+1} = R(w_n)$. Then, we have that

$$0 \le v_n \le v_{n+1} \le w_{n+1} \le w_n \le q, \qquad (2.8)$$
$$w_n - v_n \le \theta^n (q - v_n)$$

and sequence $\{v_n\}$ is bounded above by q. Hence, it converges to some r with $r \le q$. We also get by (2.8) that $w_n - v_n \to 0$ as $n \to \infty \Rightarrow w_n \to r$ as $n \to \infty$. □

We also need the auxiliary result for computing solutions of fixed point problems.

Lemma 2.5. *Let $(X, (E, K, \|\cdot\|), /\cdot/)$ be a generalized Banach space, and $P \in B(Q)$ be a bound for $Q \in L(X, X)$. Suppose there exists $y \in X$ and $q \in K$ such that*

$$Pq + /y/ \le q \quad \text{and} \quad P^k q \to 0 \quad \text{as} \quad k \to \infty. \qquad (2.9)$$

Then $z = T^\infty(0)$, $T(x) := Qx + y$ is well defined and satisfies: $z = Qz + y$ and $/z/ \le P/z/ + /y/ \le q$. Moreover, z is the unique solution in the subspace $\{x \in X | \exists\, \theta \in \mathbb{R} : \{x\} \le \theta q\}$.

Proof. The proof can be found in [14, Lemma 3.2]. □

3. Semilocal convergence

Let $(X, (E, K, \|\cdot\|), /\cdot/)$ and Y be generalized Banach spaces, $D \subset X$ an open subset, $G : D \to Y$ a continuous operator and $A(\cdot) : D \to L(X, Y)$. A zero of operator G is to be determined by a Newton-type method starting at a point $x_0 \in D$. The results are presented for an operator $F = JG$, where

$J \in L(Y, X)$. The iterates are determined through a fixed point problem:

$$x_{n+1} = x_n + y_n, \quad A(x_n) y_n + F(x_n) = 0 \tag{3.1}$$
$$\Leftrightarrow y_n = T(y_n) := (I - A(x_n)) y_n - F(x_n).$$

Let $U(x_0, r)$ stand for the ball defined by

$$U(x_0, r) := \{x \in X : /x - x_0/ \leq r\}$$

for some $r \in K$.

Next, we present the semilocal convergence analysis of Newton-type method (3.1) using the preceding notation.

Theorem 3.1. *Let $F : D \subset X \to X$, $A(\cdot) : D \to L(X, Y)$ and $x_0 \in D$ be as defined previously. Suppose:*

(H$_1$) *There exists an operator $M \in B(I - A(x))$ for each $x \in D$.*

(H$_2$) *There exists an operator $N \in L_+(E, E)$ satisfying for each $x, y \in D$*

$$/F(y) - F(x) - A(x)(y - x)/ \leq N\, /y - x/.$$

(H$_3$) *There exists a solution $r \in K$ of*

$$R_0(t) := (M + N) t + /F(x_0)/ \leq t.$$

(H$_4$) *$U(x_0, r) \subseteq D$.*

(H$_5$) *$(M + N)^k r \to 0$ as $k \to \infty$.*

Then the following hold:

(C$_1$) *The sequence $\{x_n\}$ defined by*

$$x_{n+1} = x_n + T_n^{\infty}(0), \quad T_n(y) := (I - A(x_n)) y - F(x_n) \tag{3.2}$$

is well defined, remains in $U(x_0, r)$ for each $n = 0, 1, 2, \ldots$ and converges to the unique zero of operator F in $U(x_0, r)$.

(C$_2$) *An apriori bound is given by the null-sequence $\{r_n\}$ defined by $r_0 := r$ and for each $n = 1, 2, \ldots$*

$$r_n = P_n^{\infty}(0), \quad P_n(t) = Mt + N r_{n-1}.$$

(C$_3$) *An aposteriori bound is given by the sequence $\{s_n\}$ defined by*

$$s_n := R_n^{\infty}(0), \quad R_n(t) = (M + N) t + N a_{n-1},$$

$$b_n := /x_n - x_0/ \leq r - r_n \leq r,$$

where

$$a_{n-1} := /x_n - x_{n-1}/ \text{ for each } n = 1, 2, \ldots.$$

Proof. Let us define for each $n \in \mathbb{N}$ the statement:

