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EXISTENCE OF SOLUTIONS OF NONLINEAR ABSTRACT NEUTRAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. This work is concerned with neutral integrodifferential equations of first order with infinite delay. By using the Hausdorff measure of noncompactness and fixed point theory, the existence of mild solutions is obtained without the assumption of compactness on the associated family of operators.

1. Introduction

Differential equations with delay were studied about existence and stability by Hale [8, 9], Travis and Webb [11] and Webb [12], etc. Since such equations are often more realistic to describe natural phenomena than those without delay, they have been investigated in variant aspects by many authors. Many problems in the fields of ordinary and partial differential equations can be recast as integral equations. Several existence and uniqueness results can be derived from the corresponding results of integral equations. Neutral differential equations and neutral integrodifferential equations arise in many areas of

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applied mathematics and for this reason these equations have received much attention in the last decades, see, for example, [1-11] and references therein. Usually, people study the existence of mild solutions to functional differential equations with the assumption of compactness on the associated semigroup or operators. It is well know, there are few compact semigroups in infinite dimensional space. In this paper, we study such question without the assumption of compactness on the associated semigroup.

In this paper, we investigate the existence of mild solutions for some neutral functional differential equations of first order with infinite delay

$$\begin{aligned} &\frac{d}{dt}(x(t) - g(t, x_t, \int_0^t k_0(t,s,x_s)ds)) \\ &= Ax(t) + h(t, x, x_t) + \int_0^t k_1(t,s,x_s)ds \\ &\quad + f(t, x_t, \int_0^t k_2(t,s,x_s)ds), \quad t \in J = [0,b], \end{aligned} \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B} \;\; on \;\; [-r, 0]; \tag{1.2}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators defined on a Banach space X. The history $x_t : (-r, 0] \to X, x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; g, f, h, k_0, k_1 and k_2 are appropriate functions.

In this paper, we give the existence of mild solution of the initial value problems (1.1)-(1.2) under the conditions in respect of Hausdorff's measure of noncompactness.

2. Preliminaries

Now we introduce some definitions, notations and preliminary facts which are used throughout this paper.

Let X be a Banach space and $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup of linear operators $(T(t))_{t\geq 0}$ on X, $0 \in \rho(A)$. M is a constant such that $\|T(t)\| \leq M$ for every $t \in J = [0, b]$. The notation $(-A)^\alpha, \alpha \in (0, 1)$, is a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \; x \in D((-A)^\alpha)$$

defines a norm on $D((-A)^\alpha)$. Hereafter we denote by X_α the Banach space $D((-A)^\alpha)$ norm with $\|x\|_\alpha$. Then for each $0 < \alpha \le 1$, X_α is a Banach space. For semigroup $\{T(t)_{t\ge 0}\}$, the following properties will be used.

Lemma 2.1. ([7]) *Let $0 < \beta < \alpha \le 1$. Then the following properties hold :*

(1) *X_β is a Banach space and $X_\beta \hookrightarrow X_\alpha$, and the imbedding is compact;*
(2) *there exists $M \ge 1$ such that $\|T(t)\| \le M$, for all $0 \le t \le b$;*
(3) *for any $0 \le \alpha \le 1$, there exists a positive constant C_α such that $\|(-A)^\alpha T(t)\| \le \frac{C_\alpha}{t_\alpha}$, $\quad 0 < t \le b$.*

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to the one used in [10] and it is appropriate to treat retarded integrodifferential equations.

Definition 2.2. ([10]) Let $\mathcal{B}$ be a linear space of functions mapping $(-r, 0]$ into X endowed with a seminorm $\|\cdot\|_\mathcal{B}$ and we will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-r, \sigma + b] \to X, b > 0$, such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b] : X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:
 (i) x_t is in $\mathcal{B}$,
 (ii) $\|x(t)\| \le H\|x_t\|_\mathcal{B}$,
 (iii) $\|x_t\|_\mathcal{B} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t+\sigma)\|x_\sigma\|_\mathcal{B}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + b)$.

(B) The space $\mathcal{B}$ is complete.

Definition 2.3. ([2]) The Hausdorff's measure of noncompactness χ_Y defined by $\chi_Y(B) = \inf\{r > 0, B$ can be covered by finite number of balls with radius $r\}$, for bounded set B in any Banach space Y.

