Nonlinear Functional Analysis and Applications
Vol. 17, No. 3 (2012), pp. 361-381

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

SOME COMPACT GENERALIZATIONS OF WELL-KNOWN INEQUALITIES FOR POLYNOMIALS

N. A. Rather[1] and Suhail Gulzar[2]

[1]P.G. Department of Mathematics, University of Kashmir,
Hagratbal, Srinagar-190006, India.
e-mail: dr.narather@gmail.com

[2]P.G. Department of Mathematics, University of Kashmir,
Hagratbal, Srinagar-190006, India.
e-mail: sgmattoo@gmail.com

Abstract. Let $P(z)$ be a polynomial of degree n. In this paper, we consider a problem of investigating the dependence of

$$\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right|$$

on maximum and minimum of $|P(z)|$ on $|z| = k$ for arbitrary real or complex numbers $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$, $k \geq 1$ and establish certain sharp compact generalizations of well-known Bernstien-type inequalities for polynomials, from which a variety of interesting results follows as special cases. Besides we shall first obtain an interesting result which yields a number of well-known polynomial inequalities as special cases.

1. Introduction

Let $\mathscr{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. A famous result known as Bernstein's inequality (for reference, see [9], [11] or [12]) states that if $P \in \mathscr{P}_n$, then

$$\underset{|z|=1}{Max}\left|P'(z)\right| \leq n\underset{|z|=1}{Max}\left|P(z)\right|, \tag{1.1}$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have

$$\underset{|z|=R}{Max}\left|P(z)\right| \leq R^n\underset{|z|=1}{Max}\left|P(z)\right|, \quad R \geq 1. \tag{1.2}$$

[0]Received July 24, 2012. Revised August 20, 2012.
[0]2010 Mathematics Subject Classification: 30D15, 41A17.
[0]Keywords: Polynomials; inequalities in the complex domain; Bernstein's inequality.

(for reference, see [9] or [10]).

If we restrict ourselves to the class of polynomials $P \in \mathscr{P}_n$ having no zero in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\underset{|z|=1}{Max}\left|P'(z)\right| \leq \frac{n}{2}\underset{|z|=1}{Max}\left|P(z)\right|, \tag{1.3}$$

and

$$\underset{|z|=R}{Max}\left|P(z)\right| \leq \frac{R^n+1}{2}\underset{|z|=1}{Max}\left|P(z)\right|, \quad R \geq 1. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [7], whereas inequality (1.4) is due to Ankey and Ravilin [1].

Aziz and Dawood [2] further improved inequalities (1.3) and (1.4) under the same hypothesis and proved that,

$$\underset{|z|=1}{Max}\left|P'(z)\right| \leq \frac{n}{2}\left\{\underset{|z|=1}{Max}\left|P(z)\right| - \underset{|z|=1}{Min}\left|P(z)\right|\right\}, \tag{1.5}$$

$$\underset{|z|=R}{Max}\left|P(z)\right| \leq \frac{R^n+1}{2}\underset{|z|=1}{Max}\left|P(z)\right| - \frac{R^n-1}{2}\underset{|z|=1}{Min}\left|P(z)\right|, \quad R \geq 1. \tag{1.6}$$

Jain [5] generalized both the inequalities (1.3) and (1.4) and proved that if $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$, $|z| = 1$ and $R \geq 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leq \frac{n}{2}\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\underset{|z|=R>1}{Max}\left|P(z)\right|, \tag{1.7}$$

and

$$\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| \leq \frac{1}{2}\left[\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right]\underset{|z|=R>1}{Max}\left|P(z)\right|. \tag{1.8}$$

Jain [6] obtained a result concerning minimum modulus of polynomials and proved the following:

Theorem A. *If $P \in \mathscr{P}_n$ and have all its zeros in $|z| \leq 1$, then for every real of complex β with $|\beta| \leq 1$,*

$$\underset{|z|=1}{Min}\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \geq n\left|1+\frac{\beta}{2}\right|\underset{|z|=1}{Min}\left|P(z)\right|. \tag{1.9}$$

As a refinement of inequalities (1.7) and (1.8), Jain [6] also established:

Theorem B. *If $P \in \mathscr{P}_n$ and have no zero in $|z| < 1$, then for every real of complex β with $|\beta| \leq 1$,*

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leq \frac{n}{2}\left[\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \underset{|z|=1}{Max}|P(z)| - \left\{\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\} \underset{|z|=1}{Min}|P(z)|\right], \tag{1.10}$$

and

$$\begin{aligned}\underset{|z|=1}{Max}&\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| \\ &\leq \frac{1}{2}\left[\left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|\right\}\underset{|z|=k}{Max}|P(z)| \right.\\ &\quad \left. - \left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| - \left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|\right\}\underset{|z|=k}{Min}|P(z)|\right].\end{aligned} \tag{1.11}$$

Inequalities (1.9) and (1.10) have recently appeared in [4] also.

