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A REMARK ON AN INTEGRAL INEQUALITY FOR THE B-OPERATORS

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Abstract. In this paper, an error has been pointed out in the proofs of some recent results concerning the B-operators given by Shah and Liman [Integral estimates for the family of B-operators, Operators and Matrices, Vol. 5(1),(2010), 79 - 87]. Certain sharp L_p inequalities valid for $0 < p < \infty$ for B-operators are also obtained.

1. Introduction

Let P_n denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. For $P \in P_n$, define

$$\|P\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p \right\}^{1/p}, \ 1 \le p < \infty,$$

$$\|P\|_\infty := \underset{|z|=1}{Max} \left| P(z) \right|.$$

A famous result known as Bernstein's inequality (for reference see [5],[11],[14]) states that if $P \in P_n$, then

$$\left\| P' \right\|_\infty \le n \left\| P \right\|_\infty . \tag{1.1}$$

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Inequality (1.1) can be obtained by letting $p \to \infty$ in the inequality

$$\left\|P'\right\|_p \leq n \left\|P\right\|_p, p \geq 1 \tag{1.2}$$

Inequality (1.2) is due to Zygmund [15]. Arestov [2] proved that inequality (1.2) remains true for $0 < p < 1$ as well.

Both the inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then inequalities (1.1) and (1.2) can be replaced respectively by

$$\left\|P'\right\|_\infty \leq \frac{n}{2} \left\|P\right\|_\infty \tag{1.3}$$

and

$$\left\|P'\right\|_p \leq n \frac{\left\|P\right\|_p}{\left\|1+z\right\|_p}. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [8] (see also [4]) whereas inequality (1.4) is due to Bruijn [6] (see also [3]) for $p \geq 1$. Rahman and Schmeisser [13] showed that the inequality (4) remains true for $0 < p < 1$ as well.

Rahman [12](see also [14]) introduced a class B_n of operator B which carries $P \in P_n$ into

$$B[P](z) := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \tag{1.5}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2, \;\; C(n,r) = n!/r!(n-r)! \tag{1.6}$$

lie in the half plane

$$|z| \leq |z - n/2| \tag{1.7}$$

and observed that if $P \in P_n$, then

$$|P(z)| \leq \left\|P\right\|_\infty \;\; for \;\; |z| = 1$$

implies

$$|B[P](z)| \leq |B[z^n]| \left\|P\right\|_\infty \;\; for \;\; |z| = 1. \tag{1.8}$$

And if $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then

$$|B[P](z)| \leq \frac{1}{2} \left\{|B[z^n]| + |\lambda_0|\right\} \left\|P\right\|_\infty \;\; for \;\; |z| = 1, \tag{1.9}$$

(see [12, inequality (5.2) and (5.3)]).

Recently Shah and Liman [9] extended inequality (1.8) to the L_p-norm by establishing:

Theorem A. *If* $P \in P_n$, *then for every* $R \geq 1$ *and* $p \geq 1$,

$$\|B[P](R\,\cdot)\|_p \leq R^n \left|\phi_n(\lambda_0, \lambda_1, \lambda_2)\right| \|P\|_p\,,$$

where $B \in B_n$ and

$$\phi_n(\lambda_0, \lambda_1, \lambda_2) = \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}. \tag{1.10}$$

While seeking the desired extension of inequality (1.9) to the L_p-norm, Shah and Liman [9] made an incomplete attempt and claimed to have proved the following result.

Theorem B. *If* $P \in P_n$ and $P(z)$ *does not vanish in* $|z| < 1$, *then for every* $R \geq 1$ *and* $p \geq 1$,

$$\|B[P](R\,\cdot)\|_p \leq \frac{R^n \left|\phi_n(\lambda_0, \lambda_1, \lambda_2)\right| + |\lambda_0|}{\|1+z\|_p} \|P\|_p\,,$$

where $B \in B_n$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.10).