Lemma 2.4. ([2]) *Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:*

(1) *B is pre-compact if and only if $\chi_Y(B) = 0$;*
(2) *$\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(convB)$, where $\overline{B}$ and $convB$ are the closure and the convex hull of B respectively;*

(3) $\chi_Y(B) \leq \chi_Y(C)$ *when* $B \subseteq C$*;*
(4) $\chi_Y(B+C) \leq \chi_Y(B)+\chi_Y(C)$, *where* $B+C = \{x+y : x \in B, y \in C\}$*;*
(5) $\chi_Y(B \cup C) \leq \max\{\chi_Y(B), \chi_Y(C)\}$*;*
(6) $\chi_Y(\lambda B) = |\lambda|\chi_Y(B)$ *for any* $\lambda \in R$*;*
(7) *If the map* $Q : D(Q) \subseteq Y \to Z$ *is Lipschitz continuous with constant* k *then* $\chi_Z(QB) \leq k\chi_Y(B)$ *for any bounded subset* $B \subseteq D(Q)$*, where* Z *is a Banach space;*
(8) *If* $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subset of* Y *and* $\lim_{n\to\infty} \chi_Y(W_n) = 0$*, then* $\cap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* Y*.*

Definition 2.5. ([3]) The map $Q : W \subseteq Y \to Y$ is said to be a $\chi_Y -$ *contraction* if there exists a positive constant $k < 1$ such that $\chi_Y(Q(C)) \leq k\chi_Y(C))$ for any bounded close subset $C \subseteq W$, where Y is a Banach space.

Lemma 2.6. ([1], Darbo) *If* $W \subseteq Y$ *is closed and convex and* $0 \in W$*, the continuous map* $Q : W \to W$ *is a* $\chi_Y -$*contraction, if the set* $\{x \in W : x = \lambda Qx\}$ *is bounded for* $0 < \lambda < 1$*, then the map* Q *has at least one fixed point in* W*.*

In this paper we denote χ the Hausdorff's measure of noncompactness of X and χ_C the Hausdorff's measure of noncompactness of $C([0, b]; X)$. To discuss the existence results we need the following auxiliary results.

Lemma 2.7. ([3])
(1) *If* $W \subset C([a, b]; X)$ *is bounded, then* $\chi(W(t)) \leq \chi_C(W)$*, for any* $t \in [a, b]$*, where* $W(t) = \{u(t) : u \in W\} \subseteq X$*;*
(2) *If* W *is equicontinuous on* $[a, b]$*, then* $\chi(W(t))$ *is continuous for* $t \in [a, b]$*, and*

$$\chi_C(W) = \sup\{\chi(W(t)), t \in [a, b]\};$$

(3) *If* $W \subset C([a, b]; X)$ *is bounded and equicontinuous, then* $\chi(W(t))$ *is continuous for* $t \in [a, b]$*, and*

$$\chi\Big(\int_a^t W(s)ds\Big) \leq \int_a^t \chi W(s)ds, \text{ for all } t \in [a, b],$$

where $\displaystyle\int_a^t W(s)ds = \Big\{\int_a^t x(s)ds : x \in W\Big\}$.

The following lemma is easy to get.

Lemma 2.8. *If the semigroup $S(t)$ is equicontinuous and $\eta \in L([0,b];\boldsymbol{R}^+)$, then the set* $\left\{\int_0^t S(t-s)u(s)ds, \|u(s)\| \le \eta(s) \quad \text{for a.e. } s \in [0,b]\right\}$ *is equicontinuous for $t \in [0,b]$.*

3. Main Results

Now we define the mild solution for the initial value problem(1.1)-(1.2).

Definition 3.1. A function $x : (-r,b] \to X$ is a mild solution of the initial value problem (1.1)-(1.2) if $x_0 = \varphi$, $x(\cdot)|_J \in C(J;X)$ and

$$\begin{aligned} x(t) = T(t)(\varphi(0) - g(0,\varphi,0)) + g\left(t, x_t, \int_0^t k_0(t,s,x_s)ds\right) \\ + \int_0^t AT(t-s)g\left(s, x_s, \int_0^s k_0(s,\tau,x_\tau)d\tau\right)ds \\ + \int_0^t T(t-s)h(s,x,x_s)ds + \int_0^t T(t-s)\left(\int_0^s k_1(s,\tau,x_\tau)d\tau\right)ds \\ + \int_0^t T(t-s)f\left(s, x_s, \int_0^s k_2(s,\tau,x_\tau)d\tau\right)ds. \end{aligned}$$

For the system (1.1)-(1.2), we assume that the following hypotheses are satisfied:
$(H1)$ For each $(t,s) \in J \times J$, the functions $k_i(t,s,\cdot) : \mathcal{B} \to X,\ i = 0,1,2$ are continuous and, for each $x \in \mathcal{B}$, $k_i(\cdot,\cdot,x) : J \times J \to X$ are strongly measurable, and there exist positive constants L_i, such that

$$\|k_i(t,s,x_1) - k_i(t,s,x_2)\| \le L_i \|x_1 - x_2\|_{\mathcal{B}}, \quad i = 0,1.$$