More recently, S. Mezerji *et. al* [8] proved the following generalization of inequalities (1.6) and (1.7) which also leads to a refinement of (1.8).

Theorem C. *If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geq 1$, then for $|\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned}\underset{|z|=1}{Max}&\left|P(Rk^2z) + \beta\left(\frac{Rk+1}{k+1}\right)^n P(k^2z)\right| \\ &\leq \frac{1}{2}\left[\left\{k^n\left|R^n + \beta\left(\frac{Rk+1}{k+1}\right)^n\right| + \left|1 + \beta\left(\frac{Rk+1}{k+1}\right)^n\right|\right\}\underset{|z|=k}{Max}|P(z)| \right.\\ &\quad \left. - \left\{k^n\left|R^n + \beta\left(\frac{Rk+1}{k+1}\right)^n\right| - \left|1 + \beta\left(\frac{Rk+1}{k+1}\right)^n\right|\right\}\underset{|z|=k}{Min}|P(z)|\right].\end{aligned} \tag{1.12}$$

2. Lemmas

For the proof of our theorems we need the following lemmas.

Lemma 2.1. *If $P \in \mathscr{P}_n$ and $P(z)$ have all its zeros in $|z| \leq k$ where $k \leq 1$, then for every $R \geq 1$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+k}{1+k}\right)^n |P(z)|.$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \leq k$, $k \leq 1$ we write

$$P(z) = C\prod_{j=1}^{n}\left(z - r_j e^{i\theta_j}\right),$$

where $r_j \leq k \leq 1$. Now for $0 \leq \theta < 2\pi$, $R > 1$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right| &= \left\{\frac{R^2 + r_j^2 - 2Rr_jCos(\theta - \theta_j)}{1 + r_j^2 - 2r_jCos(\theta - \theta_j)}\right\}^{1/2}, \\ &\geq \left\{\frac{R + r_j}{1 + r_j}\right\}, \\ &\geq \left\{\frac{R+k}{1+k}\right\}, \text{for } \ j = 1, 2, \cdots, n.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{P(Re^{i\theta})}{P(e^{i\theta})}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{e^{i\theta} - r_j e^{i\theta_j}}\right|, \\ &\geq \prod_{j=1}^{n}\left(\frac{R+k}{1+k}\right), \\ &= \left(\frac{R+k}{1+k}\right)^n,\end{aligned}$$

for $0 \leq \theta < 2\pi$. This implies for $|z| = 1$ and $R > 1$,

$$|P(Rz)| \geq \left(\frac{R+k}{1+k}\right)^n |P(z)|,$$

which completes the proof of Lemma 2.1. □

Lemma 2.2. *If $P \in \mathscr{P}_n$ and $P(z)$ have no zero in $|z| < k, k \geq 1$, then for $|\alpha| \leq 1, |\beta| \leq 1, R > 1$ and $|z| \geq 1$*

$$\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right|$$
$$\leq k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right|, \tag{2.1}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Proof. By hypothesis, the polynomial $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, therefore $Q(z)$ is a polynomial of degree n having all its zeros in $|z| < (1/k) \leq 1$. As

$$k^n|Q(z)| = |P(k^2z)| \quad \text{for} \quad |z| = (1/k),$$

Applying Theorem 3.1 with $F(z)$ replaced by $k^nQ(z)$ we get for arbitrary real or complex numbers α, $\beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$

$$\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right|$$
$$\leq k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right|$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *If $P \in \mathscr{P}_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$ then for $\alpha, \beta \in \mathbb{C}$,with $|\alpha| \leq 1, |\beta| \leq 1, R \geq 1$, $k \geq 1$ and $|z| \geq 1$,*

$$\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right|$$
$$+ k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right|$$
$$\leq \left[k^n|z|^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right.$$
$$\left. + \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right]\underset{|z|=k}{Max}|P(z)|. \tag{2.2}$$

.

Proof. Let $M = \underset{|z|=k}{Max}|P(z)|$, then by Rouche's theorem the polynomial $F(z) = P(z) - \mu M$ does not vanish in $|z| < k$, for every $\mu \in \mathbb{C}$ with $|\mu| > 1$. Applying

