Nonlinear Functional Analysis and Applications
Vol. 21, No. 1 (2016), pp. 145-170

http://nfaa.kyungnam.ac.kr/journal-nfaa

RANDOM HYBRID FIXED POINT THEOREMS FOR MONOTONE MAPPINGS IN PARTIALLY ORDERED POLISH SPACES WITH APPLICATIONS

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Abstract. We present some random fixed point theorems for partially contractive and continuous random mappings in a partially ordered Polish space and apply them to prove the existence and uniqueness theorems for an initial and a periodic boundary value problem of first order nonlinear random differential equations under weaker monotonic conditions. Finally, random differential inequalities and comparison principles are also established for the considered differential equations.

1. Introduction

Nonlinear random differential and integral equations is a topic of great interest in the area of stochastic or random analysis since long time. Several tools such as random fixed point theorems and random differential inequalities etc. are used to discuss various aspects of the random solutions. The most of the operator theoretical tools that exist in the literature are of existential nature and only random analogue of classical or deterministic Banach fixed point theorem provides an algorithm in terms of a sequence of successive iterations that converges to the unique random solution of the nonlinear random problem under consideration. This construction requires that the nonlinearity involved in the nonlinear equations to satisfy a Lipschitz condition which is very strong condition. In this work we replace so called Lipschitz condition

[0]Received August 1, 2015. Revised October 13, 2015.
[0]2010 Mathematics Subject Classification: 45G10, 45M99, 47H10.
[0]Keywords: Polish space, partially contraction mappings, random fixed point theorem, initial value problem, periodic boundary value problem, existence theorem.

by a weaker one-sided or partial Lipschitz condition and show that same algorithm still works for finding the random or indeterministic solutions for the nonlinear random equations. Moreover, we show that the convergence of the algorithm is monotonic and thus we get some additional information of the computation of the algorithm. In a nutshell, we prove some random fixed point theorems for contraction mappings in a partially ordered polish space and apply them to nonlinear random initial and boundary value problems of first order random differential equations for proving the existence as well as uniqueness results under certain monotonic conditions.

2. Preliminaries

Throughout the rest of the paper, let $(X, \preceq)$ denote a partially ordered set. Let there exist a metric d on X such that (X, d) is a Polish space, i.e., a complete, separable metric space. Let $(\Omega, \mathcal{A})$ denote a measurable space, where $\mathcal{A}$ is a σ-algebra of subsets of Ω. Denote by β_X the σ-algebra of all Borel subsets of X. A function $x : \Omega \to X$ is said to be measurable if

$$x^{-1}(B) = \{\omega \in \Omega \mid x(\omega) \in B\} \in \mathcal{A} \tag{2.1}$$

for all $B \in \beta_X$.

A mapping $\mathcal{T} : \Omega \times X \to X$ is called a random mapping if $\mathcal{T}(\cdot, x)$ is measurable for each $x \in X$. An image of a point $(\omega, x) \in \Omega \times X$ under the random mapping $\mathcal{T}$ is denoted by $\mathcal{T}(\omega, x)$ or simply $\mathcal{T}(\omega)x$. A random mapping $\mathcal{T}(\omega)$ is said to be continuous on X into itself if the mapping $\mathcal{T}(\omega, \cdot)$ is continuous on$X$ for each $\omega \in \Omega$. A measurable function $\xi : \Omega \to X$ is called a random fixed point of the random mapping $\mathcal{T}(\omega)$ if $\mathcal{T}(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$. The study of random fixed point theorem is initiated by Spacek [16] and Hans [12], however the article that published by Bharucha-Reid [1] is responsible for the multitude development of random fixed point theory and applications.

The following well-known result is crucial in the development of random fixed point theory in Polish spaces.

Lemma 2.1. *Let X be a Polish space. Then following statements hold in X.*

(a) *If $\{x_n(\omega)\}$ is a sequence of random variables converging to $x(\omega)$ for all $\omega \in \Omega$, then $x(\omega)$ is also a random variable.*
(b) *If $\mathcal{T}(\omega, \cdot)$ is continuous for each $\omega \in \Omega$ and $x : \Omega \to X$ is a random variable, then $\mathcal{T}(\omega)x$ is also a random variable.*

There is a considerable literature on random fixed point theory and applications to random differential and integral equations. The random fixed point theory using mixed arguments form algebra, analysis and topology is also developed under the title hybrid random fixed point theory and a nice treatment of the topic appears in Dhage [3, 4, 6], Dhage and Dhage [8] and Dhage *et. al.,* [11]. We mention that the hypothesis of continuity of the random mappings is inevitable in all these consideration. Some algebraic random fixed point theorems for the monotone random mappings have been discussed in Dhage [3, 4, 5, 7, 9, 10] in ordered separable Banach spaces. The order relation in a separable Banach space is defined through the order cones and the random fixed point theorems are deduced from different properties of the order cones which are further applied to nonlinear random integral equations for proving the maximal and minimal random solutions.

The purpose of the present paper is to obtain some hybrid random fixed point theorems for partially contraction and continuous random mappings in a partially ordered Polish space in a way which is different from that given Dhage [3, 4]. The applications of the abstract results are also discussed in relation to initial and boundary value problems of ordinary first order random differential equations. We give our main random fixed results in the following sections.

3. Partially contraction random maps

In this section we prove our main random operator theoretic results of this paper. The following definition is fundamental importance in the study of random fixed point theory in partially ordered sets and subsequently used in the rest of the paper.

Definition 3.1. A random mapping $\mathcal{T} : \Omega \times X \to X$ is said to be monotone nondecreasing if for any $x, y \in X$, $x \preceq y$ implies $\mathcal{T}(\omega)x \preceq \mathcal{T}(\omega)y$ for all $\omega \in \Omega$. Similarly, $\mathcal{T}(\omega)$ is called monotone nonincreasing if the reversed inequality is satisfied.

3.1. **Global random fixed point theory.** Firstly, we obtain the existence of random fixed point theorems in the whole of X. Our first main result along this line is as follows. the rest of the paper.

Theorem 3.2. *Suppose that $(\Omega, \mathcal{A})$ is a measurable space and $(X, \preceq, d)$ is a partially ordered Polish space. Let $\mathcal{T} : \Omega \times X \to X$ be a continuous and nondecreasing random operator satisfying for each $\omega \in \Omega$,*

$$d(\mathcal{T}(\omega)x, \mathcal{T}^2(\omega)x) \leq k(\omega)d(x, \mathcal{T}(\omega)x) \tag{3.1}$$

for all $x \in X$, $x \preceq \mathcal{T}(\omega)x$, where $k : \Omega \to \mathbb{R}$ is a measurable function such that $0 \leq k(\omega) < 1$ for all $\omega \in \Omega$. If there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \preceq \mathcal{T}(\omega)x_0$ for all $\omega \in \Omega$, then $\mathcal{T}(\omega)$ has a random fixed point $\xi(\omega)$

and the sequence $\{\mathcal{T}^n(\omega)x_0\}$ *of successive iterations converges monotonically to* $\xi(\omega)$.

Proof. Define a sequence $\{x_n\}$ of successive iterations of $\mathcal{T}(\omega)$ at x_0 as

$$x_n = \mathcal{T}(\omega)x_{n-1}, \ \ n \in \mathbb{N}. \tag{3.2}$$

Clearly, $\{x_n\}$ is a sequence of measurable functions on Ω into X. Since $\mathcal{T}(\omega)$ is nondecreasing random operator, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots . \tag{3.3}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $\xi = x_n$ is a random fixed point of $\mathcal{T}(\omega)$. Assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ one has

$$\begin{aligned} d(x_n, x_{n+1}) &= d(\mathcal{T}(\omega)x_{n-1}, \mathcal{T}(\omega)x_n) \\ &\leq k(\omega)\, d(x_{n-1}, x_n) \end{aligned} \tag{3.4}$$

for all $n = 1, 2, \ldots$. Proceeding in this way, by induction,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq k(\omega)\, d(x_{n-1}, x_n) \\ &\ \vdots \\ &\leq k^n(\omega)\, d(x_0, x_1) \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. Hence for $m > n$, one has

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + \cdots d(x_{m-1}, x_m) \\ &\leq [k^n(\omega) + k^{n+1}(\omega) + \cdots + +k^{m-1}(\omega)]\, d(x_0, x_1) \\ &\leq \frac{k^n(\omega)}{1 - k(\omega)} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.6}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. The metric space (X, d) being complete, there is a measurable function $\xi : \Omega \to X$ such that $\lim\limits_{n\to\infty} x_n = \xi$. By continuity of $\mathcal{T}(\omega)$ one has

$$\mathcal{T}(\omega)\xi = \mathcal{T}(\omega)\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \mathcal{T}(\omega)x_n = \xi.$$

This proves that $\mathcal{T}(\omega)$ has a random fixed point. This completes the proof. $\square$

Similarly, we have the following random fixed point result for monotone nondecreasing mappings satisfying partially contraction condition.