(I_n) $x_n \in X$ and $r_n \in K$ are well defined and satisfy

$$r_n + a_{n-1} \leq r_{n-1}.$$

We use induction to show (I_n). The statement (I_1) is true: By Lemma 2.4 and (H_3), (H_5) there exists $q \leq r$ such that:

$$Mq + /F(x_0)/ = q \text{ and } M^k q \leq M^k r \to 0 \text{ as } k \to \infty.$$

Hence, by Lemma 2.5, x_1 is well defined and we have $a_0 \leq q$. Then we get the estimate

$$\begin{aligned} P_1(r-q) &= M(r-q) + Nr_0 \leq Mr - Mq + Nr = R_0(r) - q \\ &\leq R_0(r) - q = r - q. \end{aligned}$$

It follows with Lemma 2.4 that r_1 is well defined and

$$r_1 + a_0 \leq r - q + q = r = r_0.$$

Suppose that (I_j) is true for each $j = 1, 2, ..., n$. We need to show the existence of x_{n+1} and to obtain a bound q for a_n. To achieve this notice that:

$$Mr_n + N(r_{n-1} - r_n) = Mr_n + Nr_{n-1} - Nr_n = P_n(r_n) - Nr_n \leq r_n.$$

Then, it follows from Lemma 2.4 that there exists $q \leq r_n$ such that

$$q = Mq + N(r_{n-1} - r_n) \text{ and } (M+N)^k q \to 0, \text{ as } k \to \infty. \tag{3.3}$$

By (I_j) it follows that

$$b_n = /x_n - x_0/ \leq \sum_{j=0}^{n-1} a_j \leq \sum_{j=0}^{n-1} (r_j - r_{j+1}) = r - r_n \leq r.$$

Hence, $x_n \in U(x_0, r) \subset D$ and by (H_1) M is a bound for $I - A(x_n)$.

We can write by (H_2) that

$$\begin{aligned} /F(x_n)/ &= /F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1})/ \\ &\leq Na_{n-1} \leq N(r_{n-1} - r_n). \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$Mq + /F(x_n)/ \leq q.$$

By Lemma 2.5, x_{n+1} is well defined and $a_n \leq q \leq r_n$. In view of the definition of r_{n+1} we have that

$$P_{n+1}(r_n - q) = P_n(r_n) - q = r_n - q,$$

so that by Lemma 2.4, r_{n+1} is well defined and

$$r_{n+1} + a_n \leq r_n - q + q = r_n,$$

which proves (I_{n+1}). The induction for (I_n) is complete. Let $m \geq n$, then we obtain in turn that

$$/x_{m+1} - x_n/ \leq \sum_{j=n}^{m} a_j \leq \sum_{j=n}^{m} (r_j - r_{j+1}) = r_n - r_{m+1} \leq r_n. \tag{3.5}$$

Moreover, we get inductively the estimate

$$r_{n+1} = P_{n+1}(r_{n+1}) \leq P_{n+1}(r_n) \leq (M+N) r_n \leq \cdots \leq (M+N)^{n+1} r.$$

It follows from (H_5) that $\{r_n\}$ is a null-sequence. Hence, $\{x_n\}$ is a complete sequence in a Banach space X by (3.5) and as such it converges to some $x^* \in X$. By letting $m \to \infty$ in (3.5) we deduce that $x^* \in U(x_n, r_n)$. Furthermore, (3.4) shows that x^* is a zero of F. Hence, (C_1) and (C_2) are proved.

In view of the estimate

$$R_n(r_n) \leq P_n(r_n) \leq r_n$$

the apriori, bound of (C_3) is well defined by Lemma 2.4. That is s_n is smaller in general than r_n. The conditions of Theorem 3.1 are satisfied for x_n replacing x_0. A solution of the inequality of (C_2) is given by s_n (see (3.4)). It follows from (3.5) that the conditions of Theorem 3.1 are easily verified. Then, it follows from (C_1) that $x^* \in U(x_n, s_n)$ which proves (C_3). □

In general the aposterior, estimate is of interest. Then, condition (H_5) can be avoided as follows:

Proposition 3.2. *Suppose that the condition (H_1) of Theorem* 3.1 *is true.*

(H_3') *There exists $s \in K$, $\theta \in (0,1)$ such that*

$$R_0(s) = (M+N)s + /F(x_0)/ \leq \theta s.$$

(H_4') $U(x_0, s) \subset D$.

