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ENERGY ESTIMATES AND LOCAL EXISTENCE RESULTS FOR A MILDLY DEGENERATE WAVE EQUATION WITH DAMPING IN UNBOUNDED DOMAINS

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Abstract. We discuss the energy decay estimates and the local existence results of the solutions for the nonlocal hyperbolic problem

$$u_{tt} + \phi(x)||\nabla u(t)||^2(-\Delta)u + \delta u_t = 0,\ x \in \mathbb{R}^N,\ t \geq 0\ ,$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3,\ \delta > 0$ and $(\phi(x))^{-1} = g(x)$ is a positive function lying in $L^{N/2}\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right)$. When the initial energy $E(u_0, u_1)$ which corresponds to the problem, is non-negative and small, there exists a unique local solution in time.

1. Introduction

In this work we study the following mildly degenerate wave equation

$$\begin{aligned} u_{tt} + \phi(x)||\nabla u(t)||^2(-\Delta)u + \delta u_t &= 0,\ \ x \in \mathbb{R}^N,\ \ t \geq 0\ , \qquad (1.1)\\ u(x,0) = u_0(x),\ u_t(x,0) &= u_1(x),\ \ x \in \mathbb{R}^N\ , \qquad (1.2) \end{aligned}$$

with initial conditions $u_0,\ u_1$ in appropriate function spaces, $N \geq 3$, and $\delta > 0$. The case of $N = 1$, equation (1.1) describes the nonlinear vibrations of

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an elastic string. Throughout the paper we assume that the functions ϕ and $g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfy the following condition

$$(\mathcal{G})\ \phi(x) > 0, \text{ for all } x \in \mathbb{R}^N \text{ and } (\phi(x))^{-1} =: g(x) \in L^{N/2}\left(\mathbb{R}^N\right) \cap L^{\infty}\left(\mathbb{R}^N\right).$$

This class will include functions of the form

$$\phi(x) \sim c_0 + \varepsilon |x|^a,\ \varepsilon > 0,\ a > 0\ ,$$

resembling phenomena of slowly varying wave speed around the constant speed c_0. Many results treat the case of $\varphi(x) =$ constant (in bounded or unbounded domains). It must be noted, that this case is proved to be totally different from the case of $\varphi(x) \to c_{\pm} > 0$, as $x \to \pm\infty$ (see [8]).

The original equation is

$$ph\frac{\partial^2 u}{\partial t^2} + \delta\frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f \tag{1.3}$$

for $0 < x < L,\ t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modulus and f the external force. When $p_0 = 0$ the equation is considered to be of *degenerate type* and the equation models an unstretched string or its higher dimensional generalization. Otherwise it is of *nondegenerate type* and the equation models an stretched string or its higher dimensional generalization. When $\delta = f = 0$, the equation was introduced by Kirchhoff [12] in the study of oscillations of stretched strings and plates. That's why equation (1.3) is called the *Kirchhoff string.*

In the case treated here the problem becomes complicated because the equation does not give rise to compact operators. The homogeneous *Sobolev* spaces combined with equivalent weighted L^p spaces, is the appropriate space to overcame these difficulties. In our paper we assume that $f(u) = 0$ (we have no external force), in order to study the behavior of the solutions for this kind of equations. This case is rather interesting in the class of the homogeneous Sobolev spaces.

In the case of *bounded domain*, when $\delta = 0$ and $f \neq 0$, the global existence is rather well studied in the class of analytic function spaces (e.g. see [5]). Crippa [3] has proved local in time solvability in the class of usual Sobolev spaces. Arosio and Garavaldi [1] have shown the existence of a unique local solution in the case of mildly degenerate type. For $\delta \geq 0$ and $f(u) = 0$, in the degenerate case, the global existence of solutions has been shown by Nishihara and Yamada [16], when the initial data are small enough. When $\delta >$

0 and $f(u) = 0$, Nakao [14] has derived decay estimates for the solutions. In particular, Kobayashi [13] constructed a unique weak solution using a Faedo-Galerkin method for a quasilinear wave equation with strong dissipation (see also [4, 15]). Nishihara [17] has derived a decay estimate from below of the potential of solutions. In the case of $\delta \geq 0$ and $f \neq 0$, Hosoya and Yamada [7] have studied the non-degenerate case with linear dissipation and proved the global existence of a unique solution under small initial data. Ikehata [9] has shown that for sufficiently small initial data, global existence can be obtained, even when the influence of the source terms is stronger than that of the damping terms.