Theorem 3.3. *Suppose that* $(\Omega, \mathcal{A})$ *is a measurable space and* $(X, \preceq, d)$ *is a partially ordered Polish space. Let* $\mathcal{T} : \Omega \times X \to X$ *be a continuous and* *nondecreasing random operator satisfying the condition of partial linear contraction* (3.1)*. If there exists a measurable function* $x_0 : \Omega \to X$ *such that* $x_0 \succeq \mathcal{T}(\omega)x_0$ *for all* $\omega \in \Omega$*, then* $\mathcal{T}(\omega)$ *has a random fixed point* $\xi(\omega)$ *and the sequence* $\{\mathcal{T}^n(\omega)x_0\}$ *of successive iterations converges monotonically to* $\xi(\omega)$*.*

Proof. The proof is similar to Theorem 3.1 and hence we omit the details. □

Theorem 3.4. *Suppose that* $(\Omega, \mathcal{A})$ *is a measurable space and* $(X, \preceq, d)$ *is a partially ordered Polish space. Let* $\mathcal{T} : \Omega \times X \to X$ *be a continuous and nondecreasing random operator satisfying for each* $\omega \in \Omega$*,*

$$d(\mathcal{T}(\omega)x, \mathcal{T}(\omega)y) \leq k(\omega)d(x, y) \tag{3.7}$$

for all comparable elements $x, y \in X$*, where* $k : \Omega \to \mathbb{R}$ *is a measurable function such that* $0 \leq k(\omega) < 1$ *for all* $\omega \in \Omega$*. If there exists a measurable function* $x_0 : \Omega \to X$ *such that* $x_0 \preceq \mathcal{T}(\omega)x_0$ *for all* $\omega \in \Omega$*, then* $\mathcal{T}(\omega)$ *has a random fixed point* $\xi(\omega)$ *and the sequence* $\{\mathcal{T}^n(\omega)x_0\}$ *of successive iterations converges monotonically to* $\xi(\omega)$*. Furthermore, if every pair of elements in* X *has a lower or an upper bound, then the random fixed point* $\xi(\omega)$ *is unique.*

Proof. Take $y = \mathcal{T}(\omega)x$ in the above contraction condition (3.7). Then we obtain the inequality (3.1). Hence, by Theorem 3.2, $\mathcal{T}(\omega)$ has a random fixed point ξ. To prove the uniqueness, let ξ^* be another random fixed point of the random mapping $\mathcal{T}(\omega)$ in X. Then by hypothesis, there exists an element $z \in X$ which is a lower or an upper bound for ξ and ξ^*. Then

$$\begin{aligned} d(\xi, \xi^*) &= d(\mathcal{T}^n(\omega)\xi, \mathcal{T}^n(\omega)\xi^*) \\ &\leq d(\mathcal{T}^n(\omega)\xi, \mathcal{T}^n(\omega)z) + d(\mathcal{T}^n(\omega)z, \mathcal{T}^n(\omega)\xi^*) \\ &\leq k^n(\omega)[d(\xi, z) + d(z, \xi^*)] \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $\xi = \xi^*$ and the proof of the theorem is complete. □

Theorem 3.5. *Suppose that* $(\Omega, \mathcal{A})$ *is a measurable space and* $(X, \preceq, d)$ *is a partially ordered Polish space. Let* $\mathcal{T} : \Omega \times X \to X$ *be a continuous and nondecreasing random operator satisfying the partial contraction condition* (3.7)*. If there exists a measurable function* $x_0 : \Omega \to X$ *such that* $x_0 \succeq \mathcal{T}(\omega)x_0$ *for all* $\omega \in \Omega$*, then* $\mathcal{T}(\omega)$ *has a random fixed point* $\xi(\omega)$ *and the sequence* $\{\mathcal{T}^n(\omega)x_0\}$ *of successive iterations converges monotonically to* $\xi(\omega)$*. Furthermore, if every pair of elements in* X *has a lower or an upper bound, then the random fixed point* $\xi(\omega)$ *is unique.*

Proof. The proof is similar to Theorem 3.4 and so we omit the details. □

The random operator $\mathcal{T}(\omega)$ satisfying the partial contraction condition (3.8) is called a partial linear contraction or is said to satisfy a condition of partially linear contraction on X. Sometimes it is possible that the random operator $\mathcal{T}(\omega)$ is not a partial linear contraction, but some iterates $\mathcal{T}^p(\omega)$ of it is a partial linear contraction on X. Then, in theses circumstances we have the following random fixed point theorem for the partial linear contraction random mappings in a partially ordered Polish space.

Theorem 3.6. *Suppose that $(\Omega, \mathcal{A})$ is a measurable space and $(X, \preceq, d)$ is a partially ordered Polish space. Let $\mathcal{T} : \Omega \times X \to X$ be a continuous and nondecreasing random operator and let there be a positive integer p satisfying for each $\omega \in \Omega$,*

$$d(\mathcal{T}^p(\omega)x, \mathcal{T}^p(\omega)y) \leq k(\omega)d(x, y) \tag{3.8}$$

for all comparable elements $x, y \in X$, where $k : \Omega \to \mathbb{R}$ is a measurable function such that $0 \leq k(\omega) < 1$ for all $\omega \in \Omega$. If there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \preceq \mathcal{T}(\omega)x_0$ for all $\omega \in \Omega$ and every pair of elements in X has a lower or an upper bound, then $\mathcal{T}(\omega)$ has a unique random fixed point $\xi(\omega)$ and the sequence $\{\mathcal{T}^n(\omega)x_0\}$ of successive iterations converges monotonically to $\xi(\omega)$.

Proof. Set $f(\omega) = \mathcal{T}^p(\omega)$. Then $f(\omega)$ satisfies all the conditions of Theorem 3.5. Indeed, if the random operator $\mathcal{T}(\omega)$ is continuous, then $f(\omega)$ is also continuous random operator from $\Omega \times X$ into X. Again, by monotonicity of $\mathcal{T}(\omega)$, we obtain

$$x_0 \preceq \mathcal{T}(\omega)x_0 \preceq \mathcal{T}^2(\omega)x_0 \preceq \cdots \preceq \mathcal{T}^p(\omega)x_0 = f(\omega)x_0.$$

Now, an application of Theorem 3.5 yields that $f(\omega)$ and consequently $\mathcal{T}(\omega)$ has a unique random fixed point ξ in X such that $\mathcal{T}(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$. This further gives

$$\mathcal{T}^p(\omega)(\mathcal{T}(\omega)\xi(\omega)) = \mathcal{T}(\omega)\xi(\omega)$$

which shows that $\mathcal{T}(\omega)\xi(\omega)$ is again a random fixed point of $\mathcal{T}^p(\omega)$. By uniqueness of ξ, we obtain $\mathcal{T}(\omega)\xi(\omega) = \xi(\omega)$. Moreover, the sequence $\{\mathcal{T}^n(\omega)x_0\}$ of successive iterations converges to $\xi(\omega)$. This completes the proof. □

Similarly, we have the following global random fixed point theorem for the random mappings satisfying the reverse inequality for the point x_0 in X.

Theorem 3.7. *Suppose that $(\Omega, \mathcal{A})$ is a measurable space and $(X, \preceq, d)$ is a partially ordered Polish space. Let $\mathcal{T} : \Omega \times X \to X$ be a continuous and nondecreasing random operator and let there be a positive integer p satisfying the partially contraction condition* (3.8). *If there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \succeq \mathcal{T}(\omega)x_0$ for all $\omega \in \Omega$ and every pair of* *elements in X has a lower or an upper bound, then $\mathcal{T}(\omega)$ has a unique random fixed point $\xi(\omega)$ and the sequence $\{\mathcal{T}^n(\omega)x_0\}$ of successive iterations converges monotonically to $\xi(\omega)$.*

3.2. **Local random fixed point theory.** Given a fixed point $x_0 \in X$ and given a real number $r > 0$, we define a closed ball $B[x_0,,r]$ in X centered at x_0 of radius r by

$$B[x_0,,r] = \{x \in X \mid d(x_0,x) \le r\}. \tag{3.9}$$

Below we prove some local random fixed point theorems for the partial linear contraction random operators in a partially ordered Polish space.