Then, there exists $r \leq s$ satisfying the conditions of Theorem 3.1. *Moreover, the zero x^* of F is unique in $U(x_0, s)$.*

Remark 3.3. (i) Notice that by Lemma 2.4 $R_n^{\infty}(0)$ is the smallest solution of $R_n(s) \leq s$. Hence any solution of this inequality yields on upper estimate for $R_n^{\infty}(0)$. Similar inequalities appear in (H_2) and (H_2').
(ii) The weak assumptions of Theorem 3.1 do not imply the existence of $A(x_n)^{-1}$. In practice the computation of $T_n^{\infty}(0)$ as a solution of a linear equation is no problem and the computation of the expensive or impossible to compute in general $A(x_n)^{-1}$ is not needed.
(iii) We can used the following result for the computation of the aposteriori estimates. The proof can be found in [14, Lemma 4.2] by simply exchanging the definitions of R.

Lemma 3.4. *Suppose that the conditions of Theorem* 3.1 *are satisfied. If* $s \in K$ *is a solution of* $R_n(s) \leq s$*, then* $q := s - a_n \in K$ *and solves* $R_{n+1}(q) \leq q$. *This solution might be improved by* $R_{n+1}^k(q) \leq q$ *for each* $k = 1, 2, \cdots$.

4. Special cases and applications

Application 4.1. *The results obtained in earlier studies such as [7, 8, 9, 14] require that operator* F *(i.e.* G*) is Fréchet-differentiable. This assumption limits the applicability of the earlier results. In the present study we only require that* F *is a continuous operator. Hence, we have extended the applicability of these methods to classes of operators that are only continuous.*

Example 4.2. The j-dimensional space $\mathbb{R}^j$ is a classical example of a generalized Banach space. The generalized norm is defined by componentwise absolute values. Then, as ordered Banach space we set $E = \mathbb{R}^j$ with componentwise ordering with e.g. the maximum norm. A bound for a linear operator (a matrix) is given by the corresponding matrix with absolute values. Similarly, we can define the "N" operators. Let $E = \mathbb{R}$. That is we consider the case of a real normed space with norm denoted by $\|\cdot\|$. Let us see how the conditions of Theorem 3.1 look like.

Theorem 4.3. (H_1) $\|I - A(x)\| \leq M$ *for some* $M \geq 0$.
(H_2) $\|F(y) - F(x) - A(x)(y - x)\| \leq N \|y - x\|$ *for some* $N \geq 0$.
(H_3) $M + N < 1$,

$$r = \frac{\|F(x_0)\|}{1 - (M + N)}. \tag{4.1}$$

(H_4) $U(x_0, r) \subseteq D$.
(H_5) $(M + N)^k r \to 0$ *as* $k \to \infty$*, where* r *is given by* (4.1).
Then the conclusions of Theorem 3.1 *hold.*

5. Applications to generalized fractional calculus

We present some applications of Theorem 4.3 in this section.

Background

We use a lot here the following generalized fractional integral.

Definition 5.1. (see also [11, p. 99]) The left generalized fractional integral of a function f with respect to given function g is defined as follows:

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a, b])$ (absolutely continuous functions) and is striclty increasing, $f \in L_\infty([a, b])$. We set

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) \, dt, \quad x \geq a, \tag{5.1}$$

clearly $\left(I_{a+;g}^{\alpha} f\right)(a) = 0$.

When g is the identity function id, we get that $I_{a+;id}^{\alpha} = I_{a+}^{\alpha}$, the ordinary left Riemann-Liouville fractional integral, where

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) \, dt, \quad x \geq a, \tag{5.2}$$

$\left(I_{a+}^{\alpha} f\right)(a) = 0$.

When $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$, we get

Definition 5.2. ([11, p. 110]) Let $0 < a < b < \infty$, $\alpha > 0$. The left Hadamard fractional integral of order α is given by

$$\left(J_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{y}\right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x \geq a, \tag{5.3}$$

where $f \in L_\infty([a, b])$.

