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A VORONOVSKAYA TYPE THEOREM FOR GENERAL SINGULAR OPERATORS WITH APPLICATIONS

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Abstract. In this article we continue with the study of approximation properties of smooth general singular integral operators over the real line. We produce Voronovskaya type results and give some quantitative results regarding the rate of convergence of these singular integral operators. We give particular applications to trigonometric singular integral operators.

1. Introduction

Here we define the smooth general singular integral operators $\Theta_{r,n,\xi}(f;x)$ of which basic approximation properties were studied in [1]-[4]. We are also motivated by [6]-[8].

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum\limits_{i=1}^{r} (-1)^{r-i}\binom{r}{i} i^{-n}, & j = 0, \end{cases} \tag{1.1}$$

that is $\sum\limits_{j=0}^{r} \alpha_j = 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. For each $\xi > 0$, μ_ξ is a probability Borel measure on $\mathbb{R}$.

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We define for $x \in \mathbb{R}$ the integrals

$$\Theta_{r,n,\xi}(f;x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) d\mu_\xi(t). \tag{1.2}$$

Assume $\Theta_{r,n,\xi}(f;x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$.

Note 1.1. The operators $\Theta_{r,n,\xi}$ are not in general positive, see [3].

2. Results

We present the main result.

Theorem 2.1. Assume $\xi^{-n} \int_{-\infty}^{\infty} |t|^n \, d\mu_\xi(t) \leq \rho$, $\forall \xi > 0$, $\rho > 0$ and $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, n-1$. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(n)}$ exists, $n \in \mathbb{N}$, and is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$\Theta_{r,n,\xi}(f;x) - f(x) = \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) c_{k,\xi} + o(\xi^{n-\gamma}). \tag{2.1}$$

When $n = 1$ the sum collapses.

Proof. We notice that $\Theta_{r,n,\xi}(c,x) = c$, for any c constant, and therefore we have

$$\Theta_{r,n,\xi}(f;x) - f(x) = \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} (f(x+jt) - f(x)) d\mu_\xi(t) \right). \tag{2.2}$$

Using Taylor's formula for f, we have

$$f(x+jt) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (jt)^k + \frac{f^{(n)}(\theta)}{n!} (jt)^n, \tag{2.3}$$

with $\theta = \theta(j)$ between x and $x + jt$. From (2.2) and (2.3), we obtain

$$\begin{aligned} &\Theta_{r,n,\xi}(f;x) - f(x) \\ &= \sum_{j=1}^{r} \alpha_j \left(\int_{-\infty}^{\infty} \left(\left[\sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (jt)^k + \frac{f^{(n)}(\theta)}{n!} (jt)^n \right] - f(x) \right) d\mu_\xi(t) \right) \\ &= \sum_{j=1}^{r} \alpha_j \left(\int_{-\infty}^{\infty} \left[\sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} j^k t^k + \frac{f^{(n)}(\theta)}{n!} j^n t^n \right] d\mu_\xi(t) \right) \qquad (2.4) \\ &= \sum_{j=1}^{r} \alpha_j \left[\sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} j^k c_{k,\xi} + \frac{j^n}{n!} \int_{-\infty}^{\infty} f^{(n)}(\theta) t^n d\mu_\xi(t) \right]. \qquad (2.5) \end{aligned}$$

Thus we have

$$\begin{aligned}\Psi &:= \Theta_{r,n,\xi}(f;x)-f(x)-\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}\left(\sum_{j=1}^{r}\alpha_j j^k\right)c_{k,\xi}\\ &= \sum_{j=1}^{r}\alpha_j\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\theta)t^n d\mu_\xi(t).\end{aligned} \tag{2.6}$$

We find

$$\begin{aligned}\Delta_\xi &:= \frac{\Psi}{\xi^n}\\ &= \frac{1}{\xi^n}\left(\sum_{j=1}^{r}\alpha_j\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\theta)t^n d\mu_\xi(t)\right)\\ &= \frac{1}{n!\xi^n}\left(\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j j^n f^{(n)}(\theta)\right)t^n d\mu_\xi(t)\right)\\ &= \frac{1}{n!\xi^n}\left(\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f^{(n)}(\theta)\right)t^n d\mu_\xi(t)\right).\end{aligned} \tag{2.7}$$

