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WEAK CONVERGENCE OF A NEW PROJECTION ALGORITHM FOR VARIATIONAL INEQUALITIES IN BANACH SPACES

Ying Liu

College of Mathematics and Information Science
Hebei University, Baoding, 071002, P.R.China
e-mail: ly_cyh2013@163.com

Abstract. Applying the generalized projection operator, we introduce a new iterative algorithm in Banach spaces for a variational inequality involving a monotone hemi-continuous operator which is more general than a inverse-strongly-monotone operator. Weak convergence of the iterative algorithm is also proved.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, and E^* be the dual of E. $\langle x, f\rangle$ denotes the duality pairing of E and E^*. Suppose that C is a nonempty, closed and convex subset of E and A is a monotone operator of C into E^*. Then we study the problem of finding a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0, \quad \forall\, v \in C. \tag{1.1}$$

This problem is called the variational inequality problem [8]. The set of solutions of the variational inequality problem is denoted by $VI(C,A)$. Variational inequality theory, as a very effective and powerful tool of the current mathematical technology, has been widely applied to mathematical programming,

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optimization and control, economics and transportation equilibrium, engineering sciences, etc. An operator A of C into E^* is said to be $\alpha-$inverse-strongly-monotone [7] if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall\, x,y \in C.$$

In order to approximate a solution of variational inequality (1.1), the inverse-strong-monotonicity of A was often assumed (see, for example, [4], [6], [7], [9]). Especially, in [7], Iiduka and Takahashi proved the following theorem.

Theorem 1.1. *Let E be a 2-uniformly convex, uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous, and C be a nonempty, closed and convex subset of E. Assume that A is an operator of C into E^* that satisfies:*

(A1) *A is $\alpha-$inverse-strongly-monotone,*
(A2) *$VI(C,A) \neq \emptyset$,*
(A3) *$\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C,A)$.*

Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \tag{1.2}$$

for every $n = 1, 2, ...,$ where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < \frac{c^2\alpha}{2}$, then the sequence $\{x_n\}$ converges weakly to some element $z \in VI(C,A)$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. Further $z = \lim_{n\to\infty} \Pi_{VI(C,A)}(x_n)$.

We know that if A is $\alpha-$inverse-strongly-monotone, then it is monotone and $\frac{1}{\alpha}-$Lipschitz continuous. But, the converse is not true. One question arises naturally: How to extend Theorem 1.1 to the more general class of monotone and continuous mappings? The aim for loosening this assumption has been achieved in [3] by using the subgradient extragradient method in Hilbert space. The purpose of this paper is to weaken the condition in Banach spaces.

In addition, we also note that:

(1) (A3) is very strong and unnatural. The necessity of this condition needs to be checked.
(2) The 2-uniform convexity of Banach space E restricts the use of variational inequality (1.1), and hence, it is interesting to extend Theorem 1.1 to spaces beyond 2-uniformly convex, uniformly smooth Banach spaces.

In order to achieve the objects mentioned above, we introduce a new iterative algorithm for the approximation to a solution of variational inequality (1.1). Based on this, we establish a weak convergence theorem which generalizes the result of [7] by loosening some assumptions on A and E.

2. Preliminaries

Throughout this paper, let E be a Banach space, and E^* be the dual space of E. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E and E^*. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$Jx := \{v \in E^* : \langle x, v \rangle = \|v\|^2 = \|x\|^2\}, \quad \forall\, x \in E.$$

The following properties of J can be found in [2].

(i) If E is strictly convex, then J is strictly monotone;
(ii) If E is uniformly smooth, then J is single-valued and uniformly norm-to-norm continuous on each bounded subset of E.

The duality mapping J from a smooth Banach space E into E^* is said to be weakly sequentially continuous [5] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup Jx$.

Let E be a smooth Banach space. Define

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \;\; \forall\, x, y \in E. \tag{2.1}$$

Clearly, we have from the definition of ϕ that

(B1) $(\|x\| - \|y\|)^2 \le \phi(y, x) \le (\|x\| + \|y\|)^2$,
(B2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,
(B3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Remark 2.1. We have from Remark 2.1 in [10] that if E is a strictly convex and smooth Banach space, then for $x, y \in E, \phi(y, x) = 0$ if and only if $x = y$.