Definition 5.3. ([5]) The left fractional exponential integral is defined as follows: Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$. We set

$$\left(I_{a+;e^x}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(e^x - e^t\right)^{\alpha-1} e^t f(t) \, dt, \quad x \geq a. \tag{5.4}$$

Definition 5.4. ([5]) Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a, b])$, $A > 1$. We give the fractional integral

$$\left(I_{a+;A^x}^{\alpha} f\right)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_a^x \left(A^x - A^t\right)^{\alpha-1} A^t f(t) \, dt, \quad x \geq a. \tag{5.5}$$

Definition 5.5. ([5]) Let $\alpha, \sigma > 0$, $0 \leq a < b < \infty$, $f \in L_\infty([a, b])$. We set

$$\left(K_{a+;x^\sigma}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_z^x (x^\sigma - t^\sigma)^{\alpha-1} f(t) \, \sigma t^{\sigma-1} dt, \quad x \geq a. \tag{5.6}$$

We mention the following generalized fractional derivatives:

Definition 5.6. ([5]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$. Consider $f \in AC^m([a, b])$ (space of functions f with $f^{(m-1)} \in AC([a, b])$). We define the left generalized fractional derivative of f of order α as follows

$$\left(D_{*a;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x (g(x)-g(t))^{m-\alpha-1} g'(t) f^{(m)}(t)\,dt, \quad (5.7)$$

for any $x\in[a,b]$, where Γ is the gamma function.

We set

$$D_{*\alpha;g}^{m}f(x)=f^{(m)}(x), \quad (5.8)$$

$$D_{*a;g}^{0}f(x)=f(x), \quad \forall\, x\in[a,b]. \quad (5.9)$$

When $g=id$, then $D_{*a}^{\alpha}f=D_{*a;id}^{\alpha}f$ is the left Caputo fractional derivative.

So we have the specific generalized left fractional derivatives.

Definition 5.7. ([5])

$$D_{*a;\ln x}^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x \left(\ln\frac{x}{y}\right)^{m-\alpha-1}\frac{f^{(m)}(y)}{y}dy, \quad x\geq a>0, \quad (5.10)$$

$$D_{*a;e^x}^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x \left(e^x-e^t\right)^{m-\alpha-1} e^t f^{(m)}(t)\,dt, \quad x\geq a, \quad (5.11)$$

and

$$D_{*a;A^x}^{\alpha}f(x)=\frac{\ln A}{\Gamma(m-\alpha)}\int_a^x \left(A^x-A^t\right)^{m-\alpha-1} A^t f^{(m)}(t)\,dt, \quad x\geq a, \quad (5.12)$$

$$\begin{aligned}&\left(D_{*a;x^\sigma}^{\alpha}f\right)(x)\\&=\frac{1}{\Gamma(m-\alpha)}\int_a^x (x^\sigma-t^\sigma)^{m-\alpha-1}\sigma t^{\sigma-1}f^{(m)}(t)\,dt, \quad x\geq a\geq 0.\end{aligned} \quad (5.13)$$

Remark 5.8. ([5]) Here $g\in AC([a,b])$ (absolutely continuous functions), g is increasing over $[a,b]$, $\alpha>0$. Then

$$\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)\,dt=\frac{(g(x)-g(a))^{\alpha}}{\alpha}, \quad \forall\, x\in[a,b]. \quad (5.14)$$

Theorem 5.9. ([5]) *Let $\alpha>0$, $\mathbb{N}\ni m=\lceil\alpha\rceil$, and $f\in C^m([a,b])$. Then $\left(D_{*a;g}^{\alpha}f\right)(x)$ is continuous in $x\in[a,b]$.*

Results. (I) We notice the following

$$\begin{aligned}\left|\left(I_{a+;g}^{\alpha}f\right)(x)\right| &\leq \frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)\,|f(t)|\,dt\\ &\leq \frac{\|f\|_\infty}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)\,dt\\ &=\frac{\|f\|_\infty}{\Gamma(\alpha)}\frac{(g(x)-g(a))^{\alpha}}{\alpha}=\frac{\|f\|_\infty}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}.\end{aligned} \quad (5.15)$$

That is

$$\begin{aligned}\left|\left(I_{a+;g}^{\alpha}f\right)(x)\right| &\leq \frac{\|f\|_{\infty}}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} \\ &\leq \|f\|_{\infty}\frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}, \quad \forall x\in[a,b].\end{aligned} \tag{5.16}$$

In particular $\left(I_{a+;g}^{\alpha}f\right)(a)=0$. Clearly $I_{a+;g}^{\alpha}$ is a bounded linear operator.