Unfortunately the proof of this result(Theorem B) including a result for self-inversive polynomials and the Lemma 4 [9] given by Shah and Liman is not correct, because the claim made by the authors on page 84 line 10, on page 85 line 19 and on page 86 line 16 is incorrect. The reason being that the authors [9] throughtout the paper make use of the argument that if $Q(z) = z^n \overline{P(1/\overline{z})}$, then for $0 \leq \theta < 2\pi$ and $R \geq 1$,

$$\left|B[Q(Re^{i\theta})]\right| = \left|B[R^n P(e^{i\theta}/R)]\right|,$$

which is not, in general, true for every $R \geq 1$ and $0 \leq \theta \leq 2\pi$, as can be easily seen by taking, in particular, the nth degree polynomial $P(z) = a_n z^n + a_0$ and $R = 2$.

In this paper, we present certain sharp L_p inequalities for B-operators, which not only validate the Theorem B and other related results in [9] for $R = 1$, but also extend them for $0 < p < 1$ as well.

2. Lemmas

For the proofs of our main results, we need the following lemmas. The first lemma follows from Corollary 18.3 on p.65 in [10].

Lemma 2.1. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then all the zeros of the polynomial $B[P](z)$ also lie in $|z| \leq 1$.*

Lemma 2.2. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then*

$$|B[P](z)| \leq |B[Q](z)|, \quad for \quad |z| \geq 1, \tag{2.1}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 2.2 is due to Rahman [12].

Next we describe a result of Arestov [2].

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n) \in \mathbb{C}^{n+1}$ *and* $P(z) = \sum_{j=0}^{n} a_j z^j$, we define

$$\Lambda_\gamma P(z) = \sum_{j=0}^{n} \gamma_j a_j z^n.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,

(ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov may now be stated as follows.

Lemma 2.3. [2] *Let $\phi(x) = \psi(logx)$ where ψ is a convex nondecreasing function on $\mathbb{R}$. Then for all $P \in P_n$ and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \phi\left(|\Lambda_\gamma P(e^{i\theta})|\right) d\theta \leq \int_0^{2\pi} \phi\left(c_{\gamma,n}|P(e^{i\theta})|\right) d\theta$$

where $c_{\gamma,n} = max\,(|\gamma_0|, |\gamma_n|)$.

In particular Lemma 2.3 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$. Therefore, we have for $0 < p < \infty$,

$$\left\{\int_0^{2\pi} |\Lambda_\gamma P(e^{i\theta})|^p d\theta\right\}^{1/p} \leq c_{\gamma,n} \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{2.2}$$

We use (2.2) to prove following interesting result.

Lemma 2.4. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every $p > 0$ and α real, $0 \le \alpha < 2\pi$,*

$$\int_0^{2\pi} \left| B[P](e^{i\theta}) + e^{i\alpha} G(e^{i\theta}) \right| d\theta \le \left| \phi_n(\lambda_0, \lambda_1, \lambda_2) + \bar{\lambda_0} e^{i\alpha} \right| \int_0^{2\pi} \left| P(e^{i\theta}) \right| d\theta \tag{2.3}$$

where $G(z)$ is the conjugate polynomial of $B[Q](z)$, $Q(z) = z^n \overline{P(1/\bar{z})}$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (1.10).

Proof. Since $Q(z) = z^n \overline{P(1/\bar{z})}$ and $P(z)$ does not vanish in $|z| < 1$, by Lemma 2.2 , we have

$$|B[P](z)| \le |B[Q](z)| \quad for \quad |z| = 1. \tag{2.4}$$

Now

$$\begin{aligned} B[Q](z) &= \lambda_0 Q(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{Q'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{Q''(z)}{2!} \\ &= \lambda_0 z^n \overline{P(1/\bar{z})} + \lambda_1 \left(\frac{nz}{2}\right) \left(nz^{n-1} \overline{P(1/\bar{z})} - z^{n-2} \overline{P'(1/\bar{z})} \right) \\ &\quad + \frac{\lambda_2}{2!} \left(\frac{nz}{2}\right)^2 (n(n-1) z^{n-2} \overline{P(1/\bar{z})} - 2(n-1) z^{n-3} \overline{P'(1/\bar{z})} \\ &\quad + z^{n-4} \overline{P''(1/\bar{z})}) \\ &= F(z) \ (say), \end{aligned}$$

