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STRONG CONVERGENCE THEORMS FOR VARIATIONAL INCLUSION PROBLEMS AND STRICT PSEUDO-CONTRACTIONS IN HILBERT SPACES

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Abstract. In this paper, we introduce a new iterative scheme for finding a common element of the set of fixed points of a $k-$strictly pseudo-contractive mapping and the set of solutions of a variational inclusion for an $\alpha-$inverse-strongly monotone mapping and a maximal monotone mapping in a real Hilbert space. Then we show that the sequence converges strongly to a common element of two sets.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $A : H \to H$ be a single-valued mapping and $M : H \to 2^H$ be a multivalued mapping. Then, we consider the following variational inclusion problem which is to find $u \in H$ such that

$$0 \in A(u) + M(u). \tag{1.1}$$

The set of solutions of the variational inclusion(1.1) is denoted by $VI(H, A, M)$. Special Cases.

(1) When M is a maximal monotone mapping and A is a strongly monotone and Lipschitz continuous mapping, problem (1.1) has been studied by Huang [8].

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(2) If $M = \partial\phi$, where $\partial\phi$ denotes the subdifferential of a proper, convex and lower semi-continuous function $\phi : H \to R\bigcup\{+\infty\}$, then problem (1.1) reduces to the following problem: find $u \in H$, such that

$$\langle A(u), v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{1.2}$$

which is called a nonlinear variational inequality and has been studied by many authors; see, for example, [1-2].

(3) If $M = \partial\delta_C$, where δ_C is the indicator function of C, then problem (1.1) reduces to the following problem: find $u \in C$, such that

$$\langle A(u), v-u\rangle \geq 0, \quad \forall v \in C, \tag{1.3}$$

which is the classical variational inequality; see,e.g., [7,9].

A mapping $A : H \to H$ is called *inverse-strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x, y \in H.$$

Such a mapping A is also called *α-inverse-strongly monotone*. If A is an α-inverse-strongly monotone mapping of H to H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in H$, and $\lambda > 0$,

$$\begin{aligned} &\|(I-\lambda A)x - (I-\lambda A)y\|^2 \\ &= \|(x-y) - \lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2. \end{aligned} \tag{1.4}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of H into H. See [9] for some examples of inverse-strongly monotone mappings.

A mapping T of C into itself is nonexpansive if $\|Tx-Ty\| \leq \|x-y\|, \forall x, y \in C$. Recently, Iiduka and Takahashi [9], Takahashi and Toyoda [17], Chen et al. [5] , Nadezhkina and Takahashi [12], Ceng and Yao [3], Yao and Yao [19] introduced many iterative methods for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality (1.3) for an α-inverse-strongly monotone mapping, they obtained some weak and strong convergence theorems.

On the other hand, Liu and Chen [10] introduced a hybrid iterative method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inclusion problem (1.1) for a maximal monotone mapping and an $\alpha-$inverse-strongly monotone mapping.

A mapping $S : C \to H$ is said to be *$k-$strictly pseudo-contractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Sx-Sy\|^2 \leq \|x-y\|^2 + k\|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in C. \tag{1.5}$$

Note that the class of $k-$strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, S is nonexpansive if and only if S is 0-strictly pseudo-contractive.

The set of fixed points of S is denoted by $F(S)$. Very recently, by using the general approximation method Qin et al. [15] obtained a strong convergence theorem for finding an element of $F(S)$.

Since variational inclusion problem (1.1) is the generalization of variational inequality (1.3) and the class of $k-$strict pseudo-contractions is the generalization of the class of nonexpansive mappings, motivated and inspired by the above results, we introduce a new iteration scheme for finding a common element of the set of fixed points of a $k-$strict pseudo-contraction and the set of solutions of variational inclusion problem (1.1) for a maximal monotone mapping and an $\alpha-$inverse-strongly monotone mapping and then obtain a strong convergence theorem.

2. Preliminaries

Throughout this paper, we always let X be a real Banach space with dual space X^*, H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by N and R the sets of positive integers and real numbers, respectively. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

Such a P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C x \quad \Leftrightarrow \quad \langle x - u, u - y\rangle \ge 0, \quad \forall y \in C.$$

A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in Mx, v \in My$ imply $\langle x - y, u - v\rangle \ge 0$. A monotone mapping $M : H \to 2^H$ is maximal if the graph $G(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, u) \in H \times H, \langle x - y, u - v\rangle \ge 0$ for every $(y, v) \in G(M)$ implies $u \in Mx$.