Call

$$\Phi_n(x,t) := \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f^{(n)}(\theta). \tag{2.8}$$

Thus

$$\Delta_\xi = \frac{1}{n!\xi^n}\left[\int_{-\infty}^{\infty}\Phi_n(x,t)t^n d\mu_\xi(t)\right]. \tag{2.9}$$

Since we assumed that $f^{(n)}$ exists and it is bounded we obtain

$$\left\|f^{(n)}\right\|_\infty < M, \text{ for some } M>0.$$

Therefore

$$|\Phi_n(x,t)| \le \left(\sum_{j=1}^{r}\binom{r}{j}\right)M = (2^r-1)M, \tag{2.10}$$

and

$$|\Delta_\xi| \le \frac{(2^r-1)M}{n!\xi^n}\left(\int_{-\infty}^{\infty}|t|^n d\mu_\xi(t)\right) \le \frac{(2^r-1)M}{n!}\rho =: \lambda. \tag{2.11}$$

Consequently, we get $(0 < \gamma \leq 1)$

$$\frac{|\Psi|}{\xi^{n-\gamma}} \leq \lambda \xi^{\gamma} \to 0, \text{ as } \xi \to 0+, \tag{2.12}$$

proving the claim. □

Corollary 2.2. (n=1 case) Let f be such that f'exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$. Here assume $\xi^{-1} \int_{-\infty}^{\infty} |t| \, d\mu_\xi(t) \leq \rho$, $\forall \xi > 0$, $\rho > 0$. Then

$$\Theta_{r,1,\xi}(f; x) - f(x) = o\left(\xi^{1-\gamma}\right). \tag{2.13}$$

Proof. Apply Theorem 2.1 for $n = 1$. □

Corollary 2.3. (n=2 case) Let f be such that f'' exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$. Here assume $\xi^{-2} \int_{-\infty}^{\infty} t^2 d\mu_\xi(t) \leq \rho$, $\forall \xi > 0$, $\rho > 0$. Then

$$\Theta_{r,2,\xi}(f; x) - f(x) = f'(x) \left(\sum_{j=1}^{r} \alpha_j j \right) c_{1,\xi} + o\left(\xi^{2-\gamma}\right). \tag{2.14}$$

Proof. Apply Theorem 2.1 for $n = 2$. □

Corollary 2.4. (n=3 case) Let f be such that $f^{(3)}$ exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$, with $\xi^{-3} \int_{-\infty}^{\infty} |t|^3 \, d\mu_\xi(t) \leq \rho$, $\forall \xi > 0$, $\rho > 0$. Then

$$\begin{aligned} &\Theta_{r,3,\xi}(f; x) - f(x) \\ &= f'(x) \left(\sum_{j=1}^{r} \alpha_j j \right) c_{1,\xi} + \frac{f''(x)}{2} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) c_{2,\xi} + o(\xi^{3-\gamma}). \end{aligned} \tag{2.15}$$

Proof. Apply Theorem 2.1 for $n = 3$. □

Corollary 2.5. (n=4 case) Let f be such that $f^{(4)}$ exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$, with $\xi^{-4} \int_{-\infty}^{\infty} t^4 d\mu_\xi(t) \leq \rho$, $\forall \xi > 0$, $\rho > 0$. Then

$$\begin{aligned} \Theta_{r,4,\xi}(f; x) - f(x) &= f'(x) \left(\sum_{j=1}^{r} \alpha_j j \right) c_{1,\xi} + \frac{f''(x)}{2} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) c_{2,\xi} \\ &\quad + \frac{f'''(x)}{6} \left(\sum_{j=1}^{r} \alpha_j j^3 \right) c_{3,\xi} + o(\xi^{4-\gamma}). \end{aligned} \tag{2.16}$$

Proof. Apply Theorem 2.1 for $n = 4$. □

3. Applications to General Trigonometric Singular Operators

We make

Remark 3.1. We need the following preliminary result.

Let p and m be integers with $1 \leq p \leq m$. We define the integral

$$I(m;p) := \int_{-\infty}^{\infty} \frac{(\sin x)^{2m}}{x^{2p}} dx = 2\int_{0}^{\infty} \frac{(\sin x)^{2m}}{x^{2p}} dx. \tag{3.1}$$

This is an (absolutely) convergent integral.