Theorem 5.10. ([6]) *Let $r>0$, $a<b$, $F\in L_{\infty}([a,b])$, $g\in AC([a,b])$ and g is strictly increasing. Consider*

$$G(s):=\int_a^s (g(s)-g(t))^{r-1}g'(t)F(t)\,dt, \quad \text{for all } s\in[a,b]. \tag{5.17}$$

Then $G\in C([a,b])$.

By Theorem 5.10, the function $\left(I_{a+;g}^{\alpha}f\right)$ is a continuous function over $[a,b]$. Consider $a<a^{*}<b$. Therefore $\left(I_{a+;g}^{\alpha}f\right)$ is also continuous over $[a^{*},b]$.

Thus, there exist $x_1,x_2\in[a^{*},b]$ such that

$$\left(I_{a+;g}^{\alpha}f\right)(x_1)=\min\left(I_{a+;g}^{\alpha}f\right)(x), \tag{5.18}$$

$$\left(I_{a+;g}^{\alpha}f\right)(x_2)=\max\left(I_{a+;g}^{\alpha}f\right)(x), \quad x\in[a^{*},b]. \tag{5.19}$$

We assume that

$$\left(I_{a+;g}^{\alpha}f\right)(x_1)>0. \tag{5.20}$$

Hence

$$\left\|I_{a+;g}^{\alpha}f\right\|_{\infty,[a^{*},b]}=\left(I_{a+;g}^{\alpha}f\right)(x_2)>0. \tag{5.21}$$

Here it is

$$J(x)=mx, \quad m\neq 0. \tag{5.22}$$

Therefore the equation

$$Jf(x)=0, \quad x\in[a^{*},b], \tag{5.23}$$

has the same solutions as the equation

$$F(x):=\frac{Jf(x)}{2\left(I_{a+;g}^{\alpha}f\right)(x_2)}=0, \quad x\in[a^{*},b]. \tag{5.24}$$

Notice that

$$I_{a+;g}^{\alpha}\left(\frac{f}{2\left(I_{a+;g}^{\alpha}f\right)(x_2)}\right)(x)=\frac{\left(I_{a+;g}^{\alpha}f\right)(x)}{2\left(I_{a+;g}^{\alpha}f\right)(x_2)}\leq\frac{1}{2}<1, \quad x\in[a^{*},b]. \tag{5.25}$$

Call

$$A(x):=\frac{\left(I_{a+;g}^{\alpha}f\right)(x)}{2\left(I_{a+;g}^{\alpha}f\right)(x_2)}, \quad \forall\, x\in[a^{*},b]. \tag{5.26}$$

We notice that

$$0<\frac{\left(I_{a+;g}^{\alpha}f\right)(x_1)}{2\left(I_{a+;g}^{\alpha}f\right)(x_2)}\leq A(x)\leq\frac{1}{2},\ \ \forall\ x\in[a^*,b]. \tag{5.27}$$

We observe

$$|1-A(x)|=1-A(x)\leq 1-\frac{\left(I_{a+;g}^{\alpha}f\right)(x_1)}{2\left(I_{a+;g}^{\alpha}f\right)(x_2)}=:\gamma_0,\ \ \forall\ x\in[a^*,b]. \tag{5.28}$$

Clearly $\gamma_0\in(0,1)$. i.e.,

$$|1-A(x)|\leq\gamma_0,\ \ \forall\ x\in[a^*,b],\ \gamma_0\in(0,1). \tag{5.29}$$

Next we assume that $F(x)$ is a contraction, i.e.

$$|F(x)-F(y)|\leq\lambda|x-y|,\ \ \forall\ x,y\in[a^*,b], \tag{5.30}$$

and $0<\lambda<\frac{1}{2}$.