In the case of *unbounded domain*, D'Ancona and Spagnolo [6] have shown the global existence of a unique C^∞ solution for the non-degenerate type with small C_0^∞ data. Todorova [20] studied the global existence and nonexistence of solutions both in the bounded and unbounded domain cases with nonlinear damping and small enough C_0^∞ initial data. Finally, Karahalios and Stavrakakis [10]-[11] have proved global existence and blow-up results for some semilinear wave equations with weak damping on all $\mathbb{R}^N$.

The presentation of this paper is as follows: In Section 2, we discuss properties of the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and some weighted L^p spaces, in order to overcome difficulties of non-compactness arising from the unboundedness of the domain. In Section 3, we show the existence of a unique local weak solution and we obtain energy decay estimates for the problem (1.1)-(1.2) with $(u_0, u_1) \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \times L_g^2\left(\mathbb{R}^N\right)$, when the initial energy $E(u_0, u_1)$ which corresponds to the problem, is non-negative and small.

Notation: We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for simplicity we use the symbols C_0^∞, $\mathcal{D}^{1,2}$, L^p, $1 \leq p \leq \infty$, for the spaces $C_0^\infty(\mathbb{R}^N)$, $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, respectively; $||.||_p$ for the norm $||.||_{L^p(\mathbb{R}^N)}$, where in case of $p = 2$ we may omit the index.

2. Preliminary results

In this section, we briefly mention some facts, notation and results, which will be used later in this paper. The space $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the "energy norm" $||u||_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} |\nabla u|^2 dx$. It is known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \nabla u \in \left(L^2(\mathbb{R}^N)\right)^N \right\}$$

and $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$ is embedded continuously in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, that is, there exists $k > 0$ such that

$$||u||_{\frac{2N}{N-2}} \leq k||u||_{\mathcal{D}^{1,2}} \; . \tag{2.1}$$

We shall frequently use the following generalized version of Poincaré's inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \alpha \int_{\mathbb{R}^N} gu^2 \, dx \ , \tag{2.2}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha =: k^{-2}||g||_{N/2}^{-1}$ (see [2, Lemma 2.1]). It is shown that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. The space $L_g^2(\mathbb{R}^N)$ is defined to be the closure of $C_0^\infty\left(\mathbb{R}^N\right)$ functions with respect to the inner product

$$(u,v)_{L_g^2(\mathbb{R}^N)} =: \int_{\mathbb{R}^N} guv \, dx \ . \tag{2.3}$$

It is clear that $L_g^2\left(\mathbb{R}^N\right)$ is a separable Hilbert space. The following Lemmas will be proved to be useful in the sequel. For the proofs we refer to [11], we note that g is a positive function.

Lemma 2.1. *Let $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty\left(\mathbb{R}^N\right)$. Then the embedding $\mathcal{D}^{1,2} \subset L_g^2$ is compact.*

Lemma 2.2. *Let $g \in L^{\frac{2N}{2N-pN+2p}}\left(\mathbb{R}^N\right)$. Then the following continuous embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^p\left(\mathbb{R}^N\right)$ is valid, for all $1 \leq p \leq 2N/(N-2)$.*

Remark 2.3. The assumption of Lemma 2.2 is satisfied under the hypothesis $(\mathcal{G})$, if $p \geq 2$.

Lemma 2.4. *Let g satisfy condition $(\mathcal{G})$. If $1 \leq q < p < p* = 2N/(N-2)$, then the following weighted inequality*

$$||u||_{L_g^p} \leq C_0||u||_{L_g^q}^{1-\theta}||u||_{\mathcal{D}^{1,2}}^{\theta} \tag{2.4}$$

is valid, for all $\theta \in (0,1)$, for which $1/p = (1-\theta)/q + \theta/p^*$, and $C_0 = k^\theta$.