According to [5], page 210, item 1033, we obtain

$$I(m;p) = \pi \frac{(-1)^p (2m)!}{4^{m-p}(2p-1)!} \sum_{k=1}^{m} (-1)^k \frac{k^{2p-1}}{(m-k)!(m+k)!}. \tag{3.2}$$

In particular, for $p = m$ the above formula becomes

$$\int_0^{\infty} \frac{(\sin x)^{2m}}{x^{2m}} dx = \pi(-1)^m m \sum_{k=1}^{m} (-1)^k \frac{k^{2m-1}}{(m-k)!(m+k)!}. \tag{3.3}$$

In this section we apply the general theory of this article to the trigonometric smooth general singular integral operators $T_{r,n,\xi}(f,x)$ defined as follows. Let $\xi > 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable and $\beta \in \mathbb{N}$. We define for $x \in \mathbb{R}$ the integrals

$$T_{r,n,\xi}(f;x) := \frac{1}{W} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) \left(\frac{\sin(t/\xi)}{t} \right)^{2\beta} dt, \tag{3.4}$$

where, from (3.3),

$$\begin{aligned} W &= \int_{-\infty}^{\infty} \left(\frac{\sin(t/\xi)}{t} \right)^{2\beta} dt \\ &= 2\xi^{1-2\beta} \int_0^{\infty} \left(\frac{\sin t}{t} \right)^{2\beta} dt. \\ &= 2\xi^{1-2\beta} \pi (-1)^{\beta} \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}. \end{aligned} \tag{3.5}$$

We assume $T_{r,n,\xi}(f;x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$.

Note 3.2. The operators $T_{r,n,\xi}$ are not in general positive, see [3].

Let $\lfloor \cdot \rfloor$ denote the integer part of a real number and let

$$c_{k,\xi} := \frac{1}{W} \int_{-\infty}^{\infty} t^k \left(\frac{\sin (t/\xi)}{t} \right)^{2\beta} dt, k = 1, \ldots, n-1.$$

We present the main result of this section.

Proposition 3.3. Let $\beta \in \mathbb{N}$, $\beta > \frac{n+1}{2}$, $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists, $n \in \mathbb{N}$, which is bounded, and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,n,\xi}(f;x) - f(x) = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{f^{(2k)}(x)}{(2k)!} \left(\sum_{j=1}^{r} \alpha_j j^{2k} \right) c_{2k,\xi} + o(\xi^{n-\gamma}). \tag{3.6}$$

When $n = 1, 2$ the sum collapses.

Proof. As in the proof of Theorem 6 from [4], inequality (54), for $\beta > \frac{j+1}{2}$ we have

$$\int_0^{\infty} t^j \left(\frac{\sin t}{t} \right)^{2\beta} dt < \infty. \tag{3.7}$$

Hence, from (3.5), we have

$$\begin{aligned}
\xi^{-n} \frac{1}{W} \int_{-\infty}^{\infty} |t|^n \left(\frac{\sin (t/\xi)}{t} \right)^{2\beta} dt &= \frac{2\xi^{-n}}{W} \int_0^{\infty} t^n \left(\frac{\sin (t/\xi)}{t} \right)^{2\beta} dt \\
&= \frac{2\xi^{-n}}{W} \int_0^{\infty} \xi^{n-2\beta+1} t^n \left(\frac{\sin t}{t} \right)^{2\beta} dt \\
&= \frac{1}{\int_0^{\infty} \left(\frac{\sin t}{t} \right)^{2\beta} dt} \int_0^{\infty} t^n \left(\frac{\sin t}{t} \right)^{2\beta} dt \\
&< \infty, \quad \forall \xi > 0.
\end{aligned}$$

Therefore, there exists some $\rho > 0$ such that

$$\xi^{-n} \frac{1}{W} \int_{-\infty}^{\infty} |t|^n \left(\frac{\sin (t/\xi)}{t} \right)^{2\beta} dt \leq \rho, \quad \forall \xi > 0. \tag{3.8}$$