Theorem 3.8. *Let $(\Omega,\mathcal{A})$ be a measurable space and let $B[x_0,r]$ be a closed ball in a partially ordered Polish space $(X,\preceq,d)$. Let $\mathcal{T}:\Omega\times X\to X$ be a continuous and nondecreasing random mapping satisfying the inequality* (3.1). *If there exists a measurable function $x_0:\Omega\to X$ such that $x_0 \preceq \mathcal{T}(\omega)x_0$ satisfying*

$$d(x_0,\mathcal{T}(\omega)x_0) \le [1-k(\omega)]r \tag{3.10}$$

for all $\omega\in\Omega$. Then $\mathcal{T}(\omega)$ has a random fixed point $\xi(\omega)\in B[x_0,r]$.

Proof. Define a sequence $\{x_n\}$ of measurable functions from Ω into X by (3.2). Now proceeding as in the proof of Theorem 3.7 it can be proved that

$$\begin{aligned} d(x_n,x_{n+1}) &\le k(\omega)\,d(x_{n-1},x_n)\\ &\vdots\\ &\le k^n(\omega)\,d(x_0,x_1)\end{aligned}$$

for all $n\in\mathbb{N}$. From the above inequality it follows that $\{x_n\}$ is a Cauchy sequence in X. We show that $\{x_n\}\subset B[x_0,r]$. Now for any $n\in\mathbb{N}$, we have

$$\begin{aligned} d(x_0,x_n) &\le \sum_{i=0}^{n} d(x_i,x_{i+1}) \le \sum_{i=0}^{n} k^i(\omega)\,d(x_0,x_1)\\ &\le \left(\sum_{i=0}^{n} k^i(\omega)\right) d(x_0,x_1) \le \left[\frac{1-k^n(\omega)}{1-k(\omega)}\right][1-k(\omega)]r\\ &\le [1-k^n(\omega)]r \le r. \end{aligned} \tag{3.11}$$

This shows that $\{x_n\}\subset B[x_0,r]$. Since $B[x_0,r]$ is a closed subset of a complete metric space, it is complete and so $\{x_n\}$ converges to a measurable function ξ in $B[x_0,r]$. The rest of the proof is similar to Theorem 3.7 and hence we omit the details. □

Theorem 3.9. *Let $(\Omega, \mathcal{A})$ be a measurable space and let $B[x_0, r]$ be a closed ball in a partially ordered Polish space $(X, \preceq, d)$. Let $\mathcal{T} : \Omega \times X \to X$ be a continuous and nondecreasing random mapping satisfying the inequality* (3.1). *If there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \succeq \mathcal{T}(\omega)x_0$ satisfying* (3.10)*, then $\mathcal{T}(\omega)$ has a random fixed point $\xi \in B[x_0, r]$.*

Proof. The proof is similar to Theorem 3.8 and we omit the details. □

Theorem 3.10. *Let $(\Omega, \mathcal{A})$ be a measurable space and let $B[x_0, r]$ be a closed ball in a partially ordered Polish space $(X, \preceq, d)$. Let $\mathcal{T} : \Omega \times X \to X$ be a continuous and nondecreasing random operator satisfying* (3.7). *If there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \preceq \mathcal{T}(\omega)x_0$ or $x_0 \succeq \mathcal{T}(\omega)x_0$ satisfying* (3.11) *for all $\omega \in \Omega$, then $\mathcal{T}(\omega)$ has a random fixed point $\xi(\omega)$ in $B[x_0, r]$ and the sequence $\{Q^n(\omega)x_0\}$ of successive iterations converges monotonically to $\xi(\omega)$. Furthermore, if every pair of elements in X has a lower or an upper bound, then the random fixed point $\xi(\omega)$ is unique.*

Proof. Take $y = \mathcal{T}(\omega)x$ in (3.7). The the inequality (3.1) is satisfied. Now an application Theorem 3.8 yields that $Q(\omega)$ has a random fixed point $\xi(\omega)$ in the closed ball $B[0, r]$ and the sequence $\{\mathcal{T}^n(\omega)x_0\}$ of successive iterations converges to $\xi(\omega)$. Further, since every pair of elements in X has a lower or an upper bound, then it can be proved as in the proof of Theorem 3.4 that the random fixed point $\xi(\omega)$ is unique. This completes the proof. □

Theorem 3.11. *Suppose that $(\Omega, \mathcal{A})$ be a measurable space and let $B[x_0, r]$ be a closed ball in a partially ordered Polish space $(X, \preceq, d)$. Let $Q : \Omega \times X \to X$ be a continuous and nondecreasing random mapping satisfying the inequality* (3.7). *Suppose also that there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \preceq \mathcal{T}(\omega)x_0$ satisfying*

$$d(x_0, \mathcal{T}^p(\omega)x_0) \leq [1 - k(\omega)]r \tag{3.12}$$

for all $\omega \in \Omega$, where p is a positive integer. If every pair of elements in X has a lower or an upper bound, then $\mathcal{T}(\omega)$ has a unique random fixed point $\xi \in B[x_0, r]$.

Proof. Let $f(\omega) = Q^p(\omega)$ for $\omega \in \Omega$. Then by an application of Theorem 3.10, $f(\omega)$ has a unique random fixed point $\xi(\omega)$ in $B[x_0, r]$. Now, by the definition of $f(\omega)$, we obtain

$$\mathcal{T}^p(\omega)(\mathcal{T}(\omega)\xi(\omega)) = \mathcal{T}(\omega)\xi(\omega)$$

for all $\omega \in \Omega$, which shows that $\mathcal{T}(\omega)\xi(\omega)$ is again a random fixed point of $Q^p(\omega)$ in $B[x_0, r]$. By uniqueness of ξ, we obtain $\mathcal{T}(\omega)\xi(\omega) = \xi(\omega)$ and the proof of the theorem is complete. □

Remark 3.12. The conclusion of above Theorem 3.11 is also true if we replace the condition $x_0 \preceq \mathcal{T}(\omega)x_0$ with $x_0 \succeq \mathcal{T}(\omega)x_0$.

In the following section we apply the abstract results of this section to nonlinear initial and periodic boundary value problems of random differential equations for proving the existence as well as uniqueness results under certain monotonic conditions.

4. Random initial value problems

Given a measurable space $(\Omega, \mathcal{A})$ and given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, the set of real numbers, consider the initial value problem (in short IVP) of nonlinear first order random differential equations (in short RDE),

$$\begin{cases} x^{'}(t,\omega) = f(t, x(t,\omega), \omega), \; t \in J, \\ x(0,\omega) = q(\omega), \end{cases} \tag{4.1}$$

for all $\omega \in \Omega$, where $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is Carathéodory and $q : \Omega \to \mathbb{R}$ is a measurable function.

By the *random solution* of the RDE (4.1) we mean a measurable function $x : \Omega \to C(J, \mathbb{R})$ that satisfies the equations in (4.1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The IVP of RDE (4.1) is well-known and extensively discussed in the literature. See Ladde and Lakshmikantham [13] and the references cited therein. The existence and uniqueness theorems for the RDE (4.1) are obtained under compactness and Lipschitz type conditions. The indeterministic fixed point theorems in polish spaces such as random versions of Schauder and Banach fixed point theorems have been employed for proving such existence and uniqueness results respectively. In the present study we discuss the existence as well as uniqueness theorem for the considered random differential equation under weaker Lipschitz condition, namely one-sided or partial Lipschitz condition. The results of this type are new to the literature and therefore, immensely contribute to the theory of random differential equations.

We seek the existence of random solutions of the RDE (4.1) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. Define a standard supremum norm $\|.\|$ and the the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{4.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \;\text{ for all }\; t \in J. \tag{4.3}$$

Clearly $C(J,\mathbb{R})$ is a Banach space with this supremum norm. Again, $C(J,\mathbb{R})$ is a partially ordered sset w.r.t. the order relation $\leq$ in it.