$(H2)$ For each $t \in J$, $h(t,\cdot,\cdot) : \mathcal{B} \times X \to X$ is continuous and, for each $x \in \mathcal{B}$, the function $h(\cdot,x,y) : J \to X$ is strongly measurable.
$(H3)$ For each $t \in J$, $f(t,\cdot,\cdot) : \mathcal{B} \times X \to X$ is continuous and, for each $(x,y) \in \mathcal{B} \times X$, the function $f(\cdot,x,y) : J \to X$ is strongly measurable. And there exist integrable functions $\eta_i : J \to [0,+\infty),\ i = 1,2$, such that

$$\chi(T(s)h(t,D_1,D_2)) \le \eta_1(t) \sup_{-r\le\theta\le0} \chi(D_2(\theta)) \quad \text{for a.e. } s,t \in J,$$

$$\chi(T(s)f(t,D_3,D_4)) \le \eta_2(t) \sup_{-r\le\theta\le0} \chi(D_3(\theta)) \quad \text{for a.e. } s,t \in J,$$

where $D_i(\theta) = \{v(\theta) : v \in D_i, i = 2,3\}$.
$(H4)$ There exists $0 < \beta < 1$, such that $g(t,v) \in X_\beta = D((-A)^\beta)$, $(-A)^\beta g(\cdot)$ is continuous for all $(t,v,w) \in J \times \mathcal{B} \times X$, and there exist positive constants $C_1,\ C_2$ and C_3, such that

$$\|(-A)^{\beta}g(t,v,w)\| \le C_1\|v\|_{\mathcal{B}} + C_2\|w\| + C_3,$$

there exists a positive constant L_g such that,

$$\|(-A)^{\beta}g(t,v_1,w_1) - (-A)^{\beta}g(t,v_2,w_2)\|$$
$$\le L_g(\|v_1-v_2\|_{\mathcal{B}} + \|w_1-w_2\|), \quad \forall (v_i\times w_i)\in\mathcal{B}\times X.$$

$(H5)$ There exist integrable functions $p_i : J\to[0,\infty)$, $i=0,1,2$, such that
$|f(t,x,y)| \le p_l(t)\Omega_1(\|x\|_{\mathcal{B}}) + p_2(t)\Omega_2(\|y\|), t\in J, z\in\mathcal{B}, y\in X,$
$|h(t,x,x_t)| \le p_0(t)\Omega_0(\|x\|_t), o\le t\le b, x_t\in\mathcal{B},$
where $\Omega_i : [0,\infty)\to(0,\infty), i=0,1,2$ are continuously differentiable nondecreasing functions, such that $\lim_{s\to\infty}\Omega_0(s)=\infty, \Omega_i^{'}, i=0,1,2$ (the first derivative of Ω_i) are also nondecreasing and $\Omega_0^{'}(M\|\phi\|)>0$.
$(H6)$ There exist functions $m_i : J\times J\to[0,+\infty)$, such that m_1 is integrable and m_0, m_2 are differentiable, a.e.,with respect to the first variable, such that $\int_0^t m_i(t,s)ds$, $\int_0^t \frac{\partial m_i(t,s)}{\partial t}ds$ are bounded on J, and $\frac{\partial m_i(t,s)}{\partial t}\ge 0$ for, a.e., $0\le s<t\le b, i=1,2$. Moreover,
$\|k_i(t,s,x)\| \le m_i(t,s)\psi_i(\|x\|_{\mathcal{B}}),\ 0\le s<t\le b,\ x\in\mathcal{B},\ i=0,1,2,$
where $\psi_i : [0,+\infty)\to(0,+\infty), i=0,1,2$ are continuous nondecreasing functions.
$(H7)$

$$(1)\ \int_0^b q(s)ds < \int_a^{+\infty}((\psi_0(s)+\psi_1(s)+\Omega_0(s)+\Omega_1(s))\left(1+\frac{\Omega_1^{'}(s)}{\Omega_0^{'}(s)}\right) + \frac{\psi_2(s)}{\Omega_0^{'}(s)}\Omega_2^{'}(L\psi_2(s))^{-1}ds,$$

where

$$\alpha(t) = \frac{K_bC_2}{1-\mu_2}\left(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta}b^{\beta}\right)\left(m_0(t,t) + \int_0^t \frac{\partial m_0(t,s)}{\partial t}ds\right),$$
$$\beta(t) = \frac{K_bM}{1-\mu_2}\int_0^t m_1(t,s)ds,$$
$$\gamma(t) = m_2(t,t) + \int_0^t \|\frac{\partial m_2(t,s)}{\partial t}\|ds,$$
$$p(t) = \max\{p_0(t), p_1(t), p_2(t))\},$$
$$q(t) = \max\left\{\alpha(t), \beta(t), \gamma(t), \frac{K_bMp(t)}{1-\mu_2}\right\},$$
$$\mu_1 = M\|(-A)^{-\beta}\|(C_1\|\varphi\|_{\mathcal{B}} + C_3) + \|(-A)^{-\beta}\|C_3 + \frac{C_{1-\beta}}{\beta}b^{\beta}C_3,$$
$$\mu_2 = \left(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta}b^{\beta}\right)C_1,$$

$\alpha_o = \dfrac{\mu_1}{1-\mu_2}$, L is a finite bound for $\int_0^t m_2(t,s)ds$
and $a = \Omega_0^{-1}(\Omega_0(\alpha_0) + \Omega_1(\alpha_0) + \Omega_2(\alpha_0))$.

$$(2)\ (L_g+L_0)\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^{\beta}\right)+Mb^2L_1+\int_0^b(\eta_1(s)+\eta_2(s))ds<1.$$

Let $y:(-r,b]\to X$ be a function defined by $y_0=\varphi$ and $y(t)=T(t)\varphi(0)$ on J. Clearly,

$$\|y_t\|_{\mathcal{B}}\le(K_bMH+M_b)\|\varphi\|_{\mathcal{B}},$$

where $K_b=\sup\limits_{0\le t\le b}K(t)$, $M_b=\sup\limits_{0\le t\le b}M(t)$.