then by hypothesis,

$$\begin{aligned} G(z) = z^n \overline{F(1/\bar{z})} &= \bar{\lambda_0} P(z) + \bar{\lambda_1} \frac{n}{2} \left(nP(z) - zP'(z) \right) \\ &\quad + \frac{\bar{\lambda_2}}{2!} \left(\frac{n}{2}\right)^2 \left(n(n-1)P(z) - 2(n-1)zP'(z) + z^2 P''(z) \right) \\ &= \left(\bar{\lambda_0} + \bar{\lambda_1} \frac{n^2}{2} + \bar{\lambda_2} \frac{n^3(n-1)}{8} \right) P(z) \\ &\quad - \left(\bar{\lambda_1} \frac{n}{2} + \bar{\lambda_2} \frac{n^2(n-1)}{4} \right) zP'(z) + \bar{\lambda_2} \frac{n^2}{8} z^2 P''(z) \end{aligned}$$

and

$$|B[Q](z)| = |F(z)| = |G(z)| \quad for \ \ |z| = 1.$$

Using this in (2.4), we get

$$|B[P](z)| \le |G(z)| \quad for \quad |z| = 1.$$

Since by Lemma 2.1, all the zeros of $F(z) = B[Q](z)$ lie in $|z| \le 1$, therefore, all the zeros of $G(z)$ lie in $|z| \ge 1$. Hence by the maximum modulus principle,

$$|B[P](z)| < |G(z)| \quad for \quad |z| < 1. \tag{2.5}$$

A direct application of Rouche's theorem shows that

$$\begin{aligned}\Lambda_\gamma P(z) &= B[P](z)+e^{i\alpha}G(z)\\ &= \left(\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}+e^{i\alpha}\bar{\lambda_0}\right)a_nz^n\\ &\quad+\cdots+\left(\bar{\lambda_0}+\bar{\lambda_1}\frac{n^2}{2}+\bar{\lambda_2}\frac{n^3(n-1)}{8}e^{i\alpha}+\lambda_0\right)a_0\end{aligned}$$

does not vanish in $|z|<1$. Therefore, Λ_γ is an admissible operator. Applying (2.2) of Lemma 2.3, the desired result follows immediately for each $p>0$. □

From Lemma 2.4, we deduce the following more general result.

Lemma 2.5. *If $P\in P_n$, then for every $p>0$ and α real, $0\le\alpha<2\pi$,*

$$\int_0^{2\pi}\left|B[P](e^{i\theta})+e^{i\alpha}G(e^{i\theta})\right|^p d\theta\le\left|\phi_n(\lambda_0,\lambda_1,\lambda_2)+\bar{\lambda_0}e^{i\alpha}\right|^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta, \tag{2.6}$$

where $G(z)$ is the conjugate polynomial of $B[Q](z)$, $Q(z)=z^n\overline{P(1/\bar{z})}$ and $\phi_n(\lambda_0,\lambda_1,\lambda_2)$ is defined by (10).

Proof. Since $P\in P_n$, we can write

$$P(z)=P_1(z)P_2(z)=\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(z-z_j),\ \ k\ge1$$

where all the zeros of $P_1(z)$ lie in $|z|\ge1$ and all the zeros of $P_2(z)$ lie in $|z|<1$. First we assume that $P_1(z)$ has no zero on $|z|=1$ so that all the zeros of $P_1(z)$ lie in $|z|>1$. Let $Q_2(z)=z^{n-k}\overline{P_2(1/\bar{z})}$. Then all the zeros of $Q_2(z)$ lie in $|z|>1$ and $|Q_2(z)|=|P_2(z)|$ for $|z|=1$. Now consider the polynomial

$$f(z)=P_1(z)Q_2(z)=\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(1-z\bar{z_j}),$$

then all the zeros of $f(z)$ lie in $|z|>1$ and for $|z|=1$,

$$|f(z)|=|P_1(z)|\,|Q_2(z)|=|P_1(z)|\,|P_2(z)|=|P(z)|\,. \tag{2.7}$$

Since $P(z)/f(z)$ is not a constant, by the Maximum Modulus Principle, it follows that