Thus the Banach space $C(J,\mathbb{R})$ together with this order relation $\leq$ becomes a partially ordered Banach space. Further, it is known that $C(J,\mathbb{R})$ is a separable and hence a Polish space. Moreover, $C(J,\mathbb{R})$ is a lattice with respect to above order relation defined in it which is clear from the real variable operations in it. So every pair of elements in $C(J,\mathbb{R})$ has a lower as well as an upper bound.

Before stating the needed hypotheses, we give a useful definition.

Definition 4.1. A mapping $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be Carathéodory if

(i) the map $(t,x) \mapsto f(t,x,\omega)$ is jointly continuous for each $\omega \in \Omega$, and
(ii) the map $\omega \mapsto f(t,x,\omega)$ is measurable for each $t \in J$ and $x \in \mathbb{R}$.

Definition 4.2. A Carathéodory function $f(t,x,\omega)$ is called $\mathcal{C}$-Carathéodory if there exists a continuous function $h : J \to \mathbb{R}$ such that

$$|f(t,x,\omega)| \leq h(t)$$

for all $x \in \mathbb{R}$ and $\omega \in \Omega$.

Definition 4.3. A Carathéodory function $f(t,x,\omega)$ is called L^1-Carathéodory if there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|f(t,x,\omega)| \leq h(t)$$

for all $x \in \mathbb{R}$ and $\omega \in \Omega$.

Note that every $\mathcal{C}$-Carathéodory function is L^1-Carathéodory and every L^1-Carathéodory function is Carathéodory but the converse may not be true.

Definition 4.4. A measurable function $\alpha : \Omega \to \mathbb{R}$ is called a lower random solution of the IVP of RDE (4.1) if

$$\begin{cases} \alpha^{'}(t,\omega) \leq f(t,\alpha(t,\omega),\omega), \\ \alpha(0,\omega) \leq q(\omega), \end{cases}$$

is satisfied for all $t \in J$ and $\omega \in \Omega$. Similarly, a measurable function $\beta : \Omega \to \mathbb{R}$ is called an upper random solution of the IVP of RDE (4.1) if

$$\begin{cases} \beta^{'}(t,\omega) \geq f(t,\beta(t,\omega),\omega), \\ \beta(0,\omega) \geq q(\omega), \end{cases}$$

is satisfied for all $t \in J$ and $\omega \in \Omega$.

Given a measurable function $x : \Omega \to C(J, \mathbb{R})$, consider the IVP of RDE,

$$\begin{cases} x^{'}(t,\omega) + \lambda(\omega)x(t,\omega) = \tilde{f}(t, x(t,\omega), \omega), \ t \in J, \\ x(0,\omega) = q(\omega), \end{cases} \tag{4.4}$$

where $\lambda : \Omega \to \mathbb{R}_+$ is a measurable function and the function $\tilde{f} : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is defined as

$$\tilde{f}(t, x, \omega) = f(t, x, \omega) + \lambda(\omega)x. \tag{4.5}$$

Remark 4.5. We remark that if the function f is Carathéodory, then the function $\tilde{f}$ is also Carathéodory on $J \times \mathbb{R}$. A random solution to the RDE (4.1) is also a random solution to the RDE (4.5) defined on J and vice-versa.

We consider the following set of hypotheses in what follows

(H_0) The function $q : \Omega \to \mathbb{R}$ is measurable.
(H_1) f is Carathéodory.
(H_2) The function $\tilde{f}$ is bounded on $J \times \mathbb{R} \times \Omega$ with bound M.
(H_3) There exist a measurable functions $\lambda, \mu : \Omega \to \mathbb{R}$ satisfying for each $\omega \in \Omega$,

$$0 \le f(t, x, \omega) + \lambda(\omega)x - f(t, y, \omega) - \lambda(\omega)y| \le \mu(\omega)(x - y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$ with $x \ge y$. Moreover, $\mu(\omega) < \lambda(\omega)$ on Ω.
(H_4) The RDE (4.1) has a lower random solution u defined on J.

Remark 4.6. From the Hypothesis (H_1) it follows that the function $t \mapsto f(t, x, \omega)$ is measurable for each $x \in \mathbb{R}$ and $\omega \in \Omega$ which is further integrable if the hypothesis (H_2) holds.

The following result which transforms the RDE (4.1) into an equivalent random integral equation (in short RIE) is frequently used in the subsequent part of the paper.

Lemma 4.7. *Assume that the hypothesis* (H_0) *holds. If* $h : \Omega \to C(J, \mathbb{R})$ *is measurable and bounded, then a measurable function* $x : \Omega \to C(J, \mathbb{R})$ *is a solution of the of the IVP of RDE*

$$\begin{cases} x^{'}(t,\omega) + \lambda(\omega)x(t,\omega) = h(t,\omega), \ \ t \in J, \\ x(0,\omega) = q(\omega), \end{cases} \tag{4.6}$$

if and only if it is a solution of the random integral equation

$$x(t,\omega) = q(\omega)e^{-\lambda(\omega)t} + e^{-\lambda(\omega)t} \int_0^t e^{\lambda(\omega)s} h(s,\omega)\, ds, \ \ t \in J, \tag{4.7}$$

for all $\omega \in \Omega$.

Proof. Since $\omega \mapsto h(t,\omega)$ is measurable, by Lemma 2.1 the map $s \mapsto h(s,\omega)$ is measurable which further in view of boundedness of h implies that it is integrable on J. Multiplying both sides by integrating factor $e^{\lambda(\omega)t}$ and applying integration from 0 to t, we obtain (4.7).

Conversely, suppose that x is a solution of the random integral equation (4.7). Then by direct differentiation of (4.7) with respect to t, we obtain RDE (4.6). □

Now we are a position to prove the main existence results for the RDE (4.1) on J.

Theorem 4.8. *Assume that hypotheses* (H_0)-(H_4) *hold. Then the RDE* (4.1) *has a unique random solution* ξ^* *defined on* J *and the sequences* $\{x_n\}$ *of successive approximations defined by*

$$x_0 = u, \quad x_n(t,\omega) = q(\omega)e^{-\lambda(\omega)t} + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x_{n-1}(s,\omega),\omega)ds, \tag{4.8}$$

for $t \in J$, *converges monotonically to* ξ^*.

Proof. By Lemma 4.1, the RDE (4.6) is equivalent to the RIE

$$\begin{aligned} x(t,\omega) &= q(\omega)e^{-\lambda(\omega)t} \\ &\quad + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds\ ,\ t \in J, \end{aligned} \tag{4.9}$$

for all $\omega \in \Omega$. Set $E = C(J,\mathbb{R})$. Define an operator $\mathcal{Q}$ on $\Omega \times E$ by

$$\begin{aligned} \mathcal{Q}(\omega)x(t,\omega) &= q(\omega)e^{-\lambda(\omega)t} \\ &\quad + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds\ ,\ t \in J. \end{aligned} \tag{4.10}$$

Since the function

$$t \mapsto \int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds$$

is continuous for all $\omega \in \Omega$, we have that $\mathcal{Q}(\omega)x \in E$ for all $\omega \in \Omega$. Further the integral $\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds$ is the limit of a finite sum of measurable functions, so the function

$$\omega \mapsto \int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds$$

is measurable. Again the sum of two measurable functions is again measurable, so that the function

$$\omega \mapsto q(\omega)e^{-\lambda(\omega)t} + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds$$

is measurable for each $t \in J$. As a result $\mathcal{Q}$ defines a random operator $\mathcal{Q} : \Omega \times E \to E$. Let $\omega \in \Omega$ be fixed and let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (H_2),

$$\begin{aligned}
\mathcal{Q}(\omega)x(t,\omega) &= q(\omega)e^{-\lambda(\omega)t} + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds \\
&= q(\omega)e^{-\lambda(\omega)t} \\
&\quad + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}[f(s, x(s,\omega),\omega) + \lambda(\omega)x(s)]ds \\
&\leq q(\omega)e^{-\lambda(\omega)t} \\
&\quad + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}[f(s, x(s,\omega),\omega) + \lambda(\omega)x(s)]ds \\
&= \mathcal{Q}(\omega)y(t,\omega)
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. In consequence, $\mathcal{Q}(\omega)x \leq \mathcal{Q}(\omega)y$ for all $\omega \in \Omega$ and so, $\mathcal{Q}(\omega)$ is a non-decreasing random operator from $\Omega \times E$ into E.