To study the properties of the operator $-\phi\Delta$, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \ x \in \mathbb{R}^N \ , \tag{2.5}$$

without boundary conditions. Since for every $u, \ v \in C_0^\infty(\mathbb{R}^N)$ we have

$$(-\phi\Delta u, \ v)_{L_g^2} = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx \ , \tag{2.6}$$

we may consider equation (2.5) as an operator equation of the form

$$A_0 u = \eta, \ \ A_0 : D(A_0) \subseteq L_g^2\left(\mathbb{R}^N\right) \to L_g^2\left(\mathbb{R}^N\right), \ \ \eta \in L_g^2\left(\mathbb{R}^N\right) \ . \tag{2.7}$$

Relation (2.6) implies that the operator $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty\left(\mathbb{R}^N\right)$, is symmetric. From (2.2) and equation (2.6) we have that

$$(A_0 u,\ u)_{L^2_g} \geq \alpha ||u||^2_{L^2_g}, \ \text{ for all } \ u \in D(A_0)\ . \tag{2.8}$$

So the operator $A_0 = -\phi\Delta$ is a symmetric, strongly monotone operator on $L^2_g\left(\mathbb{R}^N\right)$. Hence, Friedrich's extension theorem [21, Theorem 19.C] is applicable. The energetic scalar product given by (2.6) is

$$(u,\ v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx$$

and the energetic space is the completion of $D(A_0)$ with respect to $(u,\ v)_E$. It is obvious that the energetic space X_E is the homogeneous Sobolev space $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$. The *energetic extension* $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta : \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \ \to \mathcal{D}^{-1,2}\left(\mathbb{R}^N\right)\ , \tag{2.9}$$

is defined to be the duality mapping of $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$. We define $D(A)$ to be the set of all solutions of equations (2.5), for arbitrary $\eta \in L^2_g\left(\mathbb{R}^N\right)$. *Friedrich's extension* A of A_0 is the restriction of the energetic extension A_E to the set $D(A)$. The operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u,\ v)_{D(A)} = (u,\ v)_{L^2_g} + (Au,\ Av)_{L^2_g}, \ \text{ for all } \ u,\ v \in D(A)\ .$$

The norm induced by the scalar product is

$$||u||_{D(A)} = \left\{ \int_{\mathbb{R}^N} g|u|^2 \, dx + \int_{\mathbb{R}^N} \phi|\Delta u|^2 \, dx \right\}^{\frac{1}{2}}\ ,$$

which is equivalent to the norm

$$||Au||_{L^2_g} = \left\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2 \, dx \right\}^{\frac{1}{2}}\ .$$

So we have established the evolution triple

$$D(A) \subset \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \subset L^2_g\left(\mathbb{R}^N\right) \subset \mathcal{D}^{-1,2}\left(\mathbb{R}^N\right)\ , \tag{2.10}$$

where all the embeddings are dense and compact. Finally, for later use, it is necessary to remind that the eigenvalue problem

$$-\phi(x)\Delta u = \mu u,\ x \in \mathbb{R}^N\ , \tag{2.11}$$

has a complete system of eigensolutions $\{w_n,\ \mu_n\}$ satisfying the following properties

$$\begin{cases} -\phi\Delta w_j = \mu_j w_j, & j = 1,2,..., \quad w_j \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right), \\ 0 < \mu_1 \leq \mu_2 \leq ..., & \mu_j \to \infty, \quad\ \ \text{as } j \to \infty\ . \end{cases} \tag{2.12}$$

In order to clarify the kind of solutions we are going to obtain for the problem (1.1)-(1.2), we give the definition of the *weak solution* for this problem.

Definition 2.5. A *weak solution* of the problem (1.1)-(1.2) is a function u such that

(i) $u \in L^2[0,T;D(A)]$, $u_t \in L^2\left[0,T;\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)\right]$, $u_{tt} \in L^2\left[0,T;L^2_g\left(\mathbb{R}^N\right)\right]$,

(ii) for all $v \in C_0^\infty([0,T]\times\left(\mathbb{R}^N\right))$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g}\, d\tau + \int_0^T \left(||\nabla u(\tau)||^{2\gamma} \int_{\mathbb{R}^N} \nabla u(\tau)\nabla v(\tau) dx \right) d\tau$$
$$+\delta \int_0^T (u_t(\tau), v(\tau))_{L^2_g} d\tau = 0, \tag{2.13}$$