$$|P(z)|\le|f(z)|\ \ for\ \ |z|\le1. \tag{2.8}$$

We claim that the polynomial $g(z)=P(z)+\lambda f(z)$ does not vanish in $|z|\le1$ for every λ with $|\lambda|>1$. If this is not true, then $g(z_0)=0$ for some z_0 with $|z_0|\le1$. This gives

$$|P(z_0)|=|\lambda||f(z_0)|\ \ with\ \ |z_0|\le1.$$

Since $f(z_0) \neq 0$ and $|\lambda| > 1$, it follows that

$$|P(z_0)| > |f(z_0)| \ \ with \ \ |z_0| \leq 1,$$

which clearly contradicts (2.8). Thus $g(z)$ does not vanish in $|z| \leq 1$ for every λ with $|\lambda| > 1$, so that all the zeros of $g(z)$ lie in $|z| \geq t$ for some $t > 1$ and hence all the zeros of $R(z) = g(tz)$ lie in $|z| \geq 1$. Now proceeding similarly as in the proof of Lemma 2.5, we get from (2.5) with polynomial $P(z)$ replaced by $R(z)$ and $G(z)$ by $H_1(z)$,

$$|B[R](z)| < |H_1(z)| \quad for \quad |z| < 1,$$

where $H_1(z)$ is the conjugate polynomial of $B[h](z)$ and $h(z) = z^n\overline{R(1/\bar{z})}$. Taking $z = e^{i\theta}/t, 0 \leq \theta < 2\pi$, then $|z| = (1/t) < 1$ as $t > 1$ and we get for $0 \leq \theta < 2\pi$,

$$\{|B[R](z)|\}_{z=e^{i\theta}/t} < \{|H_1(z)|\}_{z=e^{i\theta}/t},$$

which after simplification leads to

$$|B[g](z)| < |H(z)| \quad for \quad |z| = 1,$$

where $H(z)$ is the conjugate polynomial of $B[S](z)$ and $S(z) = z^n\overline{g(1/\bar{z})}$. An application of Rouche's theorem shows that the polynomial

$$T(z) = B[g](z) + e^{i\alpha}H(z)$$

does not vanish in $|z| \leq 1$. Replacing $g(z)$ by $P(z) + \lambda f(z)$ and noting that B is a linear operator, it follows that the polynomial

$$T(z) = \left(B[P](z) + e^{i\alpha}G(z)\right) + \lambda\left(B[f](z) + e^{i\alpha}F(z)\right) \tag{2.9}$$

does not vanish in $|z| \leq 1$ for every λ with $|\lambda| > 1$, where $F(z)$ is the cojugate polynomial of $B[L](z)$ and $L(z) = z^n\overline{f(1/\bar{z})}$. This implies that

$$\left|B[P](z) + e^{i\alpha}G(z)\right| \leq \left|B[f](z) + e^{i\alpha}F(z)\right| \ \ for \ \ |z| \leq 1. \tag{2.10}$$

If inequality (2.10) is not true, then there a point $z = z_0$ with $|z_0| \leq 1$ such that

$$\left\{\left|B[P](z) + e^{i\alpha}G(z)\right|\right\}_{z=z_0} > \left\{\left|B[f](z) + e^{i\alpha}F(z)\right|\right\}_{z=z_0}.$$

Since $f(z)$ does not vanish in $|z| \leq 1$, it follows(as in the case of $g(z)$) that the polynomial $B[f](z) + e^{i\alpha}F(z)$ does not vanish in $|z| \leq 1$. Hence $\left\{B[f](z) + e^{i\alpha}F(z)\right\}_{z=z_0} \neq 0$ with $|z_0| \leq 1$. We take

$$\lambda = -\left\{B[P](z) + e^{i\alpha}G(z)\right\}_{z=z_0} / \left\{B[f](z) + e^{i\alpha}F(z)\right\}_{z=z_0}$$

so that λ is well-defined real or complex number with $|\lambda| > 1$ and with choice of λ, from (2.9), we get $T(z_0) = 0$ with $|z_0| \leq 1$. This is clearly a contradiction to the fact that $T(z)$ does not vanish in $|z| \leq 1$. Thus

$$\left|B[P](z) + e^{i\alpha}G(z)\right| \leq \left|B[f](z) + e^{i\alpha}F(z)\right|$$

for $|z| \leq 1$, which in particular gives for each $p > 0$ and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi} \left|B[P](e^{i\theta})] + e^{i\alpha}G(e^{i\theta})\right|^p d\theta \leq \int_0^{2\pi} \left|B[f](e^{i\theta}) + e^{i\alpha}F(e^{i\theta})\right|^p d\theta.$$