Next, by hypothesis (H_4), the IVP (4.1) has a lower random solution $u \in E$. Then,

$$\begin{cases} u'(t,\omega) \leq f(t, u(t,\omega),\omega), \\ u(0,\omega) \leq q(\omega), \end{cases}$$

which implies that

$$\begin{cases} u'(t,\omega) + \lambda(\omega)u(t,\omega) \leq f(t, u(t,\omega),\omega) + \lambda(\omega)u(t,\omega), \\ \qquad\qquad u(0,\omega) \leq q(\omega), \end{cases} \tag{4.11}$$

for all $t \in J$ and $\omega \in \Omega$.

Multiplying first equation in (4.11) by $e^{\lambda(\omega)t}$, we obtain

$$\left(e^{\lambda(\omega)t}u(t,\omega)\right)' \leq e^{\lambda(\omega)t}\tilde{f}(t, u(t,\omega),\omega)$$

which on integration from 0 to t gives

$$e^{\lambda(\omega)t}u(t,\omega) \leq q(\omega) + \int_0^t e^{\lambda(\omega)s}\tilde{f}(s, u(s,\omega),\omega)\,ds$$

or

$$\begin{aligned}u(t,\omega) &\le q(\omega)e^{-\lambda(\omega)t} + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s,u(s,\omega),\omega)\,ds\\ &= \mathcal{Q}(\omega)u(t,\omega)\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. Hence, $u \le \mathcal{Q}(\omega)u$ for all $\omega \in \Omega$.

Next, we show that $\mathcal{Q}(\omega)$ is a contraction operator on E. Let $\omega \in \Omega$ be fixed. Then, by hypothesis (H_2),

$$\begin{aligned}
&d(\mathcal{Q}(\omega)x, \mathcal{Q}(\omega)y)\\
&= \|\mathcal{Q}(\omega)x - \mathcal{Q}(\omega)y\| = \sup_{t\in J}|\mathcal{Q}(\omega)x(t) - \mathcal{Q}(\omega)y(t)|\\
&\le \sup_{t\in J} e^{-\lambda(\omega)t}\bigg|\int_0^t e^{\lambda(\omega)s}\big[f(t,x(s,\omega),\omega) + \lambda(\omega)x(s,\omega)\big]ds\\
&\qquad - \int_0^t e^{\lambda(\omega)s}\big[f(s,y(s,\omega),\omega) + \lambda(\omega)y(s,\omega)\big]ds\bigg|\\
&\le \sup_{t\in J} e^{-\lambda(\omega)t}\left|\int_0^t \mu(\omega)e^{\lambda(\omega)s}[x(s,\omega) - y(s,\omega)]\,ds\right|\\
&\le \sup_{t\in J} e^{-\lambda(\omega)t}\left|\int_0^t e^{\lambda(\omega)s}\lambda(\omega)\|x(\omega) - y(\omega)\|\,ds\right|\\
&= \sup_{t\in J} e^{-\lambda(\omega)t}\left|\int_0^t \frac{d}{ds}[e^{\lambda(\omega)s}]\,\|x(\omega) - y(\omega)\|\,ds\right|\\
&= \sup_{t\in J} e^{-\lambda(\omega)t}\left[e^{\lambda(\omega)s}\right]_0^t\,\|x(\omega) - y(\omega)\|\\
&= \sup_{t\in J}\left[1 - e^{-\lambda(\omega)t}\right]\|x(\omega) - y(\omega)\| = \sup_{t\in J}\left[1 - \frac{1}{e^{\lambda(\omega)t}}\right]\|x(\omega) - y(\omega)\|\\
&\le \left[1 - \frac{1}{e^{\lambda(\omega)T}}\right]\,\|x - y\| = k(\omega)\,d(x(\omega)\,,\,y(\omega))
\end{aligned}$$

for all $\omega \in \Omega$, where $k(\omega) = 1 - \frac{1}{e^{\lambda(\omega)T}} < 1$ for all $\omega \in \Omega$. Thus $\mathcal{Q}(\omega)$ is a partially linear contraction random operator on E into itself. Hence an application of random fixed point theorem formulated in Theorem 3.4 yields that $\mathcal{Q}(\omega)$ has a unique random fixed point ξ^* which corresponds to the unique random solution of the IVP of RDE (4.1) defined on J. The sequence $\{x_n\}$ defined by (4.8) is monotonic nondecreasing and converges to ξ^* on J. This completes the proof. □

Remark 4.9. The conclusion of Theorem 4.8 also remains true if we replace the hypothesis (H_4) with the following one:

(H_5) The PBVP (4.1) has an upper random solution v defined on J.

5. Random periodic boundary value problems

Given a measurable space $(\Omega, \mathcal{A})$ and given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$ for some $T > 0$, $\mathbb{R}$ the set of real numbers, consider the periodic boundary value problem (in short PBVP) of first order nonlinear random differential equation (in short RDE) or random PBVP of first order ordinary differential equation with periodic boundary value conditions,

$$\begin{cases} x'(t,\omega) = f(t, x(t,\omega), \omega), \ t \in J, \\ x(0,\omega) = x(T,\omega), \end{cases} \tag{5.1}$$

for all $\omega \in \Omega$, where $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a Carathéodory function.

A *random solution* to the random PBVP (5.1) is a measurable function $x : \Omega \to C(J, \mathbb{R})$ that satisfies the equations in (5.1).

The deterministic PBVP of first order nonlinear ordinary differential equations with periodic boundary value condition,

$$\begin{cases} x^{'} = f(t, x), \ t \in J, \\ x(0) = x(T), \end{cases} \tag{5.2}$$

has been discussed in the literature for different aspects of the solutions. The existence and uniqueness theorem for periodic PBVP (5.2) is discussed under one sided Lipschitz condition on the nonlinearity f together with the existence of a lower solution as well an upper random solution.

In this paper, we discuss existence and uniqueness of the random solution under Carathéodory and one-sided or partially Lipschitz condition together with the existence of a lower or an upper random solution of the PBVP of random differential equations (5.1). We need the following definition in what follows.

Definition 5.1. A measurable function $\alpha : \Omega \to C(J, \mathbb{R})$ is called a lower random solution for the PBVP of RDE (5.1) if

$$\begin{cases} \alpha^{'}(t,\omega) \le f(t, \alpha(t,\omega), \omega), \ \ t \in J, \\ \alpha(0,\omega) \le \alpha(T,\omega), \end{cases} \tag{5.3}$$

for all $\omega \in \Omega$. Similarly an upper random solution is a measurable function $\beta : \Omega \to C(J, \mathbb{R})$ satisfying for each $\omega \in \Omega$,

$$\begin{cases} \beta^{'}(t,\omega) \geq f(t,\beta(t,\omega),\omega), \quad t \in J, \\ \beta(0,\omega) \geq \beta(T,0). \end{cases} \tag{5.4}$$

Now, consider the following PBVP of random differential equation

$$\begin{cases} x^{'}(t,\omega) + \lambda(\omega)x(t,\omega) = \tilde{f}(t,x(t,\omega),\omega), \quad t \in J, \\ x(0,\omega) = x(T,\omega), \end{cases} \tag{5.5}$$

for all $\omega \in \Omega$, where

$$\tilde{f}(t,x,\omega) = f(t,x,\omega) + \lambda(\omega)x \tag{5.6}$$

and $\lambda : \Omega \to \mathbb{R}$ is a measurable function.

Remark 5.2. We note that a random solution to the PVBV of RDE (5.6) is a random solution of the PBVP of RDE (5.1) and vice versa.

We need the following result in the sequel.

Lemma 5.3. *For any integrable function $h : \Omega \to C(J, \mathbb{R})$, a function ξ is a random solution to the PBVP of RDE,*

$$\begin{cases} x^{'}(t,\omega) + \lambda(\omega)x(t,\omega) = h(t,\omega), \quad t \in J, \\ x(0,\omega) = x(T,\omega), \end{cases} \tag{5.7}$$

if and only if it is a random solution of the random RIE

$$x(t,\omega) = \int_0^T G(t,s)h(s,\omega)ds \tag{5.8}$$

where $G(t,s)$ is a Green's function given by

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda(\omega)(T+s-t)}}{e^{\lambda(\omega)T} - 1}, & 0 \leq s \leq t \leq T, \\ \\ \dfrac{e^{\lambda(\omega)(s-t)}}{e^{\lambda(\omega)T} - 1}, & 0 \leq t \leq s \leq T. \end{cases} \tag{5.9}$$

Remark 5.4. It is known that the Green's function G is continuous and nonnegative on $J \times J$.