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x) \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right),\ u_t(x,0) = u_1(x) \in L^2_g\left(\mathbb{R}^N\right) .$$

3. Existence results and energy decay estimates

In order to obtain a local existence result for the problem (1.1)-(1.2), we need information concerning the solvability of the corresponding nonhomogeneous linearized problem restricted in the sphere B_R.

$$\begin{aligned}
&u_{tt} - \phi(x)||\nabla v(t)||^2\Delta u + \delta u_t = 0, && (x,t) \in B_R \times (0,T)\ , \\
&u(x,0) = u_0(x), && u_t(x,0) = u_1(x), \qquad x \in B_R\ , \\
&u(x,t) = 0, && (x,t) \in \partial B_R \times (0,T)\ , \\
&v \in C\left(0,T;\mathcal{D}^{1,2}\right), && v_t \in C\left(0,T;L^2_g\right) .
\end{aligned} \tag{3.1}$$

Proposition 3.1. *Assume that* $u_0 \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$, $u_1 \in L^2_g\left(\mathbb{R}^N\right)$ *and* $N \geq 3$, *then the linear wave equation* (3.1) *has a unique solution such that*

$$u \in C\left(0,T;\mathcal{D}^{1,2}\right) \quad \text{and} \quad u_t \in C\left(0,T;L^2_g\right).$$

Proof. The proof follows the lines of [11, Proposition 3.1]. The Galerkin method is used, based on the information taken from the eigenvalue problem (2.11). □

Next, we will prove the following Theorem.

Theorem 3.2. *We assume that* $N \geq 3$ *and* $u_0 \neq 0$. *If* $(u_0,\ u_1) \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \times L^2_g\left(\mathbb{R}^N\right)$ *and satisfy the nondegenerate condition*

$$||\nabla u_0||^2 > 0 \ ,$$

then there exists $T = T\left(||u_0||_{\mathcal{D}^{1,2}},\ ||\nabla u_1||^2\right) > 0$ *such that the problem* (1.1)-(1.2) *admits a unique local weak solution* u *satisfying*

$$u \in \mathcal{C}(0, T; \mathcal{D}^{1,2}), \quad u_t \in \mathcal{C}\left(0, T; L^2_g\right).$$

Moreover, if $||\nabla u(t)|| > 0$ *and* $||u(t)||_{\mathcal{D}^{1,2}} + ||u_t(t)||_{L^2_g} < \infty$ *for* $t \geq 0$, *then* $T = \infty$.

Proof. For $T > 0$ and $R > 0$, we define the two parameter space of solutions

$$X_{T,R} =: \{v \in C(0;T;\mathcal{D}^{1,2}) : v_t \in C(0;T;L^2_g), v(0) = u_0,$$
$$v_t(0) = u_1, e(v(t)) \leq R^2, t \in [0;T]\},$$

where $e(u(t)) = ||u_t(t)||^2_{L^2_g} + ||u(t)||^2_{\mathcal{D}^{1,2}}$.

It is easy to see that $X_{T,R}$ can be organized as a complete metric space with the distance $d(u,v) =: \sup_{0 \leq t \leq T} e_1(u(t) - v(t))$, where

$$e_1(v) =: ||v_t||^2_{L^2_g} + ||v||^2_{\mathcal{D}^{1,2}} \ .$$

We define the non-linear mapping S in the following way. For every $v \in X_{T,R},\ u = Sv$ is the unique solution of our problem. Using the fact that $||\nabla u_0|| \equiv M_0 > 0$, we prove that there exist $T > 0,\ R > 0$ such that S maps $X_{T,R}$ into itself and S is a contraction mapping with respect to the metric $d(.,.)$. By applying the Banach contraction mapping theorem, we obtain a unique solution u belonging to $X_{T,R}$. Therefore it follows from the continuity argument for wave equations that this solution u belongs to our space. For more details we refer to [18].