Lemma 2.2 to polynomial $F(z)$, we get for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1, |\beta| \le 1$ and $|z| \ge 1$,

$$\begin{aligned}&\left|F(Rk^2z) - \alpha F(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}F(k^2z)\right|\\&\le k^n\left|H(Rz) - \alpha H(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}H(z)\right|,\end{aligned}$$

where $H(z) = z^n\overline{F(1/\overline{z})}$, replacing $F(z)$ by $P(z) - \mu M$ and $H(z)$ by $Q(z) - \overline{\mu}Mz^n$, we have for $|\alpha| \le 1, |\beta| \le 1$ and $|z| \ge 1$,

$$\begin{aligned}&\Bigg|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\\&\quad - \mu\left\{1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right\}M\Bigg|\\&\le k^n\Bigg|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\\&\quad - \overline{\mu}\left\{R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right\}Mz^n\Bigg|,\end{aligned} \tag{2.3}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Choosing argument of μ in the right hand side of inequality (2.3) such that

$$\begin{aligned}&k^n\Bigg|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\\&\quad - \overline{\mu}\left\{R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right\}Mz^n\Bigg|\\&= k^n|\overline{\mu}z^n|M\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\\&\quad - k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right|,\end{aligned} \tag{2.4}$$

which is possible by applying Corollary 3.4 to polynomial $Q(z)$,with replacing k by $\frac{1}{k}$, we get for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\begin{aligned}
&\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \\
&\quad - |\mu|\left|\left\{1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right\}M\right| \\
&\leq k^n|\overline{\mu}z^n|M\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right| \\
&\quad - k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right|.
\end{aligned}$$

Equivalently for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\begin{aligned}
&\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \\
&\quad + k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right| \\
&\leq M|\mu|\left[k^n|z|^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right. \\
&\quad \left. + \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right].
\end{aligned}$$

Letting $|\mu| \to 1$, we get the conclusion of lemma 2.3 and this completes proof of Lemma 2.3. □

3. Main Results

In this paper, we first present the following interesting result which yields a number of well-known polynomial inequalities as special cases.

Theorem 3.1. *If $F \in \mathscr{P}_n$ and $F(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for $|\alpha|\le 1, |\beta|\le 1$, $R>1$ *and* $|z|\ge 1$,

$$\begin{aligned}&\left|P(Rz)-\alpha P(z)+\beta\left\{\left(\frac{R+k}{k+1}\right)^n-|\alpha|\right\}P(z)\right|\\&\le\left|F(Rz)-\alpha F(z)+\beta\left\{\left(\frac{R+k}{k+1}\right)^n-|\alpha|\right\}F(z)\right|.\end{aligned}\tag{3.1}$$

The result is best possible and the equality holds for the polynomial $P(z)=e^{i\gamma}F(z)$ *where* $\gamma\in\mathbb{R}$ *and* $F(z)$ *is any polynomial having all its zeros in* $|z|\le k$.

Proof. Since polynomial $F(z)$ of degree n has all its zeros in $|z|\le k$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)|\le|F(z)| \text{ for } |z|=k,\tag{3.2}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z=ke^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z=ke^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then inequality (3.1) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)|<|F(z)| \text{ for } |z|>k.$$

Suppose $F(z)$ has m zeros on $|z|=k$ where $0\le m<n$, so that we can write

$$F(z)=F_1(z)F_2(z),$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z|=k$ and $F_2(z)$ is a polynomial of degree exactly $n-m$ having all its zeros in $|z|<k$. This implies with the help of inequality (3.2) that

$$P(z)=P_1(z)F_1(z),$$

where $P_1(z)$ is a polynomial of degree at most $n-m$. Again, from inequality (3.2), we have

$$|P_1(z)|\le|F_2(z)| \;\; for\; |z|=k,$$

where $F_2(z)\ne 0$ for $|z|=k$. Therefore for every real or complex number λ with $|\lambda|>1$, a direct application of Rouche's theorem shows that the zeros of the polynomial $P_1(z)-\lambda F_2(z)$ of degree $n-m\ge 1$ lie in $|z|<k$ hence the polynomial

$$G(z)=F_1(z)\left(P_1(z)-\lambda F_2(z)\right)=P(z)-\lambda F(z)$$

has all its zeros in $|z|\le k$ with at least one zero in $|z|<k$, so that we can write

$$f(z)=(z-te^{i\delta})H(z),$$

where $t < k$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \le k$. Applying Lemma 2.1 to the polynomial $H(z)$, we obtain for every $R > 1$ and $0 \le \theta < 2\pi$,

$$\begin{aligned}|G(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\ &\ge |Re^{i\theta} - te^{i\delta}| \left(\frac{R+k}{k+1}\right)^{n-1} |H(e^{i\theta})|, \\ &= \left(\frac{R+k}{k+1}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|e^{i\theta} - te^{i\delta}|} |(e^{i\theta} - te^{i\delta})H(e^{i\theta})|, \\ &\ge \left(\frac{R+k}{k+1}\right)^{n-1} \left(\frac{R+t}{1+t}\right) |G(e^{i\theta})|.\end{aligned}$$

This implies for $R > 1$ and $0 \le \theta < 2\pi$,

$$\left(\frac{1+t}{R+t}\right) |G(Re^{i\theta})| \ge \left(\frac{R+k}{k+1}\right)^{n-1} |G(e^{i\theta})|. \tag{3.3}$$