The following definitions and lemmas are useful for our paper.

Definition 2.1. ([13]) If M is a maximal monotone mapping on H, then the resolvent operator associated with M is defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}u, \quad \forall u \in H,$$

where $\lambda > 0$ is a constant and I is the identity operator.

Definition 2.2. ([6]) A single-valued operator $A : H \to H$ is said to be hemi-continuous if for any fixed $x, y, z \in H$, the function $t \to \langle A(x+ty), z\rangle$ is continuous at 0^+. It is well known that a continuous mapping must be hemi-continuous.

Definition 2.3. ([6]) A set-valued mapping $A : X \to 2^{X^*}$ is said to be bounded if $A(B)$ is bounded for every bounded subset B of X.

Lemma 2.4. ([13]) *The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive, that is,*

$$\|J_{M,\lambda}(u) - J_{M,\lambda}(v)\| \leq \|u - v\|, \quad \forall u, v \in H.$$

Lemma 2.5. ([10]) *The resolvent operator $J_{M,\lambda}$ is firmly nonexpansive, that is*

$$\langle J_{M,\lambda}u - J_{M,\lambda}v, u - v\rangle \geq \|J_{M,\lambda}u - J_{M,\lambda}v\|^2, \quad \forall u, v \in H.$$

Lemma 2.6. *Let M, $J_{M,\lambda}$ be as in Definition 2.1. Then the following holds:*

$$\|J_{M,\lambda}x - J_{M,\mu}x\|^2 \leq \frac{\mu - \lambda}{\mu}\langle J_{M,\lambda}x - J_{M,\mu}x, x - J_{M,\mu}x\rangle,$$

for all $\lambda, \mu > 0$ and $x \in H$.

Proof. For $\lambda, \mu > 0$ and $x \in H$, put $J_{M,\lambda}x = (I + \lambda M)^{-1}x = u$, $J_{M,\mu}x = (I + \mu M)^{-1}x = v$. Then we obtain that $\frac{x-u}{\lambda} \in Mu$, and $\frac{x-v}{\mu} \in Mv$. So, we have $\langle u - v, \frac{x-u}{\lambda} - \frac{x-v}{\mu}\rangle \geq 0$. Hence, $\langle u - v, u - \frac{\lambda}{\mu}v\rangle \leq \langle u - v, (1 - \frac{\lambda}{\mu})x\rangle$. That is $\langle u - v, u - v + v - \frac{\lambda}{\mu}v\rangle \leq \langle u - v, (1 - \frac{\lambda}{\mu})x\rangle$. So, we have $\|u - v\|^2 \leq (1 - \frac{\lambda}{\mu})\langle u - v, x - v\rangle$. □

Lemma 2.7. ([14]) *If $T : X \to 2^{X^*}$ is a maximal monotone mapping and $P : X \to X^*$ is a hemi-continuous bounded monotone operator with $D(P) = X$, then the sum $S = T + P$ is a maximal monotone mapping.*

Lemma 2.8. ([11]) *If $S : C \to C$ is a $k-$strict pseudo-contraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, then $(I - S)\tilde{x} = 0$.*

Lemma 2.9. ([11]) *If $S : C \to C$ is a $k-$strict pseudo-contraction, then the fixed point set $F(S)$ of S is closed and convex.*

Lemma 2.10. ([15]) *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.11. ([16]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence of $[0,1]$ such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$ for all $n \in N$ and $\limsup_{n\to\infty}\{\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\} \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.12. ([18]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.13. ([4]) *In a Hilbert space H, there holds the inequality:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, (x+y)\rangle, \quad \forall x, y \in H.$$

Lemma 2.14. *The function $u \in H$ is a solution of variational inclusion* (1.1) *if and only if $u \in H$ satisfies the relation*

$$u = J_{M,\lambda}[u - \lambda Au],$$

where $\lambda > 0$ is a constant, M is a maximal monotone mapping and $J_{M,\lambda} = (I + \lambda M)^{-1}$ is the resolvent operator.