Next, we multiply equation (1.1) by $2gu_t$ and integrate over $\mathbb{R}^N$ to get

$$2\int_{\mathbb{R}^N} gu_{tt}u_t \, dx - 2\int_{\mathbb{R}^N} ||\nabla u(t)||^2 \Delta u u_t \, dx + 2\int_{\mathbb{R}^N} g\delta u_t u_t \, dx = 0 \ .$$

So, we get

$$\frac{d}{dt}||u_t||^2_{L^2_g} + \frac{1}{2}\frac{d}{dt}||\nabla u(t)||^4 + 2\delta||u_t(t)||^2_{L^2_g} = 0 \ ,$$

thus we have

$$\frac{d}{dt}\left\{||u_t(t)||^2_{L^2_g} + \frac{1}{2}||\nabla u(t)||^4\right\} + 2\delta||u_t(t)||^2_{L^2_g} = 0 \ .$$

We define the energy for our problem

$$E(t) = ||u(t)||^2_{L^2_g} + \frac{1}{2}||\nabla u(t)||^4 \ . \tag{3.2}$$

So, we obtain the following relation

$$\frac{d}{dt}E(t) + 2\delta||u_t(t)||^2_{L^2_g} = 0\ . \tag{3.3}$$

We integrate the previous equation in $[0,t]$ to get the following

$$\begin{aligned}
\int_0^t \frac{d}{dt}E(t)\,dt + 2\delta\int_0^t ||u_t(t)||^2_{L^2_g}\,dt &= 0,\\
E(t) - E(0) + 2\delta\int_0^t ||u_t(t)||^2_{L^2_g}\,dt &= 0,\\
E(t) + 2\delta\int_0^t ||u_t(t)||^2_{L^2_g}\,dt &= E(0). \qquad (3.4)
\end{aligned}$$

Next, we multiply relation (3.2) by $2ug$ and integrate over $\mathbb{R}^N$ to get

$$2uu_{tt}g(x) - 2\phi(x)g(x)||\nabla u(t)||^2\Delta uu + 2\delta u_t ug(x) = 0$$

and

$$\int_{\mathbb{R}^N} 2uu_{tt}g\,dt - \int_{\mathbb{R}^N} 2||\nabla u(t)||^2\Delta uu\,dt + \int_{\mathbb{R}^N} 2\delta uu_t g\,dt = 0\ . \tag{3.5}$$

On the other hand we have the following relation $(uu_t)' = u_t u_t + uu_{tt}$. Thus, we get

$$uu_{tt} = (uu_t)' - u_t^2\ , \tag{3.6}$$

and

$$\int_{\mathbb{R}^N} 2uu_{tt}g\,dt = \frac{d}{dt}\int_{\mathbb{R}^N} 2guu_t\,dt - \int_{\mathbb{R}^N} 2gu_t^2\,dt\ .$$

Then, we obtain

$$\int_{\mathbb{R}^N} 2uu_{tt}g\,dt = \frac{d}{dt}2(u,u_t)_{L^2_g} - 2||u_t||^2_{L^2_g}\ . \tag{3.7}$$

Using relations (3.6) and (3.7), relation (3.5) becomes

$$\frac{d}{dt}2(u,u_t)_{L^2_g} - 2||u_t||^2_{L^2_g} - \int_{\mathbb{R}^N} 2||\nabla u(t)||^2\Delta uu\,dt + \int_{\mathbb{R}^N} 2\delta uu_t g\,dt = 0\ , \tag{3.8}$$

where we have that

$$\int_{\mathbb{R}^N} 2\delta uu_t g\,dt = \frac{1}{2}2\delta\frac{d}{dt}||u(t)||^2_{L^2_g} \tag{3.9}$$

and

$$-\int_{\mathbb{R}^N} 2||\nabla u(t)||^2\Delta uu\,dt = 2||\nabla u(t)||^2||\nabla u(t)||^2 = 2||\nabla u(t)||^4\ , \tag{3.10}$$

where we used the relation $-\int_{\mathbb{R}^N} \Delta u u \, dt = ||\nabla u(t)||^2$.