Since $R > 1 > t$ so that $G(Re^{i\theta}) \ne 0$ for $0 \le \theta < 2\pi$ and $\frac{1+k}{k+R} > \frac{1+t}{R+t}$, from inequality (3.3), we obtain

$$|G(Re^{i\theta})| > \left(\frac{R+k}{k+1}\right)^{n} |G(e^{i\theta})|, \quad R > 1 \text{ and } 0 \le \theta < 2\pi. \tag{3.4}$$

Equivalently,

$$|G(Rz)| > \left(\frac{R+k}{k+1}\right)^{n} |G(z)|$$

for $|z| = 1$ and $R > 1$. Hence for every real or complex number α with $|\alpha| \le 1$ and $R > 1$, we have

$$\begin{aligned}|G(Rz) - \alpha G(z)| &\ge |G(Rz)| - |\alpha|\,|G(z)| \qquad (3.5)\\ &> \left\{\left(\frac{R+k}{k+1}\right)^{n} - |\alpha|\right\} |G(z)|, \text{ for } |z| = 1.\end{aligned}$$

Also, inequality (3.4) can be written in the form

$$|G(e^{i\theta})| < \left(\frac{k+1}{R+k}\right)^{n} |G(Re^{i\theta})|, \tag{3.6}$$

for every $R > 1$ and $0 \le \theta < 2\pi$. Since $G(Re^{i\theta}) \ne 0$ and $\left(\frac{k+1}{R+k}\right)^{n} < 1$, from inequality (3.6), we obtain for $0 \le \theta < 2\pi$ and $R > 1$,

$$|G(e^{i\theta})| < |G(Re^{i\theta}).$$

That is,

$$|G(z)| < |G(Rz)| \text{ for } |z| = 1.$$

Since all the zeros of $G(Rz)$ lie in $|z| \leq (k/R) < 1$, a direct application of Rouche's theorem shows that the polynomial $G(Rz) - \alpha G(z)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Rouche's theorem again, it follows from (3.5) that for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$, all the zeros of the polynomial

$$\begin{aligned} T(z) &= G(Rz) - \alpha G(z) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} G(z) \\ &= \left[P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} P(z) \right] \\ &\quad - \lambda \left[F(Rz) - \alpha F(z) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} F(z) \right] \end{aligned}$$

lie in $|z| < 1$. This implies

$$\begin{aligned} &\left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} P(z) \right| \\ &\leq \left| F(Rz) - \alpha F(z) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} F(z) \right| \end{aligned} \tag{3.7}$$

for $|z| \geq 1$ and $R > 1$. If inequality (3.7) is not true, then there a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\begin{aligned} &\left| P(Rz_0) - \alpha P(z_0) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} P(z_0) \right| \\ &> \left| F(Rz_0) - \alpha F(z_0) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} F(z_0) \right| \end{aligned}$$

But all the zeros of $F(Rz)$ lie in $|z| < (k/R) < 1$, therefore, it follows (as in case of $G(z)$) that all the zeros of $F(Rz) - \alpha F(z) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} F(z)$ lie in $|z| < 1$. Hence

$$F(Rz_0) - \alpha F(z_0) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} F(z_0) \neq 0$$

with $|z_0| \geq 1$. We take

$$\lambda = \frac{P(Rz_0) - \alpha P(z_0) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} P(z_0)}{F(Rz_0) - \alpha F(z_0) + \beta \left\{ \left(\frac{R+k}{k+1} \right)^n - |\alpha| \right\} F(z_0)},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $T(z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus (3.7) holds for $|\alpha| \leq 1$, $|\beta| \leq 1$, $|z| \geq 1$, and $R > 1$. □

If we choose $\alpha = 0$ in Theorem 3.1, we get the following:

Corollary 3.2. *If $F \in \mathscr{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$\left|P(Rz) + \beta\left(\frac{R+k}{k+1}\right)^n P(z)\right| \leq \left|F(Rz) + \beta\left(\frac{R+k}{k+1}\right)^n F(z)\right|. \tag{3.8}$$

The result is sharp, and the equality holds for the polynomial $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$.

Next take $\alpha = 1$ in inequality (3.1) and divide the two sides by $R - 1$ and then make $R \to 1$, we get:

Corollary 3.3. *If $F \in \mathscr{P}_n$ and $F(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$\left|zP'(z) + \frac{n\beta}{k+1}P(z)\right| \leq \left|zF'(z) + \frac{n\beta}{k+1}F(z)\right|. \tag{3.9}$$

The result is sharp, and the equality holds for the polynomial $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$ and $F(z)$ is any polynomial having all its zeros in $|z| \leq k$.