We need the following hypothesis in what follows.

(H_6) The PBVP (5.1) has a lower random solution u defined on J.

Theorem 5.5. *Assume that hypotheses* (H_1) *through* (H_3) *and* (H_6) *hold. Then the PBVP of RDE* (5.1) *has a random solution* $\xi^*(\omega)$ *defined on* J *and the sequence* $\{x_n(\omega)\}$ *of successive approximations defined by*

$$x_0 = u, \quad x_n(t,\omega) = \int_0^T G(t,s)\tilde{f}(s, x_{n-1}(s,\omega),\omega), \;\; t \in J, \tag{5.10}$$

for $n \in \mathbb{N}$*, converges monotonically to* ξ^**, where* $\tilde{f}$ *is given by* (5.6).

Proof. Now the PBVP of RDE (5.1) is equivalent to the random PBVP (5.5) and the PBVP (5.5) is equivalent to the random RIE

$$x(t,\omega) = \int_0^T G(t,s)\tilde{f}(s, x(s,\omega),\omega)\, ds, \;\; t \in J, \tag{5.11}$$

for all $\omega \in \Omega$, where the function $\tilde{f}$ is given by the expression (5.9).

Define an operator $\mathcal{Q}$ on $\Omega \times E$ by

$$\mathcal{Q}(\omega)x(t) = \int_0^T G(t,s)\tilde{f}(s, x(s,\omega),\omega)\, ds, \;\; t \in J, \tag{5.12}$$

for all $\omega \in \Omega$.

We show that the operator $\mathcal{Q}$ satisfies all the conditions of Theorem 3.4. This will be achieved in a series of steps.

Step I: $\mathcal{Q}$ *is a random operator on* E *into itself.*

First we shall show that $\mathcal{Q}$ is a random operator on E into itself. Since the function $G(t,s)$ is continuous in t for each $s \in J$, we have that the map

$$t \mapsto G(t,s)\tilde{f}(s,x,\omega)$$

is continuous for each $\omega \in \Omega$ and $x \in \mathbb{R}$, which implies that the function $(\mathcal{Q}(\omega)x)$ is continuous on J. Hence $\mathcal{Q}$ defines a mapping $\mathcal{Q} : \Omega \times E \to E$.

Next, by Carathéodory condition, the function $\omega \mapsto f(t,x,\omega)$ is measurable for all $x \in \mathbb{R}$ and $t \in J$. Further, since $G(t,s)$ is a continuous real-valued function on $J \times J$, the map

$$\omega \mapsto G(t,s)\tilde{f}(t,x,\omega)$$

is measurable. As the integral is the limit of a finite sum of measurable functions, one has the map

$$\omega \mapsto \int_0^T G(t,s)\tilde{f}(s, x(s,\omega),\omega)\, ds$$

is measurable. Consequently, the map $\omega \mapsto \mathcal{Q}(\omega)x$ is measurable for all $x \in E$. Hence $\mathcal{Q}$ is a random operator mapping $\Omega \times E$ into E.

Step II: $\mathcal{Q}(\omega)$ *is nondecreasing on* E.

$\mathcal{Q}(\omega)$ is nondecreasing on E. Let $x, y \in E$ be such that $x \leq y$. Then,

$$
\begin{aligned}
\mathcal{Q}(\omega)x(t) &= \int_0^T G(t,s)\tilde{f}(s,x(s,\omega),\omega)ds \\
&= \int_0^T G(t,s)\left[\tilde{f}(s,x(s,\omega),\omega)+\lambda(\omega)x(s,\omega)\right]ds \\
&\leq \int_0^T G(t,s)\left[f(s,y(s,\omega),\omega)+\lambda(\omega)x(s,\omega)\right]ds \\
&= \mathcal{Q}(\omega)\,y(t)
\end{aligned}
$$

for all $t \in J$. Hence $\mathcal{Q}(\omega)$ is a nondecreasing random operator on E.

Step III: $\mathcal{Q}(\omega)$ *is a continuous random operator on* E.

Next, we show $\mathcal{Q}(\omega)$ is a continuous random operator on E. Let $\omega \in \Omega$ be fixed and let $\{x_n\}$ be a sequence in E converging to a point $x \in E$. Then, by dominated convergence theorem, we obtain,

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{Q}(\omega)x_n(t) &= \lim_{n\to\infty}\int_0^T G(t,s)\tilde{f}(s,x_n(s,\omega),\omega)\,ds \\
&= \int_0^T G(t,s)\left[\lim_{n\to\infty} f(s,x_n(s,\omega),\omega)\right]ds \\
&= \int_0^T G(t,s)\tilde{f}(s,x(s,\omega),\omega)\,ds = \mathcal{Q}(\omega)\,x(t)
\end{aligned}
$$

for all $t \in J$. This shows that $\mathcal{Q}(\omega)x_n \to \mathcal{Q}(\omega)x(\omega)$ point wise on J. To show the convergence is uniform, we show that $\{\mathcal{Q}(\omega)x_n\}$ is equi-continuous sequence of functions defined on J. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$
\begin{aligned}
&|\mathcal{Q}(\omega)x_n(t_1) - \mathcal{Q}(\omega)x_n(t_2)| \\
&= \left|\int_0^T G(t_1,s)\overline{f}(s,x(s,\omega),\omega)ds - \int_0^T G(t_2,s)\overline{f}(s,x(s,\omega),\omega)ds\right| \\
&= \left|\int_0^T \left[G(t_1,s)-G(t_2,s)\right]\overline{f}(x,x(s,\omega),\omega)ds\right| \\
&\leq \int_0^T |G(t_1,s)-G(t_2,s)|\left|\overline{f}(s,x_n(s,\omega),\omega)\right|ds \\
&\leq \int_0^T |G(t_1,s)-G(t_2,s)|\; M\,ds \quad \to \; 0 \text{ as } t_1 \to t_2
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$. Therefore, $\{\mathcal{Q}(\omega)x_n\}$ is equi-continuous sequence of functions in E which further in view of pointwise convergence implies that $\mathcal{Q}(\omega)x_n \to \mathcal{Q}(\omega)x$ uniformly. As a result $\mathcal{Q}(\omega)$ is a continuous random operator on E.

Step IV: *$\mathcal{Q}(\omega)$ is a partially contraction on $C(J,\mathbb{R})$.*

Next, we show that $\mathcal{Q}(\omega)$ is a partially linear contraction random operator on $C(J,\mathbb{R})$. Let $x, y \in C(J,\mathbb{R})$ be such that $x \geq y$ on J. Then,

$$
\begin{aligned}
d(\mathcal{Q}(\omega), \mathcal{Q}(\omega)y) &= \sup_{t\in J} |\mathcal{Q}(\omega)x(t) - \mathcal{Q}(\omega)y(t)| \\
&\leq \sup_{t\in J} \int_0^T G(t,s)|\tilde{f}(s,x(s,\omega),\omega) - \tilde{f}(s,y(s,\omega),\omega)|ds \\
&= \sup_{t\in J} \int_0^T G(t,s)\big|f(s,x(s,\omega),\omega) + \lambda(\omega)x(s) \\
&\qquad\qquad - f(s,y(s,\omega),\omega) - \lambda(\omega)y(s,\omega)\big|ds \\
&\leq \sup_{t\in J} \int_0^T G(t,s)\mu(\omega)[x(t,\omega) - y(s,\omega)]ds \\
&\leq \sup_{t\in J} \int_0^T G(t,s)\mu(\omega)|x(s,\omega) - y(s,\omega)|ds \\
&\leq \sup_{t\in J} \int_0^T G(t,s)\mu(\omega)\|x(\omega) - y(\omega)\|ds \\
&= \mu(\omega)\, d(x(\omega), y(\omega)) \sup_{t\in J} \int_0^T G(t,s)ds \\
&= \mu(\omega)\, d(x(\omega), y(\omega)) \sup_{t\in J} \frac{1}{\lambda(\omega)(e^{\lambda(\omega)T} - 1)} \times \\
&\qquad \times \left[(e^{\lambda(\omega)(T+s-t)})\Big|_0^t + e^{\lambda(\omega)(s-t)}\Big|_t^T \right] \\
&= \mu(\omega) d(x(\omega), y(\omega)) \times \frac{1}{\lambda(\omega)(e^{\lambda(\omega)T} - 1)} (e^{\lambda(\omega)T} - 1) \\
&\leq k(\omega) d(x(\omega), y(\omega)).
\end{aligned}
$$

This proves that $\mathcal{Q}(\omega)$ is a partially contraction random operator on E with $k(\omega) = \dfrac{\mu(\omega)}{s(\omega)} < 1$ for all $\omega \in \Omega$.