Now we are in a position to establish our main results.

Theorem 3.2. *If the hypotheses* $(H1)-(H7)$ *are satisfied, then the initial value problem* (1.1)-(1.2) *has at least one mild solution.*

Proof. Let $S(b)$ be the space $S(b)=\{x:(-r,b]\to X|x_0=0,x|_J\in C(J;X)\}$ endowed with supremum norm $\|\cdot\|_b$. Let $\Gamma:S(b)\to S(b)$ be the map defined by

$$(\Gamma x)(t)=\begin{cases}\qquad 0, & t\in(-\infty,0],\\ -T(t)g(0,\varphi,0)+g\left(t,x_t+y_t,\int_0^t k_0(t,s,x_s+y_s)ds\right)\\ +\int_0^t AT(t-s)g\left(s,x_s+y_s,\int_0^s k_0(s,\tau,x_\tau+y_\tau)d\tau\right)ds\\ +\int_0^t T(t-s)h(s,x,x_s+y_s)ds\\ +\int_0^t T(t-s)(\int_0^s k_1(s,\tau,x_\tau+y_\tau)d\tau)ds\\ +\int_0^t T(t-s)f\left(s,x_s+y_s,\int_0^s k_2(s,\tau,x_\tau+y_\tau)d\tau\right)ds, & t\in J.\end{cases}\tag{3.1}$$

It is easy to see that

$$\|x_t+y_t\|_{\mathcal{B}}\le(K_bMH+M_b)\|\varphi\|_{\mathcal{B}}+K_b\|x\|_t,$$

where $\|x\|_t=\sup\limits_{0\le s\le t}\|x(s)\|$. Thus, Γ is well defined and with values in $S(b)$. In addition, from the axioms of phase space, the Lebesgue dominated convergence theorem and the conditions, we can show that Γ is continuous.

Step 1. For $0<\lambda<1$, set $\{x\in C:x=\lambda\Gamma x\}$ is bounded. Let x_λ be a solution of $x=\lambda\Gamma x$ for $0<\lambda<1$. Then, we have

$$\|x_{\lambda t}+y_t\|_{\mathcal{B}}\le(K_bMH+M_b)\|\varphi\|_{\mathcal{B}}+K_b\|x_\lambda\|_t.$$

Let $v_\lambda(t)=(K_bMH+M_b)\|\varphi\|_{\mathcal{B}}+K_b\|x_\lambda\|_t$, for each $t\in J$. Then

$$\|x_\lambda(t)\|=\|\lambda\Gamma x_\lambda(t)\|$$
$$\le\|\Gamma x(t)\|$$

$$\le M\|(-A)^{-\beta}\|(C_1\|\varphi\|_{\mathcal{B}}+C_3)+\|(-A)^{-\beta}\|C_3+\frac{C_{1-\beta}}{\beta}b^{\beta}C_3$$
$$+\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^{\beta}\right)C_1v_\lambda(t)$$
$$+\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^{\beta}\right)C_2\int_0^t m_0(t,s)\psi_0(v_\lambda(s))ds$$
$$+M\int_0^t p_0(s)\Omega_0(v_\lambda(s))ds+M\int_0^t\left(\int_0^s m_1(s,\tau)\psi_1(v_\lambda(\tau))d\tau\right)ds$$
$$+M\int_0^t p_1(s)\Omega_1(v_\lambda(s))ds+M\int_0^t p_2(s)\Omega_2\left(\int_s^0 m_2(s,\tau)\psi_2(v_\lambda(\tau))d\tau\right)ds,$$

which implies that

$$\|x_\lambda\|_t\le M\|(-A)^{-\beta}\|(C_1\|\varphi\|_{\mathcal{B}}+C_3)+\|(-A)^{-\beta}\|C_3+\frac{C_{1-\beta}}{\beta}b^{\beta}C_3$$
$$+\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^{\beta}\right)C_1v_\lambda(t)$$
$$+\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^{\beta}\right)C_2\int_0^t m_0(t,s)\psi_0(v_\lambda(s))ds$$
$$+M\int_0^t p_0(s)\Omega_0(v_\lambda(s))ds+M\int_0^t\left(\int_0^s m_1(s,\tau)\psi_1(v_\lambda(\tau))d\tau\right)ds$$
$$+M\int_0^t p_1(s)\Omega_1(v_\lambda(s))ds+M\int_0^t p_2(s)\Omega_2\left(\int_0^s m_2(s,\tau)\psi_2(v_\lambda(\tau))d\tau\right)ds,$$