Setting $F(z) = z^n M/k^n$, where $M = \underset{|z|=k}{Max}|P(z)|$ in Theorem 3.1, we get the following result:

Corollary 3.4. *If $P \in \mathscr{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $k \leq 1$, $R > 1$ and $|z| \geq 1$,*

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+k}{k+1}\right)^n - |\alpha|\right\}P(z)\right|$$
$$\leq \frac{|z|^n}{k^n}\left|R^n - \alpha + \beta\left\{\left(\frac{R+k}{k+1}\right)^n - |\alpha|\right\}\right| \underset{|z|=k\leq 1}{Max}|P(z)|. \tag{3.10}$$

The result is best possible and equality in (3.10) *holds for $P(z) = az^n, a \neq 0$.*

Again, if we choose $\alpha = 1$ in Corollary 3.4, and divide the two sides of inequality (3.10) by $R - 1$ and then making $R \to 1$, we get:

Corollary 3.5. *If $P \in \mathscr{P}_n$ then for $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $k \leq 1$, $R > 1$ and $|z| \geq 1$,*

$$\left|zP'(z) + \frac{n\beta}{k+1}P(z)\right| \leq \frac{n|z|^n}{k^n}\left|1 + \frac{\beta}{k+1}\right| \underset{|z|=k}{Max}|P(z)|. \tag{3.11}$$

For $\alpha = 0$ (3.10) *reduces to*

$$\left|P(Rz) + \beta\left(\frac{R+k}{k+1}\right)^n P(z)\right| \leq \frac{|z|^n}{k^n}\left|R^n + \beta\left(\frac{R+k}{k+1}\right)^n\right| \underset{|z|=k}{Max}|P(z)| \tag{3.12}$$

for $|z| \geq 1$.

The result is sharp and equality in (3.11) and (3.12) *holds for $P(z) = az^n, a \neq 0$.*

The following compact generalization of inequalities (1.1) and (1.2) immediately follows from Theorem 3.1, by taking $k = 1$ and $\beta = 0$ in (3.10).

Corollary 3.6. *If $P \in \mathscr{P}_n$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > 1$ and $|z| \geq 1$,*

$$|P(Rz) - \alpha P(z)| \leq |z|^n |R^n - \alpha| \underset{|z|=1}{Max}|P(z)|. \tag{3.13}$$

The result is best possible as shown by $P(z) = az^n, a \neq 0$.

Remark 3.7. *For $\alpha = 0$,* (3.13) *reduces to* (1.2). *For $\alpha = 1$, if we divide the two sides of* (3.13) *by $R - 1$ and make $R \to 1$, we get inequality* (1.1).

If we take $\beta = 0$ in Theorem 3.1, then inequality (3.1) reduces to following:

Corollary 3.8. *If $F \in \mathscr{P}_n$ and $F(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for $|\alpha| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$|P(Rz) - \alpha P(z)| \leq |F(Rz) - \alpha F(z)|. \tag{3.14}$$

The result is sharp and equality holds for $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$.

Dividing the two sides of inequality (3.14) by $R-1$ with $\alpha = 1$ and making $R \to 1$, we get:

Corollary 3.9. *If $F \in \mathscr{P}_n$ and $F(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for $|z| \geq 1$,

$$\left|P'(z)\right| \leq \left|F'(z)\right|. \tag{3.15}$$

The result is sharp and equality holds for $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$.

Next, we present the following result which includes Theorem A as a special case.

Theorem 3.10. *If $P \in \mathscr{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned} \underset{|z|=1}{Min} &\left| P(Rk^2 z) - \alpha P(k^2 z) + \beta \left\{ \left(\frac{Rk+1}{k+1} \right)^n - |\alpha| \right\} P(k^2 z) \right| \\ &\geq k^n \left| R^n - \alpha + \beta \left\{ \left(\frac{Rk+1}{k+1} \right)^n - |\alpha| \right\} \right| \underset{|z|=k}{Min} |P(z)|. \end{aligned} \tag{3.16}$$

The result is best possible as shown by $P(z) = az^n, a \neq 0$.

Proof. Let $m = \underset{|z|=k}{Min}|P(z)|$. If $P(z)$ has zeros on $|z| = k$, $k \geq 1$, then the result is trivially true. Assume all the zeros of $P(z)$ lie in $|z| < k, k \geq 1$, therefore all the zeros of polynomial $P(k^2 z)$ lie in $|z| < (1/k)$ where $(1/k) \leq 1$ and $m = \underset{|z|=1/k}{Min} \left|P(k^2 z)\right| > 0$. For every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, then it follows by Rouche's theorem that the polynomial $f(z) = P(k^2 z) - \lambda m k^n z^n$ have all its zeros in lie $|z| < 1/k$, where $1/k \leq 1$, applying Lemma 2.1 to $f(z)$, we have

$$|f(Rz)| \geq \left(\tfrac{Rk+1}{1+k}\right)^n |f(z)| \quad \text{for} \quad |z| = 1.$$