Using Lemma 2.4 and (2.7), we get for each $p > 0$,

$$\begin{aligned} & \int_0^{2\pi} \left|B[P](e^{i\theta}) + e^{i\alpha}G(e^{i\theta})\right|^p d\theta \\ \leq & \left|\phi_n(\lambda_0, \lambda_1, \lambda_2) + \bar{\lambda_0}e^{i\alpha}\right|^p \int_0^{2\pi} \left|f(e^{i\theta})\right|^p d\theta \qquad (2.11) \\ = & \left|\phi_n(\lambda_0, \lambda_1, \lambda_2) + \bar{\lambda_0}e^{i\alpha}\right|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta. \end{aligned}$$

Now if $P_1(z)$ has a zero on $|z| = 1$, then applying (2.11) to the polynomial $P^*(z) = P_1(tz)P_2(z)$ where $t < 1$, we get for each $p > 0$ and α real,

$$\begin{aligned} & \int_0^{2\pi} \left|B[P^*](e^{i\theta})] + e^{i\alpha}G^*(e^{i\theta})\right|^p d\theta \qquad (2.12) \\ \leq & \left|\phi_n(\lambda_0, \lambda_1, \lambda_2) + \bar{\lambda_0}e^{i\alpha}\right|^p \int_0^{2\pi} \left|P^*(e^{i\theta})\right|^p d\theta. \end{aligned}$$

Letting $t \to 1$ in (2.12) and using continuity, the desired result follows immediately and this proves Lemma 2.5. □

Lemma 2.6. *If $P \in P_n$ and $Q(z) = z^n\overline{P(1/\bar{z})}$, then for every $p > 0$,*

$$\begin{aligned} & \int_0^{2\pi}\int_0^{2\pi} \left|B[P](e^{i\theta}) + e^{i\alpha}B[Q](e^{i\theta})\right|^p d\theta d\alpha \\ \leq & \int_0^{2\pi} \left|\phi_n(\lambda_0, \lambda_1, \lambda_2)e^{i\alpha} + \lambda_0\right|^p d\alpha \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta, \end{aligned}$$

where $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (10).
The result is best possible as shown by polynomial is $P(z) = z^n$.

Proof. By Lemma 2.5, we have for each $p > 0$,

$$\int_0^{2\pi} \left|B[P](e^{i\theta}) + e^{i\alpha}G(e^{i\theta})\right|^p d\theta \leq \left|\phi_n(\lambda_0, \lambda_1, \lambda_2) + \bar{\lambda_0}e^{i\alpha}\right|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta, \qquad (2.13)$$

where $G(z)$ is the conjugate polynomial of $B[Q](z)$ and $Q(z) = z^n\overline{P(1/\bar{z})}$. It can be easily verified that

$$|G(e^{i\theta})| = |B[Q](e^{i\theta})|, \quad 0 \leq \theta < 2\pi. \qquad (2.14)$$

Now for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $|B[P](e^{i\theta}) \neq 0$, we obtain by using (2.14) for each $p > 0$,

$$\begin{aligned}
& \int_0^{2\pi} \left| B[P](e^{i\theta}) + e^{i\alpha} B[Q](e^{i\theta}) \right|^p d\alpha \\
= \quad & |B[P](e^{i\theta})|^p \int_0^{2\pi} \left| 1 + e^{i\alpha} \frac{B[Q](e^{i\theta})}{B[P](e^{i\theta})} \right|^p d\alpha \\
= \quad & |B[P](e^{i\theta})|^p \int_0^{2\pi} \left| 1 + e^{i\alpha} \left| \frac{B[Q](e^{i\theta})}{B[P](e^{i\theta})} \right| \right|^p d\alpha \\
= \quad & \int_0^{2\pi} \left| |B[P](e^{i\theta})| + e^{i\alpha} |B[Q](e^{i\theta})| \right|^p d\alpha \qquad (2.15) \\
= \quad & \int_0^{2\pi} \left| |B[P](e^{i\theta})| + e^{i\alpha} |G(e^{i\theta})| \right|^p d\alpha.
\end{aligned}$$