Proof. Using Definition 2.1, we can obtain the desired result. □

Lemma 2.15. *Let $M : H \to 2^H$ be a maximal monotone mapping. Let $A : H \to H$ be an $\alpha-$inverse-strongly monotone mapping. Then $VI(H, A, M)$ is closed and convex.*

Proof. It follows from Lemma 2.14 that $VI(H, A, M) = F(J_{M,\lambda}(I - \lambda A))$ (the set of fixed points of $J_{M,\lambda}(I - \lambda A)$), where $\lambda \leq 2\alpha$. By Lemma 2.4 and formula (1.4), we have $J_{M,\lambda}(I - \lambda A)$ is a nonexpansive mapping of H into itself. Thus, $VI(H, A, M)$ is closed and convex. □

3. Main Results

Theorem 3.1. *Let H be a real Hilbert space. Let A be an $\alpha-$inverse-strongly monotone mapping of H into itself and $M : H \to 2^H$ be a maximal monotone mapping. Let $S : H \to H$ be a $k-$strictly pseudo-contractive mapping such that $F(S) \bigcap VI(H, A, M) \neq \emptyset$. Let $u \in H$ and $x_1 \in H$ and let $\{z_n\} \subset H$ and $\{x_n\} \subset H$ be sequences generated by*

$$\begin{cases} z_n = J_{M,\lambda_n}(x_n - \lambda_n A x_n), \\ y_n = \alpha_n u + (1 - \alpha_n) z_n, \\ v_n = \gamma_n y_n + (1 - \gamma_n) S y_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) v_n, \quad \forall n \in N, \end{cases}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1), \{\gamma_n\} \subset [0,1)$ and $\{\lambda_n\} \subset (0, 2\alpha)$ satisfy the following conditions:

(B1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(B2) $0 < c \leq \beta_n \leq d < 1, 0 < a \leq \lambda_n \leq b < 2\alpha, \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;
(B3) $0 \leq k \leq \gamma_n \leq \gamma < 1$ *and* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Then, $\{x_n\}$ converges strongly to $P_{F(S)\bigcap VI(H,A,M)} u$.

Proof. Take $p \in F(S) \bigcap VI(H, A, M)$. By $p = J_{M,\lambda_n}(p - \lambda_n A p)$, Lemma 2.4 and (1.4), we know that, for any $n \in N$,

$$\begin{aligned} \|z_n - p\|^2 &= \|J_{M,\lambda_n}(x_n - \lambda_n A x_n) - J_{M,\lambda_n}(p - \lambda_n A p)\|^2 \\ &\leq \|(x_n - \lambda_n A x_n) - (p - \lambda_n A p)\|^2 \\ &\leq \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A x_n - A p\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.1}$$

So, we obtain

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n(u - p) + (1 - \alpha_n)(z_n - p)\| \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|z_n - p\| \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|. \end{aligned}$$

By Lemma 2.10 and $(B3)$, we have

$$
\begin{aligned}
\|v_n - p\|^2 &= \|\gamma_n y_n + (1-\gamma_n)Sy_n - p\|^2 \\
&= \|\gamma_n(y_n - p) + (1-\gamma_n)(Sy_n - p)\|^2 \\
&= \gamma_n\|y_n - p\|^2 + (1-\gamma_n)\|Sy_n - p\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|y_n - Sy_n\|^2 \\
&\le \gamma_n\|y_n - p\|^2 + (1-\gamma_n)(\|y_n - p\|^2 + k\|y_n - Sy_n\|^2) \\
&\quad - \gamma_n(1-\gamma_n)\|y_n - Sy_n\|^2 \\
&= \|y_n - p\|^2 - (1-\gamma_n)(\gamma_n - k)\|y_n - Sy_n\|^2 \\
&\le \|y_n - p\|^2.
\end{aligned} \tag{3.2}
$$

So, we have that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(v_n - p)\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)(\alpha_n\|u - p\| + (1-\alpha_n)\|x_n - p\|) \\
&= (1 - \alpha_n(1-\beta_n))\|x_n - p\| + \alpha_n(1-\beta_n)\|u - p\|.
\end{aligned}
$$