Which implies,

$$|f(Rz)| > |f(z)| \quad \text{for} \quad R > 1 \quad \text{and} \quad |z| = 1.$$

Thus by Rouche's theorem for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ all the zeros of $F(z) = f(Rz) - \alpha f(z)$ lie in $|z| < 1$. and, we have

$$\begin{aligned}|f(Rz) - \alpha f(z)| &\geq |f(Rz)| - |\alpha f(z)| \\ &> \left\{ \left(\frac{Rk+1}{k+1}\right)^n - |\alpha| \right\} |f(z)|\end{aligned}$$

for $|z| = 1$ and $R > 1$. Again by Rouche's theorem for $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, the zeros of the polynomial

$$\begin{aligned}g(z) &= f(Rz) - \alpha f(z) + \beta \left\{ \left(\frac{R+k}{k+1}\right)^n - |\alpha| \right\} f(z) \\ &= \left[P(Rk^2 z) - \alpha P(k^2 z) + \beta \left\{ \left(\frac{Rk+1}{k+1}\right)^n - |\alpha| \right\} P(k^2 z) \right] \\ &\quad - \lambda m k^n z^n \left[R^n - \alpha + \beta \left\{ \left(\frac{Rk+1}{k+1}\right)^n - |\alpha| \right\} \right]\end{aligned}$$

lie in $|z| < 1$. This gives

$$\begin{aligned}&\left| P(Rk^2 z) - \alpha P(k^2 z) + \beta \left\{ \left(\frac{Rk+1}{k+1}\right)^n - |\alpha| \right\} P(k^2 z) \right| \\ &\geq k^n |z|^n \left| R^n - \alpha + \beta \left\{ \left(\frac{Rk+1}{k+1}\right)^n - |\alpha| \right\} \right| m \qquad (3.17)\end{aligned}$$

for $|z| \geq 1$.

If inequality (3.17) is not true, then there exists $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$ such that

$$\begin{aligned}&\left| P(Rk^2 z_0) - \alpha P(k^2 z_0) + \beta \left\{ \left(\frac{Rk+1}{k+1}\right)^n - |\alpha| \right\} P(k^2 z_0) \right| \\ &< k^n |z_0|^n \left| R^n - \alpha + \beta \left\{ \left(\frac{Rk+1}{k+1}\right)^n - |\alpha| \right\} \right| m\end{aligned}$$

We take

$$\lambda = \frac{\left[P(Rk^2z_0) - \alpha P(k^2z_0) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z_0)\right]}{mk^n|z_0|^n\left[R^n - \alpha + \beta\left\{\left(\frac{R+k}{k+1}\right)^n - |\alpha|\right\}\right]},$$

then $|\lambda| < 1$ and with this choice of λ, we have $g(z_0) = 0$, with $|z_0| \geq 1$, which is contradiction, since all the zeros of $g(z)$ lie in $|z| < 1$. Hence, we have

$$\begin{aligned}&\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right|\\ &\geq k^n|z|^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\underset{|z|=k}{Min}|P(z)|\end{aligned}$$

for $|z| \geq 1, R > 1$ which immediately leads to inequality (3.17) and this completes the proof of Theorem 3.10. □

If we divide the two sides of inequality (3.16) by $R-1$ with $\alpha = 1$ and then making $R \to 1$ we get:

Corollary 3.11. *If $P(z) \in \mathscr{P}_n$ and have all its zeros in $|z| \leq k$ where $k \geq 1$ then for $|\beta| \leq 1$ and $R > 1$*

$$\underset{|z|=1}{Min}\left|kzP'(k^2z) + \frac{n\beta}{k+1}P(k^2z)\right| \geq nk^{n-1}\left|1 + \frac{n\beta k}{k+1}\right|\underset{|z|=k}{Min}|P(z)| \tag{3.18}$$

. The result is sharp.

Remark 3.12. *For $k = 1$, inequality* (3.18) *reduces to Theorem A.*

Setting $\beta = 0$ in theorem 3.10, we obtain :

Corollary 3.13. *If $P \in \mathscr{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$ then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > 1$,*

$$\underset{|z|=1}{Min}\left|P(Rk^2z) - \alpha P(k^2z)\right| \geq k^n\left|R^n - \alpha\right|\underset{|z|=k}{Min}|P(z)|. \tag{3.19}$$

.

For polynomials $P \in \mathscr{P}_n$ having no zero in $|z| < k$, we establish the following result which leads to a compact generalization of inequalities (1.7) and (1.8).