Since inequality (2.15) is trivially true for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $B[P](e^{i\theta}) = 0$, it follows that

$$\int_0^{2\pi} \left| B[P](e^{i\theta})] + e^{i\alpha} B[Q](e^{i\theta}) \right|^p d\alpha = \int_0^{2\pi} \left| |B[P](e^{i\theta})| + e^{i\alpha} |G(e^{i\theta})| \right|^p d\alpha. \tag{2.16}$$

Integrating (2.16) both sides with respect to θ from 0 to 2π and using (2.6), we get

$$\begin{aligned}
& \int_0^{2\pi} \int_0^{2\pi} \left| B[P](e^{i\theta}) + e^{i\alpha} B[Q](e^{i\theta}) \right|^p d\alpha d\theta \\
= \quad & \int_0^{2\pi} \int_0^{2\pi} \left| |B[P](e^{i\theta})| + e^{i\alpha} |G(e^{i\theta})| \right|^p d\alpha d\theta \\
= \quad & \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| |B[P](e^{i\theta})| + e^{i\alpha} |G(e^{i\theta})| \right|^p d\alpha \right\} d\theta \\
= \quad & \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| B[P](e^{i\theta}) + e^{i\alpha} G(e^{i\theta}) \right|^p d\alpha \right\} d\theta \\
= \quad & \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| B[P](e^{i\theta}) + e^{i\alpha} G(e^{i\theta}) \right|^p d\theta \right\} d\alpha \\
\leq \quad & \int_0^{2\pi} \left| \phi_n(\lambda_0, \lambda_1, \lambda_2) + \bar{\lambda_0} e^{i\alpha} \right|^p d\alpha \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \\
= \quad & \int_0^{2\pi} \left| \phi_n(\lambda_0, \lambda_1, \lambda_2) e^{i\alpha} + \lambda_0 \right|^p d\alpha \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta.
\end{aligned}$$

This completes the proof of Lemma 2.6. □

3. Main results

We first prove:

Theorem 3.1. *If $P \in P_n$, then for each $p > 0$,*

$$\|B[P]\|_p \leq |\phi_n(\lambda_0, \lambda_1, \lambda_2)| \, \|P\|_p \, , \tag{3.1}$$

where $B \in B_n$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (10).
The result is best possible and equality in (3.1) holds for $P(z) = az^n, a \neq 0$.

Proof. By hypothesis $P \in P_n$, we can write

$$P(z) = P_1(z)P_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n} (z - z_j) \ \ , k \geq 1$$

where all the zeros of $P_1(z)$ lie in $|z| \leq 1$ and all the zeros of $P_2(z)$ lie in $|z| > 1$. First we suppose that all the zeros of $P_1(z)$ lie in $|z| < 1$. Let

$$Q_2(z) = z^{n-k}\overline{P_2(1/\bar{z})}.$$

Then all the zeros of $Q_2(z)$ lie in $|z| < 1$ and $|Q_2(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$F(z) = P_1(z)Q_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n} (1 - z\bar{z}_j),$$

then all the zeros of $F(z)$ lie in $|z| < 1$ and for $|z| = 1$,

$$|F(z)| = |P_1(z)| \, |Q_2(z)| = |P_1(z)| \, |P_2(z)| = |P(z)| \, . \tag{3.2}$$

Since $P(z)/F(z)$ is not a constant, by the Maximum Modulus Principle, it follows that