For k odd positive integer, $k \leq n-1$, we have that

$$c_{k,\xi} = 0.$$

Therefore, using Theorem 2.1, we obtain

$$\begin{aligned} T_{r,n,\xi}(f;x) - f(x) &= \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) c_{k,\xi} + o(\xi^{n-\gamma}) \\ &= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{f^{(2k)}(x)}{(2k)!} \left(\sum_{j=1}^{r} \alpha_j j^{2k} \right) c_{2k,\xi} + o(\xi^{n-\gamma}), \end{aligned}$$

proving the claim of the proposition. □

Corollary 3.4. (n=1 case) Let $\beta \in \mathbb{N}$, $\beta \geq 2$ and let f be such that f'exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,1,\xi}(f;x) - f(x) = o\left(\xi^{1-\gamma}\right). \tag{3.9}$$

Proof. As in the proof of Proposition 3.3, we have $\xi^{-1} \frac{1}{W} \int_{-\infty}^{\infty} |t| \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt$ $\leq \rho$, for some $\rho > 0$, $\forall \xi > 0$. Applying Proposition 3.3 for $n = 1$ we obtain the claim of the corollary. □

Corollary 3.5. (n=2 case) Let $\beta \in \mathbb{N}$, $\beta \geq 2$ and let f be such that f'' exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,2,\xi}(f;x) - f(x) = o\left(\xi^{2-\gamma}\right). \tag{3.10}$$

Proof. As in the proof of Proposition 3.3, we have $\xi^{-2} \frac{1}{W} \int_{-\infty}^{\infty} t^2 \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt$ $\leq \rho$, for some $\rho > 0$, $\forall \xi > 0$. Applying Proposition 3.3 for $n = 2$ we obtain the claim of the corollary. □

Corollary 3.6. (n=3 case) Let $\beta \in \mathbb{N}$, $\beta \geq 3$ and let f be such that $f^{(3)}$ exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,3,\xi}(f;x) - f(x) = \frac{f''(x)}{2} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) c_{2,\xi} + o(\xi^{3-\gamma}). \tag{3.11}$$

Proof. As in the proof of Proposition 3.3, we have $\xi^{-3} \frac{1}{W} \int_{-\infty}^{\infty} |t|^3 \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt$ $\leq \rho$, for some $\rho > 0$, $\forall \xi > 0$. Applying Proposition 3.3 for $n = 3$ we obtain the claim of the corollary. □

Corollary 3.7. (n=4 case) Let $\beta \in \mathbb{N}$, $\beta \geq 3$ and let f be such that $f^{(4)}$ exists and it is bounded. Let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,4,\xi}(f;x) - f(x) = \frac{f''(x)}{2} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) c_{2,\xi} + o(\xi^{4-\gamma}). \tag{3.12}$$

Proof. As in the proof of Proposition 3.3, we have $\xi^{-4}\frac{1}{W}\int_{-\infty}^{\infty} t^4 \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt \leq \rho$, for some $\rho > 0$, $\forall \xi > 0$. Applying Proposition 3.3 for $n = 4$ we obtain the claim of the corollary. □

4. Applications to Particular Trigonometric Singular Operators

In this section we work on the approximation results given in the previous section, for some particular values of n and β.

Case $\beta = 3$. We have the following results.

Corollary 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f' exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,1,\xi}(f; x) - f(x) = o(\xi^{1-\gamma}). \tag{4.1}$$

Proof. By Proposition 3.3 with $n = 1$. □

Corollary 4.2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f'' exists,which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,2,\xi}(f; x) - f(x) = o(\xi^{2-\gamma}). \tag{4.2}$$

Proof. By Proposition 3.3 with $n = 2$. □

Corollary 4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(3)}$ exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,3,\xi}(f; x) - f(x) = \frac{5}{22}\left(\sum_{j=1}^{r} \alpha_j j^2\right) f''(x)\xi^2 + o(\xi^{3-\gamma}). \tag{4.3}$$

Proof. By Proposition 3.3 with $n = 3$. □

Corollary 4.4. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(4)}$ exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,4,\xi}(f; x) - f(x) = \frac{5}{22}\left(\sum_{j=1}^{r} \alpha_j j^2\right) f''(x)\xi^2 + o(\xi^{4-\gamma}). \tag{4.4}$$

Proof. By Proposition 3.3 with $n = 4$. □

Corollary 4.5. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(3)}$ exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,3,\xi}(f;x) - f(x) = \frac{105}{604} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) f''(x)\xi^2 + o(\xi^{3-\gamma}). \tag{4.5}$$