and hence

$$\begin{aligned}
&v_\lambda(t)\\
&\le\frac{\mu_1}{1-\mu_2}\\
&\quad+\frac{K_bC_2}{1-\mu_2}\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^{\beta}\right)\int_0^t\left(\int_0^s m_0(s,\tau)\psi_o(v_\lambda(\tau))d\tau\right)ds\\
&\quad+\frac{K_bM}{1-\mu_2}\int_0^t\left(\int_0^s m_1(s,\tau)\psi_1(v_\lambda(\tau))d\tau\right)ds \qquad (3.2)\\
&\quad+\frac{K_bM}{1-\mu_2}\int_0^t p_0(s)\Omega_0(v_\lambda(s))ds+\frac{K_bM}{1-\mu_2}\int_0^t p_1(s)\Omega_1(v_\lambda(s))ds\\
&\quad+\frac{K_bM}{1-\mu_2}\int_0^t p_2(s)\Omega_2\left(\int_0^s m_2(s,\tau)\psi_2(v_\lambda(\tau))d\tau\right)ds.
\end{aligned}$$

Denoting by $u_\lambda(t)$ the right-hand side of (3.2), we get $u_\lambda(o)=\frac{\mu_1}{1-\mu_2}\equiv\alpha_0$, and

$$u_\lambda^{'}(t)=\frac{K_bC_2}{1-\mu_2}\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^{\beta}\right)\left(m_0(t,t)\psi_0(v_\lambda(t))\right.$$
$$\left.+\int_0^t\frac{\partial m_0(t,s)}{\partial t}\psi_o(v_\lambda(s))ds\right)$$

$$
\begin{aligned}
&+\frac{K_bM}{1-\mu_2}\int_0^t m_1(t,s)\psi_1(v_\lambda(s))ds+\frac{K_bM}{1-\mu_2}p_0(t)\Omega_0(v_\lambda(t))\\
&+\frac{K_bM}{1-\mu_2}p_1(t)\Omega_1(v_\lambda(t))\\
&+\frac{K_bM}{1-\mu_2}p_2(t)\Omega_2\left(\int_0^t m_2(t,s)\psi_2(v_\lambda(s))ds\right)\\
\leq&\frac{K_bC_2}{1-\mu_2}\left(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^\beta\right)\left(m_0(t,t)\psi_0(u_\lambda(t))\right.\\
&\left.+\int_0^t\frac{\partial m_0(t,s)}{\partial t}\psi_o(u_\lambda(s))ds\right)+\frac{K_bM}{1-\mu_2}\int_0^t m_1(t,s)\psi_1(u_\lambda(s))ds\\
&+\frac{K_bM}{1-\mu_2}p(t)(\Omega_0(u_\lambda(t))+\Omega_1(u_\lambda(t))\\
&+\Omega_2\left(\int_0^t m_2(t,s)\psi_2(u_\lambda(s))ds)\right).
\end{aligned}
$$

Next, let $w(t)$ be such that

$$
\Omega_0(w)=\Omega_0(u_\lambda)+\Omega_1(u_\lambda)+\Omega_2\bigg(\int_0^t m_2(t,s)\psi_2(u_\lambda)ds\bigg).
$$

We have $w\geq u_\lambda$, and by differentiation, taking into account $(H6)$, we get,

$$
\begin{aligned}
&\Omega_0'(w)w'\\
&=\Omega_0'(u_\lambda)u_\lambda'+\Omega_1'(u_\lambda)u_\lambda'\\
&\quad+\Omega_2'\bigg(\int_0^t m_2(t,s)\psi_2(u_\lambda)ds\bigg)\bigg(m_2(t,t)\psi_2(u_\lambda)\\
&\quad+\int_0^t\frac{\partial m_2(t,s)}{\partial t}\psi_2(u_\lambda)ds)\bigg)\\
&\leq\frac{\Omega_0'(w)+\Omega_1'(w)}{1-\mu_2}\bigg(K_bC_2\bigg(\|(-A)^{-\beta}\|+\frac{C_{1-\beta}}{\beta}b^\beta\bigg)\bigg(m_0(t,t)\psi_0(w)\\
&\quad+\int_0^t\frac{\partial m_0(t,s)}{\partial t}\psi_o(w)ds\bigg)\\
&\quad+K_bM\int_0^t m_1(t,s)\psi_1(w)ds+K_bMp(t)(\Omega_0(w)+\Omega_1(w))\bigg)\\
&\quad+\Omega_2'\bigg(\psi_2(w)\int_0^t m_2(t,s)ds\bigg)\psi_2(w)\bigg(m_2(t,t)+\int_0^t\|\frac{\partial m_2(t,s)}{\partial t}\|ds\bigg).
\end{aligned}
\tag{3.3}
$$

Moreover, by our assumptions on $\Omega_0^{'}$, we have

$$\Omega_0^{'}(w) \geq \Omega_0^{'}(u_\lambda) \geq \Omega_0^{'}(u_\lambda(0)) \geq \Omega_0^{'}(M\|\varphi\|_{\mathcal{B}}) > 0.$$