Equivalently we have

$$|Jf(x)-Jf(y)|\leq 2\lambda\left(I_{a+;g}^{\alpha}f\right)(x_2)|x-y|,\ \ \forall\ x,y\in[a^*,b]. \tag{5.31}$$

We observe that

$$\begin{aligned}&|F(y)-F(x)-A(x)(y-x)|\\&\leq|F(y)-F(x)|+|A(x)||y-x|\\&\leq\lambda|y-x|+|A(x)||y-x|=(\lambda+|A(x)|)|y-x|\\&=:(\psi_1),\ \ \forall\ x,y\in[a^*,b].\end{aligned} \tag{5.32}$$

By (5.16) we get

$$\left|\left(I_{a+;g}^{\alpha}f\right)(x)\right|\leq\frac{\|f\|_\infty}{\Gamma(\alpha+1)}(g(b)-g(a))^\alpha,\ \ \forall\ x\in[a^*,b]. \tag{5.33}$$

Hence

$$\begin{aligned}|A(x)|&=\frac{\left|\left(I_{a+;g}^{\alpha}f\right)(x)\right|}{2\left(I_{a+;g}^{\alpha}f\right)(x_2)}\\&\leq\frac{\|f\|_\infty(g(b)-g(a))^\alpha}{2\Gamma(\alpha+1)\left(I_{a+;g}^{\alpha}f\right)(x_2)}<\infty,\ \ \forall\ x\in[a^*,b].\end{aligned} \tag{5.34}$$

Therefore we get

$$(\psi_1)\leq\left(\lambda+\frac{\|f\|_\infty(g(b)-g(a))^a}{2\Gamma(\alpha+1)\left(I_{a+;g}^{\alpha}f\right)(x_2)}\right)|y-x|,\ \ \ \forall\ x,y\in[a^*,b]. \tag{5.35}$$

Call

$$0<\gamma_1:=\lambda+\frac{\|f\|_\infty(g(b)-g(a))^a}{2\Gamma(\alpha+1)\left(I_{a+;g}^{\alpha}f\right)(x_2)}, \tag{5.36}$$

choosing $(g(b)-g(a))$ small enough we can make $\gamma_1\in(0,1)$.

We have proved that

$$\begin{aligned} &\left|F\left(y\right)-F\left(x\right)-A\left(x\right)\left(y-x\right)\right| \\ &\leq \gamma_{1}\left|y-x\right|, \ \ \forall\, x,y\in\left[a^{*},b\right], \gamma_{1}\in\left(0,1\right). \end{aligned} \tag{5.37}$$

Next we call and we need that

$$\begin{aligned} &0<\gamma:=\gamma_{0}+\gamma_{1} \\ &=1-\frac{\left(I_{a+;g}^{\alpha}f\right)\left(x_{1}\right)}{2\left(I_{a+;g}^{\alpha}f\right)\left(x_{2}\right)}+\lambda+\frac{\left\|f\right\|_{\infty}\left(g\left(b\right)-g\left(a\right)\right)^{a}}{2\Gamma\left(\alpha+1\right)\left(I_{a+;g}^{\alpha}f\right)\left(x_{2}\right)}<1, \end{aligned} \tag{5.38}$$

$$\lambda+\frac{\left\|f\right\|_{\infty}\left(g\left(b\right)-g\left(a\right)\right)^{a}}{2\Gamma\left(\alpha+1\right)\left(I_{a+;g}^{\alpha}f\right)\left(x_{2}\right)}<\frac{\left(I_{a+;g}^{\alpha}f\right)\left(x_{1}\right)}{2\left(I_{a+;g}^{\alpha}f\right)\left(x_{2}\right)}, \tag{5.39}$$

equivalently,

$$2\lambda\left(I_{a+;g}^{\alpha}f\right)\left(x_{2}\right)+\frac{\left\|f\right\|_{\infty}\left(g\left(b\right)-g\left(a\right)\right)^{a}}{\Gamma\left(\alpha+1\right)}<\left(I_{a+;g}^{\alpha}f\right)\left(x_{1}\right), \tag{5.40}$$

which is possible for small λ, $\left(g\left(b\right)-g\left(a\right)\right)$. That is $\gamma\in(0,1)$. So our method solves (5.23).