Putting $M = \max\{\|x_1 - p\|, \|u - p\|\}$, we have that $\|x_n - p\| \le M$ for all $n \in N$. In fact, it is obvious that $\|x_1 - p\| \le M$. Suppose that $\|x_k - p\| \le M$ for some $k \in N$, then, we have that $\|x_{k+1} - p\| \le (1 - \alpha_k(1-\beta_k))M + \alpha_k(1-\beta_k)M = M$. By induction, we obtain that $\|x_n - p\| \le M$ for all $n \in N$. So, $\{x_n\}$ is bounded. Hence, $\{Ax_n\}, \{y_n\}, \{v_n\}$ and $\{z_n\}$ are also bounded. Putting $u_n = x_n - \lambda_n Ax_n$, we have

$$
\begin{aligned}
y_{n+1} - y_n &= \alpha_{n+1}u + (1-\alpha_{n+1})z_{n+1} - (\alpha_n u + (1-\alpha_n)z_n) \\
&= (\alpha_{n+1} - \alpha_n)u + (1-\alpha_{n+1})(J_{M,\lambda_{n+1}}u_{n+1} - J_{M,\lambda_{n+1}}u_n \\
&\quad + J_{M,\lambda_{n+1}}u_n - J_{M,\lambda_n}u_n + J_{M,\lambda_n}u_n) - (1-\alpha_n)J_{M,\lambda_n}u_n.
\end{aligned}
$$

So, we have that

$$
\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\le |\alpha_{n+1} - \alpha_n|\|u\| + (1-\alpha_{n+1})\|u_{n+1} - u_n\| \\
&\quad + (1-\alpha_{n+1})\|J_{M,\lambda_{n+1}}u_n - J_{M,\lambda_n}u_n\| + |\alpha_{n+1} - \alpha_n|\|J_{M,\lambda_n}u_n\| \\
&\le |\alpha_{n+1} - \alpha_n|\|u\| + \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\|J_{M,\lambda_n}u_n\| + (1-\alpha_{n+1})\|J_{M,\lambda_{n+1}}u_n - J_{M,\lambda_n}u_n\|.
\end{aligned} \tag{3.3}
$$

Note that

$$\begin{aligned}&\|v_{n+1}-v_n\|\\&=\|\gamma_{n+1}y_{n+1}+(1-\gamma_{n+1})Sy_{n+1}-\gamma_n y_n-(1-\gamma_n)Sy_n\|\\&=\|\gamma_{n+1}y_{n+1}-\gamma_{n+1}y_n+(1-\gamma_{n+1})Sy_{n+1}-(1-\gamma_{n+1})Sy_n\\&\quad+\gamma_{n+1}y_n+(1-\gamma_{n+1})Sy_n-\gamma_n y_n-(1-\gamma_n)Sy_n\|.\end{aligned}\tag{3.4}$$

It follows from Lemma 2.10 that

$$\begin{aligned}&\|\gamma_{n+1}y_{n+1}-\gamma_{n+1}y_n+(1-\gamma_{n+1})Sy_{n+1}-(1-\gamma_{n+1})Sy_n\|^2\\&=\gamma_{n+1}\|y_{n+1}-y_n\|^2+(1-\gamma_{n+1})\|Sy_{n+1}-Sy_n\|^2\\&\quad-\gamma_{n+1}(1-\gamma_{n+1})\|y_{n+1}-Sy_{n+1}-(y_n-Sy_n)\|^2\\&\le\gamma_{n+1}\|y_{n+1}-y_n\|^2+(1-\gamma_{n+1})(\|y_{n+1}-y_n\|^2\\&\quad+k\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\|^2)\\&\quad-\gamma_{n+1}(1-\gamma_{n+1})\|y_{n+1}-Sy_{n+1}-(y_n-Sy_n)\|^2\\&=\|y_{n+1}-y_n\|^2-(1-\gamma_{n+1})(\gamma_{n+1}-k)\|y_{n+1}-Sy_{n+1}-(y_n-Sy_n)\|^2\\&\le\|y_{n+1}-y_n\|^2.\end{aligned}\tag{3.5}$$

Combining (3.4) with (3.5), we obtain

$$\begin{aligned}\|v_{n+1}-v_n\|&\le\|y_{n+1}-y_n\|\\&\quad+\|\gamma_{n+1}y_n+(1-\gamma_{n+1})Sy_n-\gamma_n y_n-(1-\gamma_n)Sy_n\|\\&\le\|y_{n+1}-y_n\|+\|y_n-Sy_n\||\gamma_{n+1}-\gamma_n|.\end{aligned}\tag{3.6}$$