Therefore, inequality (3.3) implies that

$$\begin{aligned} w^{'} \leq & (\frac{\psi_0(w)}{1-\mu_2}(K_b C_2\bigg(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta}b^\beta\bigg)\bigg(m_0(t,t) + \int_0^t \frac{\partial m_0(t,s)}{\partial t}ds\bigg) \\ & + \frac{K_b M\psi_1(w)}{1-\mu_2}\int_0^t m_1(t,s)ds + \frac{K_b M p(t)}{1-\mu_2}(\Omega_0(w)+\Omega_1(w))\bigg(1+\frac{\Omega_1^{'}(w)}{\Omega_0^{'}(w)}\bigg) \\ & + \frac{\psi_2(w)}{\Omega_0^{'}(w)}\Omega_2^{'}\bigg(\psi_2(w)\int_0^t m_2(t,s)ds\bigg)\bigg(m_2(t,t) + \int_0^t \|\frac{\partial m_2(t,s)}{\partial t}\|ds\bigg) \end{aligned}$$

or, using the notation in $(H7)$, one finds

$$\begin{aligned} w^{'} \leq & \bigg(\alpha(t)\psi_0(w) + \beta(t)\psi_1(w) + \frac{K_b M p(t)}{1-\mu_2}(\Omega_0(w)+\Omega_1(w))\bigg(1+\frac{\Omega_1^{'}(w)}{\Omega_0^{'}(w)}\bigg) \\ & + \frac{\gamma(t)\psi_2(w)}{\Omega_0^{'}(w)}\Omega_2^{'}(L\psi_2(w)\bigg) \\ \leq & \, q(t)\bigg((\psi_0(w)+\psi_1(w)+\Omega_0(w)+\Omega_1(w))\bigg(1+\frac{\Omega_1^{'}(w)}{\Omega_0^{'}(w)}\bigg) \\ & + \frac{\psi_2(w)}{\Omega_0^{'}(w)}\Omega_2^{'}(L\psi_2(w))\bigg). \end{aligned}$$

Thus, by $(H9)$, for $0 \leq t \leq b$,

$$\begin{aligned} & \int_{w(0)}^{w(t)} ((\psi_0(s)+\psi_1(s)+\Omega_0(s)+\Omega_1(s))\bigg(1+\frac{\Omega_1^{'}(s)}{\Omega_0^{'}(s)}\bigg) \\ & \quad + \frac{\psi_2(s)}{\Omega_0^{'}(s)}\Omega_2^{'}(L\psi_2(s))^{-1}ds \\ & \leq \int_0^b q(s)ds \\ & < \int_a^{+\infty} ((\psi_0(s)+\psi_1(s)+\Omega_0(s)+\Omega_1(s))\bigg(1+\frac{\Omega_1^{'}(s)}{\Omega_0^{'}(s)}\bigg) \\ & \quad + \frac{\psi_2(s)}{\Omega_0^{'}(s)}\Omega_2^{'}(L\psi_2(s))^{-1}ds, \end{aligned}$$

which implies that the function $w(t)$ is bounded on J . Thus, the function $v_\lambda(t)$ is bounded on J, and $x_\lambda(\cdot)$ is also bounded on J.

Step 2. Next, we show that Γ is $\chi-$*contraction.* To clarify this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, for $t \geq 0$, where

$$\Gamma_1 x(t) = -T(t)g(0,\varphi,0) + g\Big(t, x_t + y_t, \int_0^t k_0(t,s,x_s+y_s)ds\Big) + \int_0^t AT(t-s)g\Big(s, x_s+y_s, \int_0^s k_0(s,\tau,x_\tau+y_\tau)d\tau\Big)ds + \int_0^t T(t-s)\Big(\int_0^s k_1(s,\tau,x_\tau+y_\tau)d\tau\Big)ds$$

and

$$\Gamma_2 x(t) = \int_0^t T(t-s)h(s,x,x_s+y_s)ds + \int_0^t T(t-s)f\Big(s, x_s+y_s, \int_0^s k_2(s,\tau,x_\tau+y_\tau)d\tau\Big)ds.$$

First, we show the Γ_1 is *Lipschitz* continuous. For arbitrary $x_1, x_2 \in S(b)$, from Definition 2.2 and hypothese conditions, we get

$$\begin{aligned}&\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\| \\ &\le \|(-A)^{-\beta}\|\Big(L_g\|x_{1t}-x_{2t}\|_{\mathcal{B}} + L_0\int_0^t \|x_{1s}-x_{2s}\|_{\mathcal{B}}ds\Big) \\ &+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\Big(L_g\|x_{1s}-x_{2s}\|_{\mathcal{B}} + L_0\int_0^s \|x_{1\tau}-x_{2\tau}\|_{\mathcal{B}}d\tau\Big)ds \\ &+ M\int_0^t \Big(\int_0^s L_1\|x_{1\tau}-x_{2\tau}\|_{\mathcal{B}}d\tau)\Big)ds,\end{aligned}$$

that is,

$$\begin{aligned}&\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\|_b \\ &\le \Big((L_g+L_0)\Big(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta}b^\beta\Big) + Mb^2L_1\Big)\|x_{1s}-x_{2s}\|_b.\end{aligned}$$

