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SOME NEW DIFFERENCE SEQUENCE SPACES DEFINED ON ORLIZE FUNCTION

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Abstract. In this paper we introduce $l_M\left(\Delta_u^m, M\right)$ and $l_M\left(\Delta_u^m, M, p\right)$ and study their general properties.

1. Preliminaries, background and Notation

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers respectively. Let ω denote the space of all sequences (real or complex); l_∞ and c respectively, denotes the space of all bounded sequences, the space of convergent sequences. Orlicz [9] used the idea of Orlicz function to construct the space $\left(L^M\right)$.

Lindenstrauss and Tzafriri [4] investigated Orlicz sequence spaces in more detail and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p\,(\,1 \le p < \infty\,)$. Subquently different classes of sequence spaces defined by Parashar and Choudhary [10], Mursaleen et al. [7], Bektas and Altin [1], Tripathy et al. [12], Rao an Subramanian [2] and many others.

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The difference sequence spaces, $X(\Delta) = \{x = (x_k) : \Delta x \in X\}$, where $X = l_\infty, c$ and c_0 were studied by Kizmaz [3].

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (for detail see [9]). If the convexity of Orlicz functions M is replaced by $M(x+y) \leq M(x) + M(y)$ then the function is called modulus function, introduced by Nakano [8] and further investigated by Maddox [5, 6] and many others.

Let ω denote the set of all complex sequences. An Orlicz function M is said to satisfy Δ_2- condition for all values of u, if there exsts a constant $K > 0$, such that

$$M(2u) \leq KM(u) \; (u \geq 0).$$

The Δ_2-condition is equivalent to $M(lu) \leq KlM(u)$ for all values of u and for $l > 1$. Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct Orlicz sequence space

$$l_M = \{ x \in \omega : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \}.$$

The space l_M with norm

$$\|x\| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$, $1 \leq p < \infty$, the space l_M coincide with the classical sequence spaces l_p.

We define the following sequence space.

$$l_M(\Delta_u^m, M) = \{ x \in \omega : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|u_k \Delta^m x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \}.$$

If $u_k = e = (1, 1, 1, ...)$ and $m = 1$ (fixed), it reduces to $l_M(\Delta, M)$ [11].

2. Main Results

Theorem 2.1. $l_M(\Delta_u^m, M)$ is a Banach Space with the norm

$$\|x\| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|u_k \Delta^m x_k|}{\rho}\right) \leq 1 \}.$$

Proof. Let $(x^{(i)})$ be a Cauchy sequence in $l_M(\Delta_u^m, M)$, where

$$(x^{(i)}) = (x_1^{(i)}, x_2^{(i)}, ...) \in l_M(\Delta_u^m, M)$$

for each $i \in \mathbb{N}$. Let $r, x_0 > 0$ be fixed. Then for each $\frac{|u_k|\varepsilon}{rx_0} > 0$ there exists a positive integer N such that

$$\left\| x^{(i)} - x^{(j)} \right\| < \frac{\varepsilon |u_k|}{rx_0}, \forall i, j \geq N.$$

Using the definition of norm we get

$$\sum_{k=1}^{\infty} M\left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{x^{(i)} - x^{(j)}} \right) \leq 1, \ \forall i, j \geq N.$$

This gives,

$$M\left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{x^{(i)} - x^{(j)}} \right) \leq 1$$

for each $k \geq 1$ and $\forall\, i, j \geq N$. Hence we can find r > 0 with $M\left(\frac{rx_0}{k}\right) > 1$ such that

$$M\left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{x^{(i)} - x^{(j)}} \right) \leq M\left(\frac{rx_0}{k} \right).$$

This implies that

$$\left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{x^{(i)} - x^{(j)}} \right) \leq \left(\frac{rx_0}{k} \right),$$

that is,

$$\left| \Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right| \leq \left(\frac{rx_0}{k\,|u_k|} \right) \left\| x^{(i)} - x^{(j)} \right\|.$$

So, we have

$$\left|\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right| \leq \left(\frac{rx_0}{k\,|u_k|}\right)\frac{\varepsilon\,|u_k|}{rx_0},$$

this implies that

$$\left|\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right| \leq \frac{\varepsilon}{k}.$$

Therefore, for each $\varepsilon > 0$ there exists a positive integer N such that

$$\begin{aligned} &\left|\left(\Delta^m x_1^{(i)} - \Delta^m x_1^{(j)}\right) + ... + \left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \\ \leq\; &\left|\left(\Delta^m x_1^{(i)} - \Delta^m x_1^{(j)}\right)\right| + ... + \left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \\ \leq\; & k\,\frac{\varepsilon}{k} = \varepsilon. \end{aligned}$$