It follows from (3.6) and (3.3) that

$$\begin{aligned}&\|v_{n+1}-v_n\|\le\|y_{n+1}-y_n\|+\|y_n-Sy_n\||\gamma_{n+1}-\gamma_n|\\&\le|\alpha_{n+1}-\alpha_n|\|u\|+\|x_{n+1}-x_n\|\\&\quad+|\lambda_{n+1}-\lambda_n|\|Ax_n\|+|\alpha_{n+1}-\alpha_n|\|J_{M,\lambda_n}u_n\|\\&\quad+(1-\alpha_{n+1})\|J_{M,\lambda_{n+1}}u_n-J_{M,\lambda_n}u_n\|+\|y_n-Sy_n\||\gamma_{n+1}-\gamma_n|.\end{aligned}\tag{3.7}$$

Therefore, we have

$$\begin{aligned}&\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|\\&\le|\alpha_{n+1}-\alpha_n|\|u\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|+|\alpha_{n+1}-\alpha_n|\|J_{M,\lambda_n}u_n\|\\&\quad+(1-\alpha_{n+1})\|J_{M,\lambda_{n+1}}u_n-J_{M,\lambda_n}u_n\|+\|y_n-Sy_n\||\gamma_{n+1}-\gamma_n|.\end{aligned}$$

It follows from Lemma 2.6 , $(B1)$, $(B2)$, $(B3)$ that

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|)\le 0.$$

From Lemma 2.11, we get

$$v_n-x_n\to 0,\quad (n\to\infty).\tag{3.8}$$

Consequently, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|v_n - x_n\| = 0. \tag{3.9}$$

Using (3.2) and (3.1), we also have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\beta_n(x_n - p) + (1-\beta_n)(v_n - p)\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\alpha_n(u - p) + (1-\alpha_n)(z_n - p)\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\alpha_n\|u - p\|^2 + (1-\alpha_n)\|z_n - p\|^2) \qquad (3.10)\\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\alpha_n\|u - p\|^2 \\
&\quad + (1-\alpha_n)(\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ap\|^2)) \\
&\le \|x_n - p\|^2 + (1-\beta_n)\alpha_n\|u - p\|^2 \\
&\quad + (1-\beta_n)(1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ap\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
&(1-d)(1-\alpha_n)a(2\alpha - b)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\beta_n)\alpha_n\|u - p\|^2.
\end{aligned}$$

It follows from $(B1)$, $(B2)$ and (3.9) that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.11}$$

Using Lemma 2.5, we have

$$\begin{aligned}
&\|z_n - p\|^2 \\
&= \|J_{M,\lambda_n}(x_n - \lambda_n Ax_n) - J_{M,\lambda_n}(p - \lambda_n Ap)\|^2 \\
&\le \langle (x_n - \lambda_n Ax_n) - (p - \lambda_n Ap), z_n - p\rangle \\
&= \frac{1}{2}\big(\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\|^2 + \|z_n - p\|^2 \\
&\quad - \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap) - (z_n - p)\|^2\big) \\
&\le \frac{1}{2}\big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Ap)\|^2\big) \\
&= \frac{1}{2}\big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2 \\
&\quad - \lambda_n^2\|Ax_n - Ap\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle\big).
\end{aligned}$$

So, we have

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2 - \lambda_n^2\|Ax_n - Ap\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle. \tag{3.12}$$

Then, from (3.10) and (3.12), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(v_n - p)\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\alpha_n\|u - p\|^2 + (1-\alpha_n)\|z_n - p\|^2) \\
&\le \beta_n\|x_n - p\|^2 + \alpha_n\|u - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + \alpha_n\|u - p\|^2 + (1-\beta_n)(\|x_n - p\|^2 \\
&\quad - \|x_n - z_n\|^2 - \lambda_n^2\|Ax_n - Ap\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle) \\
&\le \|x_n - p\|^2 + \alpha_n\|u - p\|^2 - (1-\beta_n)\|x_n - z_n\|^2 \\
&\quad + 2(1-\beta_n)\lambda_n\|x_n - z_n\|\|Ax_n - Ap\|,
\end{aligned}$$

and hence,

$$(1-d)\|x_n - z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|u - p\|^2 + 2(1-\beta_n)\lambda_n\|x_n - z_n\|\|Ax_n - Ap\|.$$