Proof. By Proposition 3.3 with $n = 3$. □

Corollary 4.6. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(4)}$ exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,4,\xi}(f;x) - f(x) = \frac{105}{604} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) f''(x)\xi^2 + o(\xi^{4-\gamma}). \tag{4.6}$$

Proof. By Proposition 3.3 with $n = 4$. □

Corollary 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(5)}$ exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$\begin{aligned} T_{r,5,\xi}(f;x) - f(x) &= \frac{105}{604} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) f''(x)\xi^2 + \frac{35}{2416} \left(\sum_{j=1}^{r} \alpha_j j^4 \right) f^{(4)}(x)\xi^4 \\ &\quad + o(\xi^{5-\gamma}). \end{aligned} \tag{4.7}$$

Proof. By Proposition 3.3 with $n = 5$. □

Corollary 4.8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(6)}$ exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$\begin{aligned} T_{r,6,\xi}(f;x) - f(x) &= \frac{105}{604} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) f''(x)\xi^2 + \frac{35}{2416} \left(\sum_{j=1}^{r} \alpha_j j^4 \right) f^{(4)}(x)\xi^4 \\ &\quad + o(\xi^{6-\gamma}). \end{aligned} \tag{4.8}$$

Proof. By Proposition 3.3 with $n = 6$. □

Case $\beta = 6$. We have the following results.

Corollary 4.9. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(3)}$ exists, which is bounded and let $\xi \to 0+$, $0 < \gamma \leq 1$. Then

$$T_{r,3,\xi}(f;x) - f(x) = \frac{311465}{2620708} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) f''(x)\xi^2 + o(\xi^{3-\gamma}). \tag{4.9}$$

Proof. By Proposition 3.3 with $n = 3$. □

Corollary 4.10. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(5)}$ exists, which is bounded and let $\xi\to 0+$, $0<\gamma\le 1$. Then

$$\begin{aligned} T_{r,5,\xi}(f;x)-f(x) &= \frac{311465}{2620708}\left(\sum_{j=1}^{r}\alpha_j j^2\right) f''(x)\xi^2 \\ &+\frac{35805}{5241416}\left(\sum_{j=1}^{r}\alpha_j j^4\right) f^{(4)}(x)\xi^4+o(\xi^{5-\gamma}). \end{aligned} \tag{4.10}$$

Proof. By Proposition 3.3 with $n=5$. □

Corollary 4.11. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(7)}$ exists, which is bounded and let $\xi\to 0+$, $0<\gamma\le 1$. Then

$$\begin{aligned} T_{r,7,\xi}(f;x)-f(x) &= \frac{311465}{2620708}\left(\sum_{j=1}^{r}\alpha_j j^2\right) f''(x)\xi^2 \\ &+\frac{35805}{5241416}\left(\sum_{j=1}^{r}\alpha_j j^4\right) f^{(4)}(x)\xi^4 \\ &+\frac{5313}{20965664}\left(\sum_{j=1}^{r}\alpha_j j^6\right) f^{(6)}(x)\xi^6+o(\xi^{7-\gamma}). \end{aligned} \tag{4.11}$$

Proof. By Proposition 3.3 with $n=7$. □

Corollary 4.12. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(9)}$ exists, which is bounded and let $\xi\to 0+$, $0<\gamma\le 1$. Then

$$\begin{aligned} T_{r,9,\xi}(f;x)-f(x) &= \frac{311465}{2620708}\left(\sum_{j=1}^{r}\alpha_j j^2\right) f''(x)\xi^2 \\ &+\frac{35805}{5241416}\left(\sum_{j=1}^{r}\alpha_j j^4\right) f^{(4)}(x)\xi^4 \\ &+\frac{5313}{20965664}\left(\sum_{j=1}^{r}\alpha_j j^6\right) f^{(6)}(x)\xi^6 \\ &+\frac{8085}{1174077184}\left(\sum_{j=1}^{r}\alpha_j j^8\right) f^{(8)}(x)\xi^8+o(\xi^{9-\gamma}). \end{aligned} \tag{4.12}$$

Proof. By Proposition 3.3 with $n=9$. □

For similar studies as here see also [1], [2].

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