Step V: *u is a lower random solution of the random equation $\mathcal{Q}(\omega)x = x$.*

Let $\omega \in \Omega$ be fixed and we prove that $u(\omega)$ is a random function such that $u \leq \mathcal{Q}(\omega)u$ for all $\omega \in \Omega$. Now, by hypothesis (H_6), we obtain

$$u'(t,\omega) + \lambda(\omega)u(t,\omega) \leq f(t, u(t,\omega), \omega) + \lambda(\omega)u(t,\omega), \ \ t \in J. \tag{5.13}$$

Multiplying the above inequality by $e^{\lambda(\omega)\,t}$, we obtain

$$\big(u(t,\omega)e^{\lambda(\omega)\,t}\big)' \leq [f(t, u(t,\omega), \omega) + \lambda(\omega)u(t,\omega)]e^{\lambda(\omega)\,t}, \ \ t \in J,$$

which on integration gives

$$u(t,\omega)e^{\lambda(\omega)\,t} \leq u(0,\omega) + \int_0^t [f(s, u(s,\omega), \omega) + \lambda(\omega)u(s,\omega)]e^{\lambda(\omega)\,s}\,ds. \tag{5.14}$$

This further implies that

$$u(0,\omega) \leq \int_0^T \frac{e^{\lambda(\omega)\,s}}{e^{\lambda(\omega)T} - 1}[f(s, u(s,\omega), \omega) + \lambda(\omega)u(s,\omega)]e^{\lambda(\omega)\,s}\,ds.$$

From this inequality and (5.14), we obtain

$$\begin{aligned} u(0,\omega) \leq & \int_0^t \frac{e^{\lambda(\omega)\,s}}{e^{\lambda(\omega)T} - 1}[f(s, u(s,\omega), \omega) + \lambda(\omega)u(s,\omega)]e^{\lambda(\omega)\,s}\,ds \\ & + \int_t^T \frac{e^{\lambda(\omega)\,s}}{e^{\lambda(\omega)T} - 1}[f(s, u(s,\omega), \omega) + \lambda(\omega)u(s,\omega)]e^{\lambda(\omega)\,s}\,ds \end{aligned}$$

and hence

$$\begin{aligned} u(t,\omega) &\leq \int_0^T G(t,s)[f(s, u(s,\omega), \omega) + \lambda(\omega)u(s,\omega)]\,ds \\ &= \mathcal{Q}(\omega)u(t,\omega) \end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$.

Now, the desired existence and uniqueness of the random solution follows by an application of Theorem 3.7. This completes the proof. □

Remark 5.6. In above Theorem 5.5, the uniqueness of the random solution ξ of the PBVP of RDE (5.1) can be obtained as $\lim_{n\to\infty} \mathcal{Q}^n(\omega)x = \xi(\omega)$ for every $x \in C(J, \mathbb{R})$. If we choose $x = u$, then $\{\mathcal{Q}^n(\omega)u\}$ is a monotone nondecreasing sequence which converges uniformly to the unique random solution of the PBVP of RDE (5.1) on J.

Theorem 5.7. *Assume that hypotheses (H_1) through (H_4) hold and suppose that the hypothesis*

(H_7) *The PBVP* (5.1) *has an upper random solution v defined on J.*

is satisfied. Then the PBVP of RDE (5.1) *has a random solution* $\xi^*(\omega)$ *defined on* J *and the sequence* $\{x_n(\omega)\}$ *of successive approximations defined by*

$$x_0 = v, \quad x_n(t,\omega) = \int_0^T G(t,s)\overline{f}(s, x_{n-1}(s,\omega),\omega), \ \ t \in J, \tag{5.15}$$

for $n \in \mathbb{N}$, *converges monotonically to* ξ^*, *where* $\tilde{f}$ *is given by* (5.6).

Proof. We follow the proof of Theorem 5.5 and check that all conditions of Theorem 3.5 are satisfied. If v is an upper random solution of the PBVP of RDE (5.1), with an analogous procedure that exposed for the case of the lower random solution, we check that

$$v(t) \geq \int_0^T G(t,s)[f(s, v(s,\omega),\omega) + \lambda v(s,\omega)]\, ds = \mathcal{Q}(\omega)v(t,\omega)$$

for all $t \in J$ and $\omega \in \Omega$. Now, an application of Theorem 3.6 provides the existence of a random fixed point $\xi(\omega)$ for the random operator $\mathcal{Q}(\omega)$ and which is unique by Theorem 3.6. Therefore there exists a unique random solution of the PBVP of RDE (5.1) defined on J. This completes the proof. □

6. Random differential inequalities

The main problem of the differential inequalities is to obtain the information about the behavior of the solutions of inequalities related to given differential equations and finding the bounds for such solutions of the inequalities has widely been discussed in the literature since long time. Firstly, in the following we discuss the comparison of two random solutions of the related given an IVP and a PBVP of random differential equations (4.1) and (5.1) satisfying the conditions of opposite inequalities which are called non-strict random differential inequalities. We mention that Theorems 3.5 and 3.6 are useful to derive the random differential inequalities under suitable conditions.

Theorem 6.1. *Assume that hypotheses* (H_0)-(H_3) *are satisfied. Suppose that there exist measurable functions* $y, z \in\colon \Omega \to C(J,\mathbb{R})$ *satisfying for each* $\omega \in \Omega$,

$$y'(t,\omega) \leq f(t, y(t,\omega),\omega) \tag{6.1}$$

and

$$z'(t,\omega) \geq f(t, z(t,\omega),\omega) \tag{6.2}$$

for all $t \in J$. *Further if*

$$y(0,\omega) \leq q(\omega) \leq z(0,\omega) \tag{6.3}$$

for all $\omega \in \Omega$, *then*

$$y(t,\omega) \leq z(t,\omega) \tag{6.4}$$

for all $t \in J$ *and* $\omega \in \Omega$.

Proof. Set $E = C(J, \mathbb{R})$ and define the order relation $\leq$ in E as

$$x \leq y \quad \iff \quad x(t) \leq y(t) \quad \text{for all } \ t \in J.$$

The above relation is same as the order relation defined by the order cone $\mathcal{K}$ in E defined by

$$\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \quad \text{for all } \ t \in J\}. \tag{6.5}$$

Define the operator $\mathcal{Q} : \Omega \times E \to E$ by

$$\begin{aligned}\mathcal{Q}(\omega)x(t,\omega) &= q(\omega)e^{-\lambda(\omega)t} \\ &\quad + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, x(s,\omega),\omega)\,ds\ ,\ t \in J,\end{aligned}$$

for each $\omega \in \Omega$. Then it can be shown as in the proof of theorem 5.5 that $\mathcal{Q}(\omega)$ is a continuous, nondecreasing and contraction random operator on E into itself. From the inequalities (6.1) and (6.2) it follows that

$$\begin{aligned}y(t,\omega) &\leq q(\omega)e^{-\lambda(\omega)t} \\ &\quad + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, y(s,\omega),\omega)\,ds = \mathcal{Q}(\omega)y(t,\omega)\end{aligned}$$

and

$$\begin{aligned}z(t,\omega) &\geq q(\omega)e^{-\lambda(\omega)t} \\ &\quad + e^{-\lambda(\omega)t}\int_0^t e^{\lambda(\omega)s}\tilde{f}(s, z(s,\omega),\omega)\,ds = \mathcal{Q}(\omega)z(t,\omega)\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$.