(II) Let $\alpha\notin\mathbb{N}$, $\alpha>0$ and $\lceil\alpha\rceil=m$, $a<a^{*}<b$, $G\in AC^{m}\left([a,b]\right)$, with $0\neq G^{(m)}\in L_{\infty}\left([a,b]\right)$. Here we consider the left generalized (Caputo type) fractional derivative:

$$\left(D_{*a;g}^{\alpha}G\right)\left(x\right)=\frac{1}{\Gamma\left(m-\alpha\right)}\int_{a}^{x}\left(g\left(x\right)-g\left(t\right)\right)^{m-\alpha-1}g'\left(t\right)G^{(m)}\left(t\right)dt, \tag{5.41}$$

for any $x\in\left[a,b\right]$. By Theorem 5.10, we get that $\left(D_{*a;g}^{\alpha}G\right)\in C\left([a,b]\right)$, in particular $\left(D_{*a;g}^{\alpha}G\right)\in C\left([a^{*},b]\right)$. Here notice that $\left(D_{*a;g}^{\alpha}G\right)\left(a\right)=0$. Therefore there exist $x_{1},x_{2}\in[a^{*},b]$ such that $D_{*a;g}^{\alpha}G\left(x_{1}\right)=\min D_{*a;g}^{\alpha}G\left(x\right)$, and $D_{*a;g}^{\alpha}G\left(x_{2}\right)=\max D_{*a;g}^{\alpha}G\left(x\right)$, for $x\in[a^{*},b]$.

We assume that

$$D_{*a;g}^{\alpha}G\left(x_{1}\right)>0. \quad (\text{i.e., } D_{*a;g}^{\alpha}G\left(x\right)>0, \ \forall\, x\in[a^{*},b]). \tag{5.42}$$

Furthermore

$$\left\|D_{*a;g}^{\alpha}G\right\|_{\infty,[a^{*},b]}=D_{*a;g}^{\alpha}G\left(x_{2}\right). \tag{5.43}$$

Here it is

$$J\left(x\right)=mx,\ m\neq0. \tag{5.44}$$

The equation

$$JG\left(x\right)=0, \quad x\in\left[a^{*},b\right], \tag{5.45}$$

has the same set of solutions as the equation

$$F\left(x\right):=\frac{JG\left(x\right)}{2D_{*a;g}^{\alpha}G\left(x_{2}\right)}=0, \quad x\in\left[a^{*},b\right]. \tag{5.46}$$

Notice that

$$D^{\alpha}_{*a;g}\left(\frac{G(x)}{2D^{\alpha}_{*a;g}G(x_2)}\right)=\frac{D^{\alpha}_{*a;g}G(x)}{2D^{\alpha}_{*a;g}G(x_2)}\leq\frac{1}{2}<1,\quad\forall\, x\in[a^*,b]. \tag{5.47}$$

We call

$$A(x):=\frac{D^{\alpha}_{*a;g}G(x)}{2D^{\alpha}_{*a;g}G(x_2)},\quad\forall\, x\in[a^*,b]. \tag{5.48}$$

We notice that

$$0<\frac{D^{\alpha}_{*a;g}G(x_1)}{2D^{\alpha}_{*a;g}G(x_2)}\leq A(x)\leq\frac{1}{2}. \tag{5.49}$$

Hence it holds

$$|1-A(x)|=1-A(x)\leq 1-\frac{D^{\alpha}_{*a;g}G(x_1)}{2D^{\alpha}_{*a;g}G(x_2)}=:\gamma_0,\quad\forall\, x\in[a^*,b]. \tag{5.50}$$

Clearly $\gamma_0\in(0,1)$. We have proved that

$$|1-A(x)|\leq\gamma_0\in(0,1),\quad\forall\, x\in[a^*,b]. \tag{5.51}$$

Next we assume that $F(x)$ is a contraction over $[a^*,b]$, i.e.,

$$|F(x)-F(y)|\leq\lambda|x-y|,\quad\forall\, x,y\in[a^*,b], \tag{5.52}$$

and $0<\lambda<\frac{1}{2}$. Equivalently we have

$$|JG(x)-JG(y)|\leq 2\lambda\left(D^{\alpha}_{*a;g}G(x_2)\right)|x-y|,\quad\forall\, x,y\in[a^*,b]. \tag{5.53}$$

We observe that

$$\begin{aligned}&|F(y)-F(x)-A(x)(y-x)|\\&\leq|F(y)-F(x)|+|A(x)||y-x|\\&\leq\lambda|y-x|+|A(x)||y-x|=(\lambda+|A(x)|)|y-x|\\&=:(\xi_2),\quad\forall\, x,y\in[a^*,b].\end{aligned} \tag{5.54}$$