Theorem 3.14. *If $P(z) \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < k, k \geq 1$ then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,*

$$\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \leq \frac{1}{2}\left[\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right| + k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||z|^n\right] \underset{|z|=k}{Max}|P(z)| \tag{3.20}$$

Proof. Since $P(z)$ does not vanish in $|z| < k,\ k \geq 1$, by Lemma 2.2, we have for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1$ and $R > 1$,

$$\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \leq k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right| \tag{3.21}$$

for $|z| \geq 1$ and where $Q(z) = z^n\overline{P(1/\overline{z})}$. Inequality (3.21) in conjunction with Lemma 2.3 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1$ and $R > 1$,

$$\begin{aligned} 2&\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \\ &\leq \left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \\ &\quad + k^n\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right| \\ &\leq \left[k^n|z|^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right. \\ &\quad \left. + \left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right] \underset{|z|=k}{Max}|P(z)|, \end{aligned}$$

for $|z| \geq 1$.. This completes the proof. □

Remark 3.15. *If we take $\alpha = k = 1$ in Theorem 3.14 and divide two sides of inequality* (3.20) *by $R-1$ and then make $R \to 1$, we get inequality* (1.7)*, whereas inequality* (1.8) *follows from Theorem 3.14, when $\alpha = 0$ and $k = 1$.*

Remark 3.16. *If we choose $k = 1$ in inequality* (3.20)*, then Theorem 3.14 reduces to the result proved by Aziz and Rather* [3].

As a refinement of Theorem 3.14 and a generalization of Theorem C, we finally prove the following result, which provides a compact generalization of inequalities (1.7), (1.8) and (1.12) as well.

Theorem 3.17. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < k, k \geq 1$ then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned}
&\underset{|z|=1}{Max}\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \\
&\leq \frac{1}{2}\left[\left\{k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right.\right. \\
&\quad \left. + \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right\}\underset{|z|=k}{Max}|P(z)| \qquad (3.22)\\
&\quad - \left\{k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right. \\
&\quad \left.\left. - \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right\}\underset{|z|=k}{Min}|P(z)|\right].
\end{aligned}$$

Proof. Let $m = \underset{|z|=k}{Min}|P(z)|$. If $P(z)$ has a zero on $|z| = k$, then the result follows from Theorem 3.14. Therefore, we assume that $P(z)$ has all its zeros in $|z| > k$ where $k \geq 1$ so that $m > 0$. Now for every λ with $|\lambda| < 1$, it follows by Rouche's theorem, that the polynomial $h(z) = P(z) - \lambda m$ does not vanish in $|z| < k$. Applying Lemma 2.2 to the polynomial $h(z)$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$, and $|z| \geq 1$

$$\begin{aligned}
&\left|h(Rk^2z) - \alpha h(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}h(k^2z)\right| \\
&\leq k^n\left|Q_1(Rz) - \alpha Q_1(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q_1(z)\right|
\end{aligned}$$

where $Q_1(z) = z^n\overline{h(1/\overline{z})}$. Equivalently,

$$\begin{aligned}&\Bigg|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\\
&\quad - \lambda\left[1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right]m\Bigg|\\
&\leq k^n\Bigg|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\\
&\quad - \overline{\lambda}\left[R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right]m\Bigg| \text{ for } |z| = 1. \qquad (3.23)\end{aligned}$$

Since all the zeros of $Q(z/k^2)$ lie in $|z| \leq k$, $k \geq 1$, then by Theorem 3.10 applied to $Q(z/k^2)$, we have for $R > 1$,

$$\begin{aligned}&\underset{|z|=1}{Min}\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right|\\
&\geq k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\underset{|z|=k}{Min}|Q(z/k^2)|\\
&= \left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\underset{|z|=k}{Min}|P(z)|\\
&= \left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|m. \qquad (3.24)\end{aligned}$$

Now, choosing the argument of λ on the right hand side of inequality 3.23 such that

$$\begin{aligned}&k^n\Bigg|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\\
&\quad - \overline{\lambda}\left\{R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right\}mz^n\Bigg|\\
&= k^n\Bigg[\left|Q(Rz) - \alpha Q(z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}Q(z)\right|\\
&\quad - |\overline{\lambda}|\left|\left\{R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right\}m\right|\Bigg],\end{aligned}$$

for $|z|=1$, which is possible by inequality (3.24). We get for $|z|=1$,

$$\begin{aligned}&\left|P(Rk^2z)-\alpha P(k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}P(k^2z)\right|\\&\quad-|\lambda|\left|\left\{1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right\}m\right|\\&\le k^n\left|Q(Rz)-\alpha Q(z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}Q(z)\right|\\&\quad-|\lambda|\,k^n\left|\left\{R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right\}m\right|\end{aligned}$$

Equivalently for $|z|=1, R>1$, we have

$$\begin{aligned}&\left|P(Rk^2z)-\alpha P(k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}P(k^2z)\right|\\&\quad-k^n\left|Q(Rk^2z)-\alpha Q(k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}Q(k^2z)\right|\\&\le|\lambda|\left[\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right.\\&\quad\left.-\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right]m.\end{aligned}\tag{3.25}$$