Hence, Γ_1 is *Lipschitz* continuous with *Lipschitz* constant

$$L' = (L_g+L_0)(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta}b^\beta) + Mb^2L_1.$$

Next, taking $W \subset S(b)$. Since $T(t)$ is equicontinuous, thus, $T(t-s)f(s, W_s + y_s, \int_0^s k_2(s,\tau,W_\tau+y_\tau)d\tau)$ is equicontinuous. By $\chi_C(W) = \sup\{\chi(W(t)), t \in J\}$, we have

$$\begin{aligned}&\chi(\Gamma_2 w(t)) \\ &= \chi\Big(\int_0^t T(t-s)h(s,x,W_s+y_s)ds \\ &+ \int_0^t T(t-s)f\Big(s, W_s+y_s, \int_0^s k_2(s,\tau,W_\tau+y_\tau)d\tau\Big)ds\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \chi\Big(\int_0^t T(t-s)h(s,x,W_s+y_s)ds\Big)\\
&\quad+\chi\Big(\int_0^t T(t-s)f\Big(s,W_s+y_s,\int_0^s k_2(s,\tau,W_\tau+y_\tau)d\tau\Big)ds\Big)\\
&\leq \int_0^t(\eta_1(s)+\eta_2(s))\sup_{-r<\theta\leq 0}\chi(W(s+\theta)+y(s+\theta))ds\\
&\leq \int_0^t(\eta_1(s)+\eta_2(s))\sup_{0\leq\tau\leq s}\chi W(\tau)ds\\
&\leq \chi_C(W)\int_0^t(\eta_1(s)+\eta_2(s))ds,
\end{aligned}
$$

for each bounded set $W \in C(J;X)$. Since

$$
\begin{aligned}
\chi_C(\Gamma W) &= \chi_C(\Gamma_1 W+\Gamma_2 W)\\
&\leq \chi_C(\Gamma_1 W)+\chi_C(\Gamma_2 W)\\
&\leq \Big(L^{'}+\int_0^t(\eta_1(s)+\eta_2(s))ds\Big)\chi_C(W)\\
&\leq \chi_C(W),
\end{aligned}
$$

Γ is $\chi-$ *contraction*. In view of Lemma 2.6, that is, Darbo-Sadovskii fixed point theorem, we conclude that Γ has at least one fixed point in W. Let x is a fixed point of Γ on $S(b)$. Then $z = x+y$ is a mild solution of (1.1)-(1.2). So we deduce the existence of a mild solution of (1.1)-(1.2). □

Remark 3.3. The result above improves and generalizes, the Theorem 3.1 in [4].

4. Application

In the next application, $\mathcal{B}$ will be the phase space $C_0 \times L^2(h,X)$. (see [7])

As an application of this theorem, we shall consider the system with a control parameter, such as

$$
\begin{aligned}
&\frac{d}{dt}\Big(x(t)-g\Big(t,x_t,\int_0^t k_0(t,s,x_s)ds\Big)\Big)\\
&= Ax(t)+Bu(t)+h(t,x,x_t) \qquad (4.1)\\
&\quad+\int_0^t k_1(t,s,x_s)ds+f\Big(t,x_t,\int_0^t k_2(t,s,x_s)ds\Big),\quad t\in J=[0,b],
\end{aligned}
$$

$$x_0=\varphi\in\mathcal{B},\quad \text{on } [-r,0]; \qquad (4.2)$$

where B is a bounded linear operator from a Banach space U into X and $u \in L^2(J,U)$. The mild solution is given by

$$\begin{aligned} x(t) &= T(t)(\varphi(0) - g(0,\varphi,0)) + g\left(t, x_t, \int_0^t k_0(t,s,x_s)ds\right) \\ &+ \int_0^t AT(t-s)g\left(s, x_s, \int_0^s k_0(s,\tau,x_\tau)d\tau\right)ds \\ &+ \int_0^t T(t-s)(Bu(s) + h(s,x,x_s))ds \\ &+ \int_0^t T(t-s)\left(\int_0^s k_1(s,\tau,x_\tau)d\tau\right)ds \\ &+ \int_0^t T(t-s)f\left(s, x_s, \int_0^s k_2(s,\tau,x_\tau)d\tau\right)ds. \end{aligned}$$

Definition 4.1. System (4.1)-(4.2) is controllable to the origin on the interval J if, for every continuous initial function $\varphi \in \mathcal{B}$, there exists a control $u \in L^2(J,U)$, such that the mild solution $x(t)$ of (4,1)-(4.2) satisfies $x(b) = 0$.