$$|P(z)| \leq |F(z)| \ \ for \ \ |z| \geq 1. \tag{3.3}$$

Since $F(z) \neq 0$ for $|z| \geq 1$, a direct application of Rouche's theorem shows that the polynomial $H(z) = P(z) + \lambda F(z)$ has all its zeros in $|z| < 1$ for every λ with $|\lambda| > 1$. Applying Lemma 1.1 to the polynomial $H(z)$ and noting that B is a linear operator, it follows that all the zeros of

$$T(z) = B[H](z) = B[P](z) + \lambda B[F](z)$$

lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$|B[P](z)| \leq |B[F](z)| \ \ for \ \ |z| \geq 1,$$

which, in particular, gives for each $p > 0$ and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi} \left|B[P](e^{i\theta})\right|^p d\theta \leq \int_0^{2\pi} \left|B[F](e^{i\theta})\right|^p d\theta. \tag{3.4}$$

Again, since all the zeros of $F(z)$ lie in $|z| < 1$, by Lemma 1.1, all the zeros of $B[F(z)]$ also lie in $|z| < 1$. Therefore, the operator Λ_γ defined by

$$\Lambda_\gamma F(z) = B[F](z) = \left(\lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}\right) b_n z^n + \cdots + \lambda_0 b_0$$

is admissible. Hence by (2.2) of Lemma 2.3, for each $p > 0$, we have

$$\begin{aligned}\int_0^{2\pi} \left|B[F](e^{i\theta})\right|^p d\theta &\leq \left|\lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}\right|^p \int_0^{2\pi} \left|F(e^{i\theta})\right|^p d\theta \\ &= |\phi_n(\lambda_0, \lambda_1, \lambda_2)|^p \int_0^{2\pi} \left|F(e^{i\theta})\right|^p d\theta \qquad (3.5)\end{aligned}$$

Combining inequalities (3.4) and (3.5) and noting that $|F(e^{i\theta})| = |P(e^{i\theta})|$, we obtainfor each $p > 0$,

$$\left\{\int_0^{2\pi} \left|B[P](e^{i\theta})\right|^p d\theta\right\}^{1/p} \leq |\phi_n(\lambda_0, \lambda_1, \lambda_2)| \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \qquad (3.6)$$

In case $P_1(z)$ has a zero on $|z| = 1$, then the inequality (3.6) follows by using similar argument as in the case of Lemma 2.5. This completes the proof of Theorem 3.1. □

Remark 3.2. *For $\lambda_0 = \lambda_2 = 0$, inequality (3.1) reduces to inequality (1.2) for each $p > 0$. Next if we choose $\lambda_0 = \lambda_1 = 0$ in (3.1), we immediately get*

$$\left\|P''\right\|_p \leq n(n-1) \left\|P\right\|_p, \quad p > 0. \qquad (3.7)$$

Inequality (1.8) also follows from Theorem 3.1 by letting $p \to \infty$ in (3.1).

Theorem 1.1 can be sharpened if we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$. In this direction, we next present the following interesting L_p extension of the inequality (1.9) for each $p > 0$.

Theorem 3.3. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $p > 0$,*

$$\left\|B[P]\right\|_p \leq \frac{\left\|\phi_n(\lambda_0, \lambda_1, \lambda_2)z + \lambda_0\right\|_p}{\left\|1 + z\right\|_p} \left\|P\right\|_p, \qquad (3.8)$$

where $B \in B_n$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (10).
The result is best possible and equality in (3.8) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

Proof. By hypothesis $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, therefore, if $Q(z) = z^n \overline{P(1/\bar{z})}$, then by Lemma 2.2, we have for $0 \leq \theta < 2\pi$,

$$|B[P](e^{i\theta})| \leq |B[Q](e^{i\theta})| \qquad (3.9)$$

Also, by Lemma 2.6, for each $p>0$ and α real,

$$\int_0^{2\pi}\int_0^{2\pi}\left|B[P](e^{i\theta})+e^{i\alpha}B[Q](e^{i\theta})\right|^p d\theta d\alpha$$
$$\leq \int_0^{2\pi}\left|\phi_n(\lambda_0,\lambda_1,\lambda_2)e^{i\alpha}+\lambda_0\right|^p d\alpha\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta. \tag{3.10}$$