Let E be a reflexive, strictly convex, and smooth Banach space. K denotes a nonempty, closed, and convex subset of E. For each $x \in E$, there exists a unique element $x_0 \in K$ (denoted by $\Pi_K(x)$) such that

$$\phi(x_0, x) = \min_{y \in K} \phi(y, x).$$

The mapping $\Pi_K : E \to K$ defined by $\Pi_K(x) = x_0$ is called the generalized projection operator from E onto K. Moreover, x_0 is called the generalized projection of x. See [1] for some properties of Π_K.

Lemma 2.2. ([1]) *Let E be a reflexive, strictly convex, and smooth Banach space. Let C be a nonempty, closed, and convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \forall\, y \in C.$$

Lemma 2.3. ([1]) *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and let $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \ge 0, \quad \forall\, y \in C.$$

Lemma 2.4. ([10]) *Let E be a uniformly convex and smooth Banach space. Let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Definition 2.5. A multi-valued mapping $M : E \to E^*$ with domain $D(M) = \{z \in E : Mz \neq \emptyset\}$ and range $R(M) = \bigcup\{Mz \in E^* : z \in D(M)\}$ is said to be monotone if $\langle x_1 - x_2, u_1 - u_2\rangle \geq 0$ for each $x_i \in D(M)$ and $u_i \in M(x_i)$, $i = 1, 2$.

Definition 2.6. A monotone mapping M is said to be maximal if its graph $G(M) = \{(x, u) : u \in Mx\}$ is not properly contained in the graph of any other monotone operator.

It is known that a monotone mapping M is maximal if and only if for $(x, u) \in E \times E^*, \langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in G(M)$ implies $u \in Mx$.

Definition 2.7. An operator A of C into E^* is said to be hemi-continuous if for all $x, y \in C$, the mapping f of $[0, 1]$ into E^* defined by $f(t) = A(tx+(1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is,

$$N_C(v) := \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \text{ for all } y \in C\}.$$

Lemma 2.8. ([12]) *Let C be a nonempty, closed, and convex subset of a Banach space E and let A be a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.9. ([13]) *Let C be a nonempty, closed, and convex subset of a Banach space E and let A be a monotone, hemicontinuous operator of C into E^*. Then*

$$VI(C, A) = \{u \in C : \langle v - u, Av\rangle \geq 0 \textit{ for all } v \in C\}.$$

It is obvious from Lemma 2.9 that the set $VI(C, A)$ is a closed convex subset of C.

Lemma 2.10. ([13]) *Let C be a nonempty, compact, convex subset of a Banach space E and let A be a monotone, hemicontinuous operator of C into E^*. Then the set $VI(C, A)$ is nonempty.*

Lemma 2.11. ([11]) *Let E be a reflexive Banach space and λ be a positive number. If $T : E \to 2^{E^*}$ is a maximal monotone mapping, then $R(J + \lambda T) = E^*$ and $(J + \lambda T)^{-1} : E^* \to E$ is a demi-continuous single-valued maximal monotone mapping.*

Lemma 2.12. ([7]) *Let S be a nonempty, closed, and convex subset of a uniformly convex, smooth Banach space E. Let $\{x_n\}$ be a sequence in E. Suppose that, for all $u \in S$,*

$$\phi(u, x_{n+1}) \leq \phi(u, x_n),$$

for every $n = 1, 2, \cdots$. Then $\{\Pi_S x_n\}$ is a Cauchy sequence.

3. Main results

In this section, we construct the following iterative algorithm for solving variational inequality (1.1) involving a monotone hemi-continuous operator A.

Algorithm 3.1.
Step 0. Arbitrarily select initial $x_0 \in E$ and set $k = 0$.
Step 1. Find $y_k \in C$ such that

$$y_k = \Pi_C(J^{-1}[J(x_k) - \lambda_k A(y_k)]), \tag{3.1}$$

where the positive sequence $\{\lambda_k\}$ satisfies

$$\alpha_1 := \inf_{k \geq 0} \lambda_k > 0. \tag{3.2}$$

Step 2. Set $C_k = \{w \in E : \langle w - y_k, J(x_k) - J(y_k)\rangle \leq 0\}$. If $x_k = y_k$, then stop; otherwise, take x_{k+1} such that

$$x_{k+1} = \Pi_{C_k}(x_k). \tag{3.3}$$

Step 3. Let $k = k + 1$ and return to Step 1.