For the controllability of neutral systems, one can refer the book [15] and the survey paper [16]. To establish the controllability result, we need the following additional hypotheses.
$(H8)$ The linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)ds,$$

has induced an inverse operator $\widetilde{W}^{-1}$, which takes values in $L^2(J,U)/kerW$, and there exists a positive constant M_1, such that

$$\|B\widetilde{W}^{-1}\| \leq M_1.$$

$(H9)$

$$\begin{aligned} \int_0^b q(s)ds < \int_c^{+\infty} &((\psi_0(s) + \psi_1(s) + \Omega_0(s) + \Omega_1(s))\left(1 + \frac{\Omega_1'(s)}{\Omega_0'(s)}\right) \\ &+ \frac{\psi_2(s)}{\Omega_0'(s)}\Omega_2'(L\psi_2(s))^{-1}ds, \end{aligned}$$

where $q(t) = \max\left\{\alpha(t), \beta(t), \gamma(t), \dfrac{K_b M p(t)}{1-\mu_2}\right\}$. $c = \dfrac{\mu_1 + MM_1Nb}{1-\mu_2}$, where

$$\mu_1 = M\|(-A)^{-\beta}\|(C_1\|\varphi\|_{\mathcal{B}} + C_3) + \|(-A)^{-\beta}\|C_3 + \frac{C_{1-\beta}}{\beta}b^\beta C_3,$$

$$\mu_2 = (\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta} b^\beta) C_1,$$

and

$$\begin{aligned} N = {} & M\|(-A)^{-\beta}\|(C_1\|\varphi\|_{\mathcal{B}} + C_3) + \|(-A)^{-\beta}\|C_3 + \frac{C_{1-\beta}}{\beta} b^\beta C_3 \\ & + \left(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta} b^\beta\right) C_1 \|x\|_b \\ & + \left(\|(-A)^{-\beta}\| + \frac{C_{1-\beta}}{\beta} b^\beta\right) C_2 \int_0^b m_0(t,s)\psi_0(\|x_s\|_{\mathcal{B}})ds \\ & + M\int_0^b p_0(s)\Omega_0(\|x_s\|_{\mathcal{B}})ds \\ & + M\int_0^b \left(\int_0^s m_1(s,\tau)\psi_1(\|x_\tau\|_{\mathcal{B}})d\tau\right)ds \\ & + M\int_0^b p_1(s)\Omega_1(\|x_s\|_{\mathcal{B}})ds \\ & + M\int_0^b p_2(s)\Omega_2\left(\int_0^s m_2(s,\tau)\psi_2(\|x_\tau\|_{\mathcal{B}})d\tau\right)ds. \end{aligned}$$

Theorem 4.2. *If Hypotheses* $(H1)$-$(H7)$, $(H8)$ *and* $(H9)$ *are satisfied, then, system* (4.1)-(4.2) *is controllable.*

Proof. Using Hypoghesis $(H8)$, for an arbitrary function $X(\cdot)$ define the control

$$\begin{aligned} u(t) = {} & -\widetilde{W}^{-1}\bigg(T(b)(\varphi(0) - g(0,\varphi,0)) + g\left(b, x_b, \int_0^b k_0(b,s,x_s)ds\right) \\ & + \int_0^b AT(b-s)g\left(s, x_s, \int_0^s k_0(s,\tau,x_\tau)d\tau\right)ds \\ & + \int_0^b T(b-s)h(s,x,x_s)ds + \int_0^b \left(T(b-s)\int_0^s k_1(s,\tau,x_\tau)d\tau\right)ds \\ & + \int_0^b T(b-s)f\left(s, x_s, \int_0^s k_2(s,\tau,x_\tau)d\tau\right)ds\bigg)(t). \end{aligned}$$

Let $y : (-\infty, b] \to X$ be a function defined by $y_0 = \varphi$ and $y(t) = T(t)\varphi(0)$ on J. Let $S(b)$ be the space $S(b) = \{x : (-\infty, b] \to X | x_0 = 0, x|_J \in C(J; X)\}$ endowed with supremum norm $\|\cdot\|_b$. We shall show that when using this control, the operator $\Gamma : S(b) \to S(b)$ defined by

$$(\Gamma x)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ -T(t)g(0,\varphi,0) + g\left(t, x_t + y_t, \int_0^t k_0(t,s,x_s+y_s)ds\right) \\ + \int_0^t AT(t-s)g\left(s, x_s + y_s, \int_0^s k_0(s,\tau,x_\tau + y_\tau)d\tau\right)ds \\ + \int_0^t T(t-s)h(s,x,x_s+y_s)ds \\ + \int_0^t T(t-s)\left(\int_0^s k_1(s,\tau,x_\tau+y_\tau)d\tau\right)ds \\ + \int_0^t T(t-s)f\left(s, x_s+y_s, \int_0^s k_2(s,\tau,x_\tau+y_\tau)d\tau\right)ds, & t \in J \end{cases} \tag{4.3}$$

has a fixed point. Let x is a fixed of Γ on $S(b)$. Then $z = x + y$ is a solution of (4.1)-(4.2). Substituting u(t) in the above equation, we get $(\Gamma x)(b) = 0$, which means that the control u steers system (4.1)-(4.2) from the given initial function φ to the origin in time b, provided we can obtain a fixed point of the nonlinear operator Γ. The remaining part of the proof is similar to Theorem 3.2, and hence, it is omitted. □

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