Next, using relations (3.9) and (3.10), we obtain from relation (3.8) the following

$$\frac{d}{dt}2(u,u_t)_{L_g^2} - 2||u_t||_{L_g^2}^2 + 2||\nabla u(t)||^4 + \delta\frac{d}{dt}||u(t)||_{L_g^2}^2 = 0 \; .$$

Thus we get the following equality

$$\frac{d}{dt}\left\{\delta||u(t)||_{L_g^2}^2 + 2(u,u_t)_{L_g^2}\right\} + 2||\nabla u(t)||^4 = 2||u_t(t)||_{L_g^2}^2 \; . \tag{3.11}$$

We integrate relation (3.11) in $[0,t]$ and we get

$$\int_0^t \frac{d}{dt}\delta||u(t)||_{L_g^2}^2 \, dt + 2\frac{d}{dt}\int_0^t (u(t),u_t(t))_{L_g^2} \, dt + 2\int_0^t ||\nabla u(t)||^4 \, dt$$
$$= 2\int_0^t ||u_t(t)||_{L_g^2}^2 \, dt \; .$$

So, we have that

$$\delta(||u||_{L_g^2}^2 - ||u_0||_{L_g^2}^2) + 2(u,u_t)_{L_g^2} - 2(u_0,u_1)_{L_g^2} + 2\int_0^t ||\nabla u(t)||^4 \, dt$$
$$= 2\int_0^t ||u_t||_{L_g^2}^2 \, dt.$$

Thus, we obtain the following estimate

$$\begin{aligned} &\delta||u(t)||_{L_g^2}^2 + 2\int_0^t ||\nabla u(t)||^4 \, dt \\ &\le \delta||u_0(t)||_{L_g^2}^2 + 2(u_0(t),u_1(t))_{L_g^2} \\ &\quad +2||u(t)||_{L_g^2}||u_t(t)||_{L_g^2} + 2\int_0^t ||u_t(t)||_{L_g^2}^2 \, dt \; . \end{aligned} \tag{3.12}$$

From relations (3.2) and (3.4), we get the following equality

$$||u_t(t)||_{L_g^2}^2 + \frac{1}{2}||\nabla u(t)||^4 + 2\delta\int_0^t ||u_t(t)||_{L_g^2}^2 \, dt = E(0) \; .$$

Thus we have that

$$||u_t(t)||_{L_g^2}^2 \le E(0) \tag{3.13}$$

and

$$\frac{1}{2}||\nabla u(t)||^4 \le E(0) \Rightarrow ||\nabla u(t)||^2 \le (2E(0))^{1/2}\ . \tag{3.14}$$

We obtain from relation (3.12) that

$$\begin{aligned}
&\delta||u(t)||^2_{L^2_g} + 4\int_0^t ||\nabla u(t)||^4\ dt \\
&\le \delta||u_0(t)||^2_{L^2_g} + 2(u_0(t),u_1(t))_{L^2_g} + 2||u(t)||_{L^2_g}||u_t(t)||_{L^2_g} \\
&\quad +2\int_0^t ||u_t(t)||^2_{L^2_g}\ dt + 2\int_0^t ||\nabla u(t)||^4\ dt \\
&\le \delta||u_0||^2_{L^2_g} + 2(u_0,u_1)_{L^2_g} + E(0) + E(0).
\end{aligned}$$

So, we have (using Young's inequality)

$$\begin{aligned}
&\delta||u(t)||^2_{L^2_g} + 4\int_0^t ||\nabla u(t)||^4 dt \\
&\le \delta||u_0||^2_{L^2_g} + 2(u_0,u_1)_{L^2_g} + 2E(0) \\
&\le 2\{\delta||u_0||^2_{L^2_g} + 2(u_0,u_1)_{L^2_g}\} + 2\cdot 2E(0) \\
&\le 2\{\delta||u_0||^2_{L^2_g} + 2(u_0,u_1)_{L^2_g} + 2E(0)\} \\
&\le I_0^2,
\end{aligned} \tag{3.15}$$

where

$$I_0^2 = 2\{\delta||u_0||^2_{L^2_g} + 2(u_0,u_1)_{L^2_g} + 2E(0)\}\ . \tag{3.16}$$

Let $\rho = \max\{\delta, 4\}$, then

$$||u(t)||^2_{L^2_g} + \int_0^t ||\nabla u(t)||^4\ dt \le \rho^{-1} I_0^2\ . \tag{3.17}$$

For later use, we introduce the following function $H(t)$, where

$$H(t) = \frac{||\nabla u_t(t)||^2_{L^2_g}}{||\nabla u(t)||^2} + ||\Delta u(t)||^2\ . \tag{3.18}$$