Letting $|\lambda|\to 1$ in inequality (3.25), we obtain for all $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\le 1, |\beta|\le 1, R>1$ and $|z|=1$,

$$\begin{aligned}&\left|P(Rk^2z)-\alpha P(k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}P(k^2z)\right|\\&\quad-k^n\left|Q(Rk^2z)-\alpha Q(k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}Q(k^2z)\right|\\&\le\left[\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right.\\&\quad\left.-\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right]m.\end{aligned}\tag{3.26}$$

Inequality (3.26) in conjunction with Lemma (2.3) gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1, R > 1$ and $|z| = 1$,

$$\begin{aligned} 2\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\} P(k^2z)\right| \\ \leq \Bigg[\left\{k^n \left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right. \\ \left. + \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right\} \underset{|z|=k}{Max}|P(z)| \\ - \left\{k^n \left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right. \\ \left. - \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right\} \underset{|z|=k}{Min}|P(z)|\Bigg]. \end{aligned}$$

Which is equivalent to inequality (3.22) and thus completes the proof of theorem 3.17.

□

For $\alpha = 0$ Theorem 3.17 reduces to Theorem C.

If we take $\alpha = 1$ divide the two sides of inequality (3.22) by $R - 1$ and then letting $R \to 1$, we get:

Corollary 3.18. *If $P(z) \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| \leq k$ where $k \geq 1$, then for $|\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned} \underset{|z|=1}{Max}\left|kzP'(k^2z) + \frac{n\beta}{k+1}P(k^2z)\right| \\ \leq \frac{n}{2}\Bigg[\left\{k^{n-1}\left|1 + \frac{\beta k}{k+1}\right| + \left|\frac{\beta}{k+1}\right|\right\} \underset{|z|=k}{Max}|P(z)| \\ - \left\{k^{n-1}\left|1 + \frac{\beta k}{k+1}\right| - \left|\frac{\beta}{k+1}\right|\right\} \underset{|z|=k}{Min}|P(z)|\Bigg]. \end{aligned} \tag{3.27}$$

Remark 3.19. *For $k = 1$, inequality (3.27) reduces to Theorem B.*

The following result immediately follows from Theorem 3.17 by taking $\beta = 0$ and $k = 1$.

Corollary 3.20. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1, |\beta| \le 1$ and $R > 1$,*

$$\underset{|z|=1}{Max}\,|P(Rz) - \alpha P(z)| \le \left(\frac{|R^n - \alpha| + |1-\alpha|}{2}\right)\underset{|z|=1}{Max}\,|P(z)| - \left(\frac{|R^n - \alpha| - |1-\alpha|}{2}\right)\underset{|z|=1}{Min}\,|P(z)|\,. \tag{3.28}$$

The result is sharp and extremal polynomial is $P(z) = az^n + b, |a| = |b| \neq 0$.

Remark 3.21. *For $\alpha = 0$, inequality (3.28) reduces to inequality (1.6). Also if we divide the two sides if inequality (3.28) by $R - 1$ with $\alpha = 1$ and let $R \to 1$, we get inequality (1.5).*

References

[1] N. C. Ankeny and T. J. Rivlin, *On a theorm of S.Bernstein,* Pacific J. Math., **5** (1955), 849 - 852.

[2] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivatives,* J. Approx. Theory **54** (1998), 306 -311.

[3] A. Aziz and N. A. Rather, *Some compact generalization of Bernstien-type inequalities for polynomials,* Math. Inequal. Appl., **7** (3) (2004), 393 - 403.

[4] K. K. Dewan and Hans, *Generalization of certain well-known polynomial inequalities,* J. Math. Anal. Appl., **363** (2010) 3841.

[5] V. K. Jain, *Generalization of certain well-known inequalities for polynomials,* Glas. Mat. Ser. III, **32** (1997), 45 - 52.

[6] V. K. Jain, *Inequalities for a polynomial and its derivative,* Proc. Indian Acad. Sci. Math. Sci., **110**, No. 2, May (2000), 137-146.

[7] P. D. Lax, *Proof of a conjecture of P.Erdös on the derivative of a polynomial,* Bull. Amer. Math. Soc., **50** (1944), 509-513

[8] S. Mazerji, A. Baseri and A. Zireh, *Generalization of certain inequalities for a polynomial and its derivative,* Lobachevskii J. Math., **33** (2012),68-74.

[9] G. V. Milovanovic, D. S. Mitrinovic and Th. M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities*, Zeros, World scientific Publishing Co., Singapore, (1994).

[10] G. Pólya an G. Szegö, *Aufgaben und lehrsätze aus der Analysis,* Springer-Verlag, Berlin (1925).

[11] Q. I. Rahman and G. Schmessier, *Analytic theory of polynomials,* Claredon Press, Oxford, 2002.

[12] A. C. Schaffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions,* Bull. Amer. Math. Soc., **47**(1941), 565-579.