Remark 3.1. (i) y_k is solvable for all $k = 0, 1, 2, ...$ Indeed, let $T \subset E \times E^*$ be an operator as follows:

$$Tv := \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.4}$$

By Lemma 2.8, T is maximal monotone. Hence, it follows from Lemma 2.11 that $R(J + \lambda T) = E^*$, for all $\lambda > 0$. Therefore, for any $Jx_k \in E^*$, there exists $y_k \in E$, such that $Jx_k - Jy_k \in \lambda_k T y_k$. We have from the definition of T that $Jx_k - Jy_k - \lambda_k Ay_k \in \lambda_k N_C(y_k)$. In view of the definition of $N_C(\cdot)$, we have that

$$\langle y_k - w, \frac{1}{\lambda_k}(Jx_k - Jy_k - \lambda_k Ay_k)\rangle \geq 0, \ \ \forall\, w \in C.$$

It follows from Lemma 2.3 that $y_k = \Pi_C[J^{-1}(Jx_k - \lambda_k Ay_k)]$. This implies that Step 1 of Algorithm 3.1 is well defined.

(ii) If $x_k = y_k$, then $x_k \in VI(C, A)$, which implies that the iterative sequence $\{x_k\}$ is finite, and the last term is a solution of variational inequality (1.1). Otherwise, $x_k \notin C_k$. Therefore Algorithm 3.1 is well-defined.

(iii) In Algorithm 3.1, the step 1 is used to construct a half-space, the next iterate x_{k+1} is then obtained by a generalized projection of x_k, which is not expensive at all from a numerical point of view.

Now we show the convergence of the iterative sequence generated by Algorithm 3.1 in the Banach space E.

Theorem 3.2. *Let E be a uniformly convex, uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous, C be a nonempty, closed and convex subset of E. Assume that $A : C \to E^*$ is a hemi-continuous monotone operator and $VI(C, A) \neq \emptyset$. Then, the iterative sequence $\{x_k\}$ generated by Algorithm* 3.1 *converges weakly to an element $\hat{x} \in VI(C, A)$. Further, $\hat{x} = \lim_{k\to\infty} \Pi_{VI(C,A)}(x_k)$.*

Proof. The proof will be split into four steps.

Step 1. Show that C_k is a nonempty, closed and convex subset of C for every $k = 0, 1, 2, ...$ It is obvious that C_k is closed and convex. Next, we show that $VI(C, A) \subset C_k$ for all $k = 0, 1, 2, ...$

Suppose $x^* \in VI(C, A)$.Then we have $\langle y_k - x^*, \lambda_k Ax^*\rangle \geq 0$. On the other hand, we have from $y_k = \Pi_C(J^{-1}[J(x_k) - \lambda_k A(y_k)])$ that

$$\langle y_k - x^*, Jx_k - \lambda_k A(y_k) - Jy_k\rangle \geq 0.$$

Hence,

$$\langle y_k - x^*, Jx_k - Jy_k + \lambda_k Ax^* - \lambda_k A(y_k)\rangle \geq 0.$$

It follows from the monotonicity of A that

$$\langle y_k - x^*, Jx_k - Jy_k\rangle \geq \lambda_k\langle y_k - x^*, A(y_k) - Ax^*\rangle \geq 0,$$

which implies that $VI(C, A) \subset C_k$ for all $k = 0, 1, 2, ...$

Step 2. Show that $\{x_k\}$ and $\{y_k\}$ have the same weak accumulation points. Since $x_{k+1} = \Pi_{C_k} x_k$, by Lemma 2.2, we deduce that

$$\phi(x^*, x_{k+1}) \leq \phi(x^*, x_k) - \phi(x_{k+1}, x_k). \tag{3.5}$$

Thus,

$$\phi(x^*, x_{k+1}) \leq \phi(x^*, x_k), \tag{3.6}$$

which yields that the sequence $\{\phi(x^*, x_k)\}$ is convergent. We know from (B1) that $\{x_k\}$ is bounded. It follows from (3.5) that

$$\phi(x_{k+1}, x_k) \leq \phi(x^*, x_k) - \phi(x^*, x_{k+1}).$$

Since $\{\phi(x^*, x_k)\}$ is convergent, we have that

$$\lim_{n\to\infty} \phi(x_{k+1}, x_k) = 0. \tag{3.7}$$

It follows from the construction of C_k that $y_k = \Pi_{C_k} x_k$, and hence, we deduce from $x_{k+1} = \Pi_{C_k} x_k \in C_k$ that

$$\phi(y_k, x_k) \leq \phi(x_{k+1}, x_k). \tag{3.8}$$

Consequently, (3.7) and (3.8) imply that

$$\lim_{n\to\infty} \phi(y_k, x_k) = 0. \tag{3.9}$$

It follows from (3.9) and Lemma 2.4 that

$$\lim_{n\to\infty} \|y_k - x_k\| = 0, \tag{3.10}$$

which leads to $\{x_k\}$ and $\{y_k\}$ have the same weak accumulation points.