Now an application of Theorem 3.4 provides that there is a unique measurable function $\xi : \Omega \to E$ satisfying for each $\omega \in \Omega$,

$$y \leq \mathcal{Q}(\omega)y \leq \mathcal{Q}^2(\omega)y \leq \cdots \leq \mathcal{Q}^n(\omega)y \leq \cdots \tag{6.6}$$

and

$$\lim_{n\to\infty} \mathcal{Q}^n(\omega)y = \xi(\omega) = \mathcal{Q}(\omega)\xi(\omega)$$

for all $\omega \in \Omega$. Similarly, we have for each $\omega \in \Omega$,

$$z \geq \mathcal{Q}(\omega)z \geq \mathcal{Q}^2(\omega)z \geq \cdots \geq \mathcal{Q}^n(\omega)z \geq \cdots \tag{6.7}$$

and

$$\lim_{n\to\infty} \mathcal{Q}^n(\omega)z = \xi(\omega) = \mathcal{Q}(\omega)\xi(\omega).$$

Since the order cone $\mathcal{K}$ is a closed set in E, one has

$$\mathcal{Q}^n(\omega)y \leq \xi(\omega) \leq \mathcal{Q}^n(\omega)z$$

for all $n \in \mathbb{N}$. From (6.6) and (6.7) it follows that

$$y(t,\omega) \leq z(t,\omega)$$

for all $t \in J$ and $\omega \in \Omega$. This completes the proof. $\square$

Theorem 6.2. *Assume that hypotheses* (H_1) *through* (H_3) *are satisfied. Suppose that there exist measurable functions* $y, z : \Omega \to E$ *satisfying*

$$y'(t,\omega) \leq f(t, y(t,\omega), \omega) \tag{6.8}$$

and

$$z'(t,\omega) \geq f(t, z(t,\omega), \omega) \tag{6.9}$$

for all $t \in J$. *Furthermore, if*

$$y(0,\omega) \leq y(T,\omega) \tag{6.10}$$

and

$$z(0,\omega) \geq z(T,\omega) \tag{6.11}$$

for all $\omega \in \Omega$, *then*

$$y(t,\omega) \leq z(t,\omega)$$

for all $t \in J$ *and* $\omega \in \Omega$.

Proof. The proof is similar to Theorem 6.1 with appropriate modifications. we omit the details. □

Below we prove the comparison principles for the random IVP (4.1) and random PBVP (5.1) on J.

Theorem 6.3. *Assume that hypotheses* (H_0) *through* (H_3) *are satisfied. Suppose that there exists a measurable function* $m : \Omega \to E$ *satisfying for each* $\omega \in \Omega$,

$$\begin{cases} m'(t,\omega) \leq f(t, m(t,\omega), \omega), \\ m(0,\omega) \leq q(\omega), \end{cases} \tag{6.12}$$

for all $t \in J$, *then*

$$m(t,\omega) \leq \xi(t,\omega) \tag{6.13}$$

for all $t \in J$ *and* $\omega \in \Omega$, *where* ξ *is a unique random solution of the random IVP* (4.1) *defined on* J.

Again, we have

Theorem 6.4. *Assume that hypotheses* (H_0) *through* (H_3) *are satisfied. Suppose that there exist measurable functions* $v : \Omega \to E$ *satisfying for each* $\omega \in \Omega$

$$\begin{cases} v'(t,\omega) \geq f(t, v(t,\omega), \omega), \\ v(0,\omega) \geq q(\omega), \end{cases} \tag{6.14}$$

for all $t \in J$, *then*

$$v(t,\omega) \geq \xi(t,\omega) \tag{6.15}$$

for all $t \in J$ *and* $\omega \in \Omega$, *where* ξ *is a unique random IVP* (5.1) *defined on* J.

Similarly, we can prove the comparison principles for the random PBVP (5.1) on J.

Theorem 6.5. *Assume that hypotheses* (H_1) *through* (H_3) *are satisfied. Suppose that there exists a measurable function* $m : \Omega \to E$ *satisfying for each* $\omega \in \Omega$,

$$\begin{cases} m^{'}(t,\omega) \leq f(t, m(t,\omega), \omega), \\ m(0,\omega) \leq m(0,T), \end{cases} \tag{6.16}$$

for all $t \in J$. *Then*

$$m(t,\omega) \leq \xi(t,\omega)$$

where ξ *is a unique random solution of the random PBVP* (6.1) *defined on* J.

Theorem 6.6. *Assume that hypotheses* (H_1) *through* (H_3) *are satisfied. Suppose that there exists a measurable function* $v : \Omega \to E$ *satisfying for each* $\omega \in \Omega$,

$$\begin{cases} v^{'}(t,\omega) \geq f(t, v(t,\omega), \omega), \\ v(0,\omega) \geq v(0,T), \end{cases} \tag{6.17}$$

for all $t \in J$. *Then*

$$v(t,\omega) \geq \xi(t,\omega)$$

where ξ *is a unique random solution of the random PBVP* (5.1) *defined on* J.

The proofs of above theorems are the easy consequence of Theorems 3.4 and 3.5. Hence we omit the details. Below we show that the comparison principles formulated in Theorems 6.3 and 6.5 have some nice applications to the Perron type uniqueness results for the initial and boundary value problems of first order random differential equations.

For a given measurable function $u : \Omega \to C(J, \mathbb{R}_+)$, consider the scalar random IVP

$$\begin{cases} u'(t,\omega) = g(t, u(t,\omega), \omega), \\ u(0,\omega) = 0, \end{cases} \tag{6.18}$$

for all $t \in J$ and $\omega \in \Omega$, where the nonlinearity $g : J \times \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ satisfies the hypotheses (H_1) through (H_3) with f replaced by g and has the only random solution 0.

Theorem 6.7. *Suppose that the function* $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ *satisfies for each* $\omega \in \Omega$,

$$0 \leq f(t,x,\omega) - f(t,y,\omega) \leq g(t, x-y, \omega) \tag{6.19}$$

for all $t \in J$ *and for all* $x, y \in \mathbb{R}$, $x \geq y$. *Then the random IVP* (4.1) *has at most one random solution defined on* J.

Proof. Let $x(t,\omega)$ and $y(t,\omega)$ be two random solutions of the random IVP (5.1) such that $x(t,\omega) \geq y(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. Denote

$$m(t,\omega) = x(t,\omega) - y(t,\omega) = |x(t,\omega) - y(t,\omega)|, \ t \in J,$$

for all $\omega \in \Omega$. Then, we have

$$\begin{aligned} m'(t,\omega) &= \frac{d}{dt}[|x(t,\omega) - y(t,\omega)|] \\ &= \big|x'(t,\omega) - y'(t,\omega)\big| \\ &= |f(t,x(t,\omega),\omega) - f(t,y(t,\omega),\omega)| \\ &\leq g\big(t, |x(t,\omega) - y(t,\omega)|, \omega\big) \\ &= g(t,m(t,\omega),\omega) \end{aligned} \tag{6.20}$$

for all $t \in J$ and $\omega \in \Omega$ satisfying

$$m(0,\omega) = |x(0,\omega) - y(0,\omega)| = 0.$$

Now an application of Theorem 6.3 provides that there is a unique random solution $\xi^*(\omega)$ of the random IVP (4.1) such that

$$m(t,\omega) \leq \xi^*(t,\omega) \tag{6.21}$$

for all $t \in J$ and $\omega \in \Omega$. But the identically zero function is the only random solution of the random IVP (4.1) defined on J, so by uniqueness, $\xi^*(t,\omega) = 0$ for all $t \in J$ and $\omega \in \Omega$. Therefore, from (6.21), it follows that $m(t,\omega) = 0$, that is, $x(t,\omega) = y(t,\omega)$ on $J \times \Omega$. This completes the proof. $\square$

Next, for a given measurable function $u : \Omega \to C(J, \mathbb{R}_+)$, we consider the scalar random PBVP

$$\begin{cases} u'(t,\omega) = g(t,u(t,\omega),\omega), \\ u(0,\omega) = u(T,\omega), \end{cases} \tag{6.22}$$

for all $t \in J$ and $\omega \in \Omega$, where the nonlinearity $g : J \times \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ satisfies the hypotheses (H_1) through (H_3) with f replaced by g and has the only random solution 0.

Theorem 6.8. *Suppose that the function $f : J \times \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies* (6.19). *Then the random PBVP* (5.1) *has at most one random solution defined on J.*

Proof. The proof is similar to Theorem 6.7 and we omit the details. $\square$

Remark 6.9. We remark that the function $g(t,u,\omega) = L(t,\omega)\,u$ is admissible in above uniqueness Theorems 6.7 and 6.8, where $L : \Omega \to C(J, \mathbb{R}_+)$ is measurable.

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