But,

$$\begin{aligned} &\left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \\ \leq\; &\left|\left(\Delta^m x_1^{(i)} - \Delta^m x_1^{(j)}\right)\right| + ... + \left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right|, \end{aligned}$$

that is,

$$\left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \leq \varepsilon,\ \forall\, i,j \geq N.$$

Therefore, $\left(\Delta^m x_k^{(i)}\right)$ is a Cauchy sequence in $\mathbb{R}$, for each fixed k. Using the continuity of M, we can find that

$$\sum_{k=1}^{N} M\left(\frac{\left|u_k\left[\Delta^m x_k^{(i)} - \lim_{j\to\infty}\Delta^m x_k^{(j)}\right]\right|}{\rho}\right) \leq 1.$$

That is,

$$\sum_{k=1}^{N} M\left(\frac{\left|u_k\left[\Delta^m x_k^{(i)} - \Delta^m x_k\right]\right|}{\rho}\right) \leq 1.$$

Taking infimum of such $\rho's$, we get

$$\inf\left\{\rho\ :\ \sum_{k=1}^{N} M\left(\frac{\left|u_k\left[\Delta^m x_k^{(i)}-\Delta^m x_k\right]\right|}{\rho}\right)\ \leq\ 1\right\}\ <\ \varepsilon,$$

for all $i\geq\ N$ and $j\to\infty$. Since $(\,x^{(i)})\in\ l_M\left(\Delta_u^m,M\right)$ and M is Orlicz function (hence continuous), $\Delta x\ \in\ l_M\left(\Delta_u^m,M\right)$. This completes the proof. □

3. Paranormed Sequence Spaces

A linear topological space X over the field of real numbers $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $h\,:\,X\to\mathbb{R}$ such that $h(\theta)=0,\ h(-x)=h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n-\alpha|\to 0$ and $h(x_n-x)\to 0$ imply $h(\alpha_n x_n-\alpha x)\to 0$ for all $\alpha's$ in $\mathbb{R}$ and $x's$ in X, where θ is a zero vector in the linear space X.

Let $p=(p_k)$ be any sequence of positive real numbers. Then in the same way we can also define the following sequence spaces for an Orlicz function M as l were extended to $l(p)$.

$$l_M\left(\Delta_u^m,M,p\right)\ =\ \left\{\rho\ >\ 0\ :\ x\ =\ (x_k)\ :\sum_{k=1}^{\infty} M\left(\frac{|\,u_k\Delta^m x_k\,|}{\rho}\right)^{p_k}\ <\ \infty\right\}.$$

If $(u_k)\ =\ (\,p_k)\ =\ e\ =\ (1,1,1,...)$, then $l_M(u,\Delta,M,p)$ reduces to $l_M(\Delta,M\,)$ [12].

The sequence spaces are paranormed spaces with

$$G^*(x)=\inf\left\{\rho^{\frac{p_n}{H}}\ :\ \left[\sum_{k=1}^{\infty} M\left(\frac{|\,u_k\Delta^m x_k\,|}{\rho}\right)^{p_k}\right]^{\frac{1}{H}}\ \leq\ 1\right\},$$

where $H\ =\ Max\left\{1\,,\,\sup_k p_k\right\}$.

Theorem 3.1. $l_M\left(\Delta_u^m,M,p\right)$ is a complete paranormed space with

$$G^*(x)\ =\inf\left\{\rho^{\frac{p_n}{H}}\ :\ \left[\sum_{k=1}^{\infty} M\left(\frac{|\,u_k\Delta^m x_k\,|}{\rho}\right)^{p_k}\right]^{\frac{1}{H}}\ \leq\ 1\right\}.$$

Proof. Let $(x^{(i)})$ be a Cauchy sequence in $l_M\left(\Delta_u^m,M,p\right)$, where

$$(x^{(i)})=\ (x_1^{(i)},x_2^{(i)},...)\ \in\ l(u,\Delta,M,p)$$

for each $i \in \mathbb{N}$. Let $r, x_0 > 0$ be fixed. Then for each $\frac{|u_k|\varepsilon}{rx_0} > 0$, there exists a positive integer N such that

$$\left\| x^{(i)} - x^{(j)} \right\| < \frac{\varepsilon\,|u_k|}{rx_0}, \forall\, i, j \geq N.$$

Using the definition of paranorm, we get

$$\left\{ \sum_{k=1}^{\infty} M \left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{G^*\left(x^{(i)} - x^{(j)} \right)} \right)^{p_k} \right\}^{\frac{1}{H}} \leq 1, \forall\, i, j \geq N \;.$$