Next, we multiply equation (1.1) by $-\Delta u_t g$ and integrate over $\mathbb{R}^N$ to get

$$\begin{aligned}
&\int_{\mathbb{R}^N} -\Delta u_t u_{tt} g\ dt + \int_{\mathbb{R}^N} ||\nabla u(t)||^2 \Delta u \Delta u_t\ dt - \int_{\mathbb{R}^N} \delta g u_t \Delta u_t\ dt = 0 \\
&\Rightarrow \frac{d}{dt}\frac{1}{2}||\nabla u_t(t)||^2_{L^2_g} + ||\nabla u(t)||^2 \frac{d}{dt}\frac{1}{2}||\Delta u(t)||^2 + \delta||\nabla u_t(t)||^2_{L^2_g} = 0 \\
&\Rightarrow \frac{d}{dt}||\nabla u_t(t)||^2_{L^2_g} + ||\nabla u(t)||^2 \frac{d}{dt}||\Delta u(t)||^2 + 2\delta||\nabla u_t(t)||^2_{L^2_g} = 0\ .
\end{aligned} \tag{3.19}$$

Since we have that $||\nabla u_0|| > 0$, for $u_0 \neq 0$, we see that $||\nabla u(t)|| > 0$ near $t = 0$. Let

$$T \equiv \sup \{t \in [0, +\infty) : ||\nabla u(s)|| > 0, \ \ 0 \leq s \leq t\} \ .$$

If $T < +\infty$, we have that $||\nabla u(T)|| = 0$. We multiply relation (3.19) by $||\nabla u(t)||^{-2}$ for $0 \leq t < T$ and we get the following equality

$$\frac{d}{dt}H(t) + 2\left(\delta + \frac{(\nabla u(t), \nabla u_t(t))}{||\nabla u(t)||^2}\right)\frac{||\nabla u_t(t)||^2_{L^2_g}}{||\nabla u(t)||^2} = 0 \ . \tag{3.20}$$

Since

$$H(0) = \frac{||\nabla u_1||^2_{L^2_g}}{||\nabla u_0||^2} + ||\Delta u_0||^2 < 1 \tag{3.21}$$

and

$$\frac{|(\nabla u(t), \nabla u_t(t))|}{||\nabla u(t)||^2} < H(t)^{1/2} \tag{3.22}$$

we observe that

$$\frac{d}{dt}\ H(t) \leq 0, \quad \ H(t) \leq H(0) \ , \tag{3.23}$$

for some $t > 0$, which means that relation (3.23) holds for $0 \leq t < T$, because of contradiction. On the other hand, if $||\nabla u(T)|| = 0$, we get from (3.23) that $\lim_{t \to T} ||\nabla u_t(t)|| = 0$. Then, from the uniqueness of the solution (see [19], Proposition 4.1, p.125) for equation (1.1), we remark that (1.1) has a trivial solution on $[0, T]$, with $\{u(T), u_t(T)\} = \{0, 0\}$. This contradicts the hypothesis that $u_0 \neq 0$. Finally, we conclude that $T = \infty$, that is $||\nabla u(t)|| > 0$ for $t \geq 0$. Thus we get, after all these calculations, that equation(1.1) gives a unique local solution u, which belongs to $\cap_{k=0}^{2} C^k\left([0, T); H^{2-k}(R^N)\right)$.

Moreover from (3.20) and (3.23) we obtain that

$$\frac{d}{dt}H(t) + \delta^2\frac{||\nabla u_t(t)||^2_{L^2_g}}{||\nabla u(t)||^2} \leq 0, \quad t \geq 0 \tag{3.24}$$

and

$$H(t) + \delta^2 \int_0^t \frac{||\nabla u_t(t)||^2_{L^2_g}}{||\nabla u(t)||^2}\ dt \leq H(0), \quad t \geq 0 \ . \tag{3.25}$$

Then we have that from relations (3.4), (3.17) and (3.24), we obtain that

$$||u(t)||_{\mathcal{D}^{1,2}} + ||u_t(t)||_{L^2_g} \leq C < \infty \quad for \ \ t \geq 0 \ .$$

That completes the proof of the theorem. □

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