Step 3. Show that each weak accumulation point of the sequence $\{x_k\}$ is a solution of variational inequality (1.1).

Since J is uniformly norm-to-norm continuous on bounded sets, from (3.10), we have that

$$\lim_{k\to\infty} \|Jx_k - Jy_k\| = 0. \tag{3.11}$$

Let $\hat{x}$ be a weak accumulation point of $\{x_k\}$. We can extract a subsequence that weakly converges to $\hat{x}$. Without loss of generality, let us suppose that $x_k \rightharpoonup \hat{x}$ as $k \to \infty$. Then it follows from (3.10) that $y_k \rightharpoonup \hat{x}$ as $k \to \infty$. We next prove that $\hat{x} \in VI(C, A)$. Let $T \subset E \times E^*$ be an operator as (3.4). By Lemma 2.8, T is maximal monotone and $T^{-1}0 = VI(C, A)$. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $y_k \in C$, we get

$$\langle v - y_k, w - Av \rangle \geq 0. \tag{3.12}$$

On the other hand, from $y_k = \Pi_C(J^{-1}[J(x_k) - \lambda_k A(y_k)])$ and Lemma 2.3, we have $\langle v - y_k, Jy_k - J(x_k) + \lambda_k A(y_k)\rangle \geq 0$, and hence

$$\left\langle v - y_k, \frac{Jx_k - Jy_k}{\lambda_k} - A(y_k) \right\rangle \leq 0. \tag{3.13}$$

Then it holds from (3.12) and (3.13) that

$$\begin{aligned}\langle v-y_k, w\rangle &\geq \langle v-y_k, Av\rangle \\ &\geq \langle v-y_k, Av\rangle + \langle v-y_k, \tfrac{Jx_k-Jy_k}{\lambda_k} - A(y_k)\rangle \\ &= \langle v-y_k, Av-Ay_k\rangle + \langle v-y_k, \tfrac{Jx_k-Jy_k}{\lambda_k}\rangle \\ &\geq \langle v-y_k, \tfrac{Jx_k-Jy_k}{\lambda_k}\rangle.\end{aligned}$$

Since $y_k \rightharpoonup \hat{x}$, we have from (3.11) that $\langle v-\hat{x}, w\rangle \geq 0$. By the maximality of T, we obtain $\hat{x} \in T^{-1}0$ and hence $\hat{x} \in VI(C,A)$.

Step 4. Show that $x_k \rightharpoonup \hat{x}$, as $k \to \infty$ and $\hat{x} = \lim_{k\to\infty} \Pi_{VI(C,A)}(x_k)$.

Put $u_k = \Pi_{VI(C,A)}(x_k)$. It holds from (3.6) and Lemma 2.12 that $\{u_k\}$ is a Cauchy sequence. Since $VI(C,A)$ is closed, we have that $\{u_k\}$ converges strongly to $z \in VI(C,A)$. By the uniform smoothness of E, we also have that $\lim\limits_{n\to\infty} \|Ju_k - Jz\| = 0$. Now, we prove that $z = \hat{x}$. In fact, it follows from Lemma 2.3, $u_k = \Pi_{VI(C,A)}x_k$ and $\hat{x} \in VI(C,A)$ that $\langle \hat{x}-u_k, Ju_k - Jx_k\rangle \geq 0$. By the weakly sequential continuity of J, we infer that $\langle \hat{x}-z, Jz - J\hat{x}\rangle \geq 0$. Hence we have from the monotonicity of J that $\langle \hat{x}-z, Jz-J\hat{x}\rangle = 0$. Since E is strictly convex, we have that $z = \hat{x}$. Therefore, the sequence $\{x_k\}$ converges weakly to $\hat{x} = \lim\limits_{k\to\infty} \Pi_{VI(C,A)}(x_k)$. □

Remark 3.3. Theorem 3.2 improves Theorem 1.1 in the following senses.

(i) The assumptions (A1) and (A3) in Theorem 1.1 are removed, we only require that A is monotone and hemi-continuous.

(ii) Theorem 3.2 generalizes Theorem 1.1 from a $2-$uniformly convex, uniformly smooth Banach space to a uniformly convex, uniformly smooth Banach space.

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