Using (3.9), $\alpha_n \to 0$ and (3.11), we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.13}$$

Since $y_n = \alpha_n u + (1-\alpha_n)z_n$, we have

$$\|y_n - z_n\| = \alpha_n\|u - z_n\| \to 0, \quad (n \to \infty). \tag{3.14}$$

Since $\|v_n - y_n\| \le \|v_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\|$, from (3.8), (3.13) and (3.14), we have

$$\lim_{n\to\infty}\|v_n - y_n\| = 0. \tag{3.15}$$

Next, putting $z_0 = P_{F(S)\bigcap VI(H,A,M)}u$, we shall show that

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0. \tag{3.16}$$

Take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle. \tag{3.17}$$

Without loss of generality, we may assume that $y_{n_i} \rightharpoonup w$. Let us show $w \in F(S)\bigcap VI(H,A,M)$. From (3.14), we have $z_{n_i} \rightharpoonup w$. Since A is $\frac{1}{\alpha}-$Lipschitz continuous monotone and $D(A) = H$, by Lemma 2.7, $M + A$ is a maximal monotone mapping. Let $(v, f) \in G(M+A)$. Since $f - Av \in Mv$ and $\frac{1}{\lambda_{n_i}}(x_{n_i} - z_{n_i} - \lambda_{n_i} Ax_{n_i}) \in Mz_{n_i}$, we have

$$\langle v - z_{n_i}, (f - Av) - \frac{1}{\lambda_{n_i}}(x_{n_i} - z_{n_i} - \lambda_{n_i} Ax_{n_i})\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - z_{n_i}, f\rangle &\geq \langle v - z_{n_i}, Av + \frac{1}{\lambda_{n_i}}(x_{n_i} - z_{n_i} - \lambda_{n_i} Ax_{n_i})\rangle \\
&= \langle v - z_{n_i}, Av - Ax_{n_i}\rangle + \langle v - z_{n_i}, \frac{1}{\lambda_{n_i}}(x_{n_i} - z_{n_i})\rangle \\
&= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle \\
&\quad + \langle v - z_{n_i}, \frac{1}{\lambda_{n_i}}(x_{n_i} - z_{n_i})\rangle \\
&\geq \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle + \langle v - z_{n_i}, \frac{1}{\lambda_{n_i}}(x_{n_i} - z_{n_i})\rangle.
\end{aligned}$$

Let $i \to \infty$, we obtain $\langle v - w, f\rangle \geq 0$. Since $A + M$ is maximal monotone, we have $0 \in Aw + Mw$ and hence $w \in VI(H, A, M)$. Next, we prove $w \in F(S)$. It follows from (3.15) and $(B3)$ that

$$\|Sy_n - y_n\| = \frac{1}{1-\gamma_n}\|v_n - y_n\| \leq \frac{1}{1-\gamma}\|v_n - y_n\| \to 0, \quad (n \to \infty). \tag{3.18}$$

By Lemma 2.8, we have $w \in F(S)$. Therefore, we have $w \in F(S) \bigcap VI(H, A, M)$.

From (3.17) and the property of metric projection, we have

$$\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle = \lim_{n\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle = \langle u - z_0, w - z_0\rangle \leq 0.$$

Finally, we prove $x_n \to z_0$. In fact, since $y_n - z_0 = \alpha_n(u - z_0) + (1-\alpha_n)(z_n - z_0)$, from (3.2), Lemma 2.13 and (3.1), we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|v_n - z_0\|^2 \\
&\leq \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2 \\
&\leq \beta_n\|x_n - z_0\|^2 + (1-\beta_n)((1-\alpha_n)\|z_n - z_0\|^2 \\
&\quad + 2\alpha_n\langle u - z_0, y_n - z_0\rangle) \\
&\leq (1 - (1-\beta_n)\alpha_n)\|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n\langle u - z_0, y_n - z_0\rangle.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$ and $\limsup_{n\to\infty} 2\langle u - z_0, y_n - z_0\rangle \leq 0$, from Lemma 2.12, we have

$$\lim_{n\to\infty}\|x_n - z_0\| = 0.$$

This completes the proof. $\square$

Remark 3.2. Take $k = 0$, $\{\gamma_n\} = \{0\}$ in Theorem 3.1, we can obtain a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inclusion for a maximal monotone mapping and an $\alpha-$inverse-strongly monotone mapping.

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