Now it can be easily verified that for every real number α and $r\geq 1$,

$$\left|r+e^{i\alpha}\right|\geq\left|1+e^{i\alpha}\right|.$$

This implies for each $p>0$,

$$\int_0^{2\pi}\left|r+e^{i\alpha}\right|^p d\alpha\geq\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha. \tag{3.11}$$

If $B[P](e^{i\theta})\neq 0$, we take $r=|B[Q](e^{i\theta})|/|B[P](e^{i\theta})|$, then by (3.9), $r\geq 1$ and we get

$$\begin{aligned}
&\int_0^{2\pi}\left|B[P](e^{i\theta})+e^{i\alpha}B[Q](e^{i\theta})\right|^p d\alpha\\
=&\left|B[P](e^{i\theta})\right|^p\int_0^{2\pi}\left|1+e^{i\alpha}\frac{B[Q](e^{i\theta})}{B[P](e^{i\theta})}\right|^p d\alpha \qquad (3.12)\\
=&\left|B[P](e^{i\theta})\right|^p\int_0^{2\pi}\left|e^{i\alpha}+\left|\frac{B[Q](e^{i\theta})}{B[P](e^{i\theta})}\right|\right|^p d\alpha\\
\geq&\left|B[P](e^{i\theta})\right|^p\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha.
\end{aligned}$$

For $B[P](e^{i\theta})=0$, this inequality is trivially true. Using this in (3.10), we conclude that for each $p>0$,

$$\int_0^{2\pi}\left|B[P](e^{i\theta})\right|^p d\theta\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha$$
$$\leq\int_0^{2\pi}\left|\phi_n(\lambda_0,\lambda_1,\lambda_2)e^{i\alpha}+\lambda_0\right|^p d\alpha\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta,$$

which is equivalent to (3.8) and this completes the proof. □

Remark 3.4. *Taking $\lambda_0=\lambda_2=0$ in (3.8), one gets inequality (1.4) for each $p>0$. Next if we choose $\lambda_0=\lambda_1=0$ in (3.8), it follows that if $P(z)\neq 0$ in $|z|<1$, then for each $p>0$*

$$\left\|P''(z)\right\|_p\leq\frac{n(n-1)}{\|1+z\|_p}\|P\|_p. \tag{3.13}$$

The extremal polynomial is $P(z)=az^n+b, |a|=|b|=1$.

Inequality (1.9) can be obtained from Theorem 3.2 by letting $p \to \infty$ in (3.8). Moreover, by using triangle inequality, one can easily deduce Theorem B for $R = 1$ from Theorem 3.2.

A polynomial $P \in P_n$ is said be self-inversive if $P(z) = Q(z)$ where $Q(z)$ is the conjugate polynomial of $P(z)$, that is, $Q(z) = z^n \overline{P(1/\bar{z})}$.

Finally in this paper, the following result is established for self-inversive polynomials.

Theorem 3.5. *If $P \in P_n$ is a self-inversive polynomial, then for each $p > 0$,*

$$\|B[P]\|_p \leq \frac{\|\phi_n(\lambda_0, \lambda_1, \lambda_2)z + \lambda_0\|_p}{\|1 + z\|_p} \|P\|_p . \tag{3.14}$$

where $B \in B_n$ and $\phi_n(\lambda_0, \lambda_1, \lambda_2)$ is defined by (10).
The result is best possible and equality in (3.13) holds for $P(z) = z^n + 1$.

Proof. Since $P \in P_n$ is self-inversive polynomial, we have $P(z) = Q(z)$ for all $z \in C$ where $Q(z) = z^n \overline{P(1/\bar{z})}$. This gives,

$$|B[P](z)| = |B[Q](z)| \quad for\ all \quad z \in \mathbb{C}$$

so that

$$|B[Q](e^{i\theta})/B[P](e^{i\theta})| = 1, \ 0 \leq \theta < 2\pi.$$

Using this in place of (3.9) and proceeding similarly as in the proof of Theorem 3.2, we get the desired result. This completes the proof of Theorem 3.3. □

Concluding Remark. The best possible estimates analogous to Theorem A, Theorem B and other related results for the case $R > 1$ will appear somewhere else.

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