We observe that

$$\begin{aligned}\left|D^{\alpha}_{*a;g}G(x)\right|&\leq\frac{1}{\Gamma(m-\alpha)}\int_a^x(g(x)-g(t))^{m-\alpha-1}g'(t)\left|G^{(m)}(t)\right|dt\\&\leq\frac{1}{\Gamma(m-\alpha)}\left(\int_a^x(g(x)-g(t))^{m-\alpha-1}g'(t)\,dt\right)\left\|G^{(m)}\right\|_{\infty}\\&=\frac{1}{\Gamma(m-\alpha)}\frac{(g(x)-g(a))^{m-\alpha}}{(m-\alpha)}\left\|G^{(m)}\right\|_{\infty}\\&=\frac{1}{\Gamma(m-\alpha+1)}(g(x)-g(a))^{m-\alpha}\left\|G^{(m)}\right\|_{\infty}\\&\leq\frac{(g(b)-g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)}\left\|G^{(m)}\right\|_{\infty}.\end{aligned} \tag{5.55}$$

That is

$$\left|D^{\alpha}_{*a;g}G(x)\right| \leq \frac{(g(b)-g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)}\left\|G^{(m)}\right\|_{\infty} < \infty, \quad \forall\, x \in [a,b]. \tag{5.56}$$

Hence, for $x \in [a^*, b]$, we get that

$$|A(x)| = \frac{\left|D^{\alpha}_{*a;g}G(x)\right|}{2D^{\alpha}_{*a;g}G(x_2)} \leq \frac{(g(b)-g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)}\frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{*a;g}G(x_2)} < \infty. \tag{5.57}$$

Consequently we observe

$$(\xi_2) \leq \left(\lambda + \frac{(g(b)-g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)}\frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{*a;g}G(x_2)}\right)|y-x|, \quad \forall\, x,y \in [a^*,b]. \tag{5.58}$$

Call

$$0 < \gamma_1 := \lambda + \frac{(g(b)-g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)}\frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{*a;g}G(x_2)}, \tag{5.59}$$

choosing $(g(b)-g(a))$ small enough we can make $\gamma_1 \in (0,1)$.

We proved that

$$|F(y) - F(x) - A(x)(y-x)| \leq \gamma_1 |y-x|, \ \forall\, x,y \in [a^*,b], \tag{5.60}$$

where $\gamma_1 \in (0,1)$. Next we call and need

$$\begin{aligned} 0 < \gamma &:= \gamma_0 + \gamma_1 \\ &= 1 - \frac{D^{\alpha}_{*a;g}G(x_1)}{2D^{\alpha}_{*a;g}G(x_2)} + \lambda + \frac{(g(b)-g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)}\frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{*a;g}G(x_2)} < 1, \end{aligned} \tag{5.61}$$

equivalently we find,

$$\lambda + \frac{(g(b)-g(a))^{m-\alpha}}{2\Gamma(m-\alpha+1)}\frac{\left\|G^{(m)}\right\|_{\infty}}{D^{\alpha}_{*a;g}G(x_2)} < \frac{D^{\alpha}_{*a;g}G(x_1)}{2D^{\alpha}_{*a;g}G(x_2)}, \tag{5.62}$$

equivalently,

$$2\lambda D^{\alpha}_{*a;g}G(x_2) + \frac{(g(b)-g(a))^{m-\alpha}}{\Gamma(m-\alpha+1)}\left\|G^{(m)}\right\|_{\infty} < D^{\alpha}_{*a;g}G(x_1), \tag{5.63}$$

which is possible for small λ, $(g(b)-g(a))$.

That is $\gamma \in (0,1)$. Hence equation (5.45) can be solved with our presented numerical methods.

Conclusion. Our presented earlier semilocal convergence Newton-type general methods, see Theorem 4.3, can apply in the above two generalized fractional settings since the following inequalities have been fulfilled:

$$\|1 - A(x)\|_{\infty} \leq \gamma_0 \tag{5.64}$$

and

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \tag{5.65}$$

where $\gamma_0, \gamma_1 \in (0,1)$, furthermore it holds

$$\gamma = \gamma_0 + \gamma_1 \in (0,1), \tag{5.66}$$

for all $x, y \in [a^*, b]$, where $a < a^* < b$.

The specific functions $A(x)$, $F(x)$ have been described above.

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