Since $1 \leq p_k < \infty$, it follows that

$$M \left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{G^*\left(x^{(i)} - x^{(j)} \right)} \right) \leq 1$$

for each $k \geq 1$ and $\forall\, i, j \geq N$. Hence we can find r > 0 with $M\left(\frac{rx_0}{k}\right) > 1$ such that

$$M \left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{G^*\left(x^{(i)} - x^{(j)} \right)} \right) \leq M\left(\frac{rx_0}{k} \right).$$

This implies that

$$\left(\frac{\left| u_k \left[\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right] \right|}{G^*\left(x^{(i)} - x^{(j)} \right)} \right) \leq \left(\frac{rx_0}{k} \right),$$

that is,

$$\left| \Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right| \leq \left(\frac{rx_0}{k\,|u_k|} \right) G^*\left(x^{(i)} - x^{(j)} \right).$$

And hence

$$\left| \Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right| \leq \left(\frac{rx_0}{k\,|u_k|} \right) \frac{\varepsilon\,|u_k|}{rx_0}.$$

This implies that

$$\left| \Delta^m x_k^{(i)} - \Delta^m x_k^{(j)} \right| \leq \frac{\varepsilon}{k}.$$

Therefore, for each $\varepsilon > 0$ there exists a positive integer N such that

$$\begin{aligned} & \left|\left(\Delta^m x_1^{(i)} - \Delta^m x_1^{(j)}\right) + ... + \left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \\ \leq\ & \left|\left(\Delta^m x_1^{(i)} - \Delta^m x_1^{(j)}\right)\right| + ... + \left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \\ \leq\ & k\,\frac{\varepsilon}{k} = \varepsilon. \end{aligned}$$

But,

$$\left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \leq \left\{ \left|\left(\Delta x_1^{(i)} - \Delta x_1^{(j)}\right)\right| + ... + \left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \right\},$$

that is,

$$\left|\left(\Delta^m x_k^{(i)} - \Delta^m x_k^{(j)}\right)\right| \leq \varepsilon,\ \forall\ i,j \geq N.$$

Therefore, $\left(\Delta^m x_k^{(i)}\right)$ is a Cauchy sequence in $\mathbb{R}$, for each fixed k. Using the continuity of M, we can find that

$$\sum_{k=1}^{N} M\left(\frac{\left|u_k\left[\Delta^m x_k^{(i)} - \lim_{j\to\infty} \Delta^m x_k^{(j)}\right]\right|}{\rho}\right) \leq 1.$$

That is,

$$\left(\sum_{k=1}^{N} M\left(\frac{\left|u_k\left[\Delta^m x_k^{(i)} - \Delta^m x_k\right]\right|}{\rho}\right)^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Taking infimum of such $\rho' s$ we get

$$\inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{N}\left(M\left(\frac{\left|u_k\left[\Delta^m x_k^{(i)} - \Delta^m x_k\right]\right|}{\rho}\right)^{p_k}\right) \leq 1\right\} < \varepsilon,$$

for all $i \geq N$ and $j \to \infty$. Since $x^{(i)} \in l_M(u, \Delta^m, M, p)$ and M is Orlicz function (hence continuous), $\Delta x \in l_M(u, \Delta, M, p)$. This completes the proof. □

Theorem 3.2. Let $0 < p_k < q_k < \infty$ for each k. Then we have

$$l_M\left(\Delta_u^m, M, p\right) \subseteq l_M\left(\Delta_u^m, M, q\right).$$

Proof. Let $x \in l_M(\Delta_u^m, M, p)$. Then there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}\left(M\left(\frac{|u_k\Delta^m x_k|}{\rho}\right)\right)^{p_k} < \infty.$$

This implies that

$$M\left(\frac{|u_k\Delta^m x_k|}{\rho}\right) \leq 1,$$

for sufficiently large k. Since M is non-decreasing, we get

$$\sum_{k=1}^{\infty}\left(M\left(\frac{|u_k\Delta^m x_k|}{\rho}\right)\right)^{q_k} \leq \sum_{k=1}^{\infty}\left(M\left(\frac{|u_k\Delta^m x_k|}{\rho}\right)\right)^{p_k}.$$

Therefore, we have

$$\sum_{k=1}^{\infty}\left(M\left(\frac{|u_k\Delta^m x_k|}{\rho}\right)\right)^{q_k} < \infty.$$

That is, $x \in l_M(\Delta_u^m, M, q)$. This